AIRY WAVE PACKETS AS QUANTUM SOLUTIONS FOR RECOVERING CLASSICAL TRAJECTORIES

by

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A senior thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Bachelor of Science

Department of Physics and Astronomy

Brigham Young University

August 2006

BRIGHAM YOUNG UNIVERSITY

DEPARTMENT APPROVAL

of a senior thesis submitted by

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This thesis has been reviewed by the research advisor, research coordinator, and department chair and has been found to be satisfactory.

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ABSTRACT

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We use quantum mechanics to describe J.J. Thomson's experiment for determining the mass-to-charge ratio $\frac{m}{e}$ of the electron. We review the derivation of Thomson's classical trajectories in the electrostatic and magnetic fields. We model Thomson's capacitor and obtain the stationary wave mechanical solutions. We construct wave packets representing the transverse probability amplitude of the beam. We allow this packet to evolve in time to observe the motion of the peak and compare it with the classical trajectory. As time progresses the packet disperses and the peak delocalizes. We discuss physical and numerical causes for this phenomenon. We also use the analytic solution in momentum space to characterize the wave packets. We consider the problem in the Heisenberg representation and confirm the correspondence between quantum expectation values and classical trajectories. The uncertainties of the Thomson trajectories are identical to those of the undeflected beam.

ACKNOWLEDGMENTS

I would like to thank my family and friends for their invaluable support and encouragement during the entirety of this project. I would like to thank Jacque Jackson for the help and support she gave me. I would also like to thank Dr. Jean-Francois Van Huele for the countless hours he spent assisting and guiding me throughout the course of my research and for his valued suggestions while writing my thesis. I would especially like to thank my Mother for the irreplaceable source of strength and encouragement she has been during the entire course of my life and for always being there for me when I needed her to be. Thank you again to all those who assisted me in this endeavor.

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Chapter 1

Introduction

Classical mechanics is an approximate description of physical reality. Quantum mechanics was developed to improve it and to provide a more complete representation of physical phenomena. It also explains and predicts new phenomena like the stability of matter and its interaction with radiation.

There exist several fundamental differences between these two theories. Their application domains are typically vastly different. As a result, systems are commonly treated in the literature as either quantum or classical. Specific systems that exhibit strong classical or quantum behavior are better understood in their respective theory. Quantum theory, however, claims to be the more accurate theory for describing all physical systems. We should thus be able to describe any physical system using a quantum approach, even those where the classical description is typically sufficient. Achieving an agreement between these two descriptions of the same system is the goal of this project.

Quantum mechanics was also assisted in its development by some of the early experiments of the 20th century. One such experiment was the determination of the charge-to-mass ratio $\frac{e}{m}$ of the electron by J. J. Thomson in 1897. This discovery of the

electron was crucial in understanding the nature of the atom as well as in achieving the ability to view the universe from a quantized perspective. It is this experiment that we choose to study.

Our goal is to apply quantum theory to the Thomson experiment and give its purely quantum mechanical description. Our survey of descriptions of the Thomson experiment in the literature has led us to classical models exclusively. We have been unable to find a quantum mechanical treatment of this experiment.

Our research is exploring to what extent quantum mechanics can be used to understand the Thomson experiment. We model the capacitor through a potential and use the time-independent Schrödinger equation to find stationary solutions. We then superpose those solutions to construct wave packets. We use the wave packets to represent the cathode ray beam in Thomson's experiment and allow them to evolve in time. We deal with quantum effects such as dispersion and localization. We compare our results with the classical solutions, namely, the trajectories of the electron beam in the electric field.

In Chapter 2 we give the history of the Thomson experiment and describe the quantum mechanical model of the capacitor. In Chapter 3 we derive the classical trajectories from the force law. Chapter 4 explores the nature of the quantum solutions and describes the construction of the wave packet and its normalization. Chapter 5 addresses the time evolution of the packet, discusses methods for analyzing and controlling dispersion, and gives an analytic solution based on momentum space for comparison to the numeric results. We conclude in Chapter 6. Appendix A derives another method for obtaining the $E - B$ relationship. Appendix B gives the Maple code for the construction of the wave packet. Appendix C discusses methods for localizing the wave packet and Appendix D gives the quantum mechanical justification for treating the Thomson experiment as a quantum rather than a classical system.

Chapter 2

The Thomson Experiment

In this chapter we review some of Thomson's work. We discuss the electric and magnetic fields and their potentials and we comment on the extent that we will be modeling Thomson's setup.

2.1 J. J. Thomson's $\frac{e}{m}$ measurements

Thomson had been experimenting with the electrostatic discharge properties of various gases. For each different gas the velocity of the electrons would vary slightly. This would change the way that the ray was affected by the field because it would increase or decrease the exposure time. In turn it would change the amount of deflection that was observed. The length of the field would also change the exposure time and would therefore affect the deflection.[1]

Thomson conducted his experiment numerous times using different tubes and different gases. Although the field length and initial velocity would vary with each experiment, they would remain constant during a given experiment.[1] Thomson needed something that could be controlled during each experiment and would lead to the ratio

he was looking for. This was the purpose of the two critical variables E and B. By changing the amount of current flowing through the coil, Thomson could control the strength of the magnetic field B. Thomson could also vary the voltage between the plates and could then adjust the electric field strength E.[1]

Once a given system had been established, Thomson could make the adjustments he needed in order to balance the forces. He would obtain slightly different values each time and would eventually average his results in order to obtain the most accurate value for $\frac{e}{m}$.[1] The equations for the electrostatic and electrodynamic forces are different and are affected by different physical stipulations. They would be adjusted in different ways during the course of the experiment. The question could then be raised as to what type of relationship exists between these two field strengths. We are concerned with creating a balanced force and it might be helpful to understand this relationship in detail.

2.2 J. J. Thomson's equations

In conducting his experiment for the determination of the charge-to-mass ratio ($\frac{e}{m}$) for an electron, Thomson introduced both electric and magnetic fields. His objective was to obtain two different forces that would cancel. This cancelation leads to a lack of deflection in a cathode ray. Electrons feel an electric force from two oppositely charged conducting plates. The ray bends toward the positive and away from the negative. Electrons also feel a magnetic force. If the two forces are equal, they cancel and the cathode ray travels undisturbed.

Let us consider the point where the forces are balanced. This means that the electric and magnetic forces are equal in magnitude. We can then set the two forces

on the particle equal to each other and solve. Starting from the Lorentz force law[2]

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{2.1}$$

we specialize in the electric

$$\vec{F_1} = \vec{E}q \tag{2.2}$$

and magnetic forces

$$\vec{F_2} = q(\vec{v} \times \vec{B}). \tag{2.3}$$

The forces are equal in magnitude.

$$|\vec{F_1}| = |\vec{F_2}| \tag{2.4}$$

$$|\vec{E}|q = q|\vec{v} \times \vec{B}| \tag{2.5}$$

$$|\vec{E}| = |\vec{v} \times \vec{B}| \tag{2.6}$$

Recall that the force vectors are actually opposing each other. Thomson chose the initial velocity to be perpendicular to the magnetic field. The resultant force is perpendicular to both. The magnitude of the cross product of two orthogonal vectors gives us the product of their magnitudes.

$$|\vec{v} \times \vec{B}| = |\vec{v}||\bar{B}| \sin\theta = |\vec{v}||\vec{B}| \tag{2.7}$$

since

$$\theta = \frac{\pi}{2}. \tag{2.8}$$

We now have a simple expression for the relationship between the two field strengths. It is a function of the velocity of the particle (a very simple one)

$$\frac{E}{B} = v. \tag{2.9}$$

Thomson used this result to find the velocity of the electrons in his cathode ray. He was able to measure the strengths of the E and B fields when there was no deflection

of the beam and could thus find a value for v. This value was important because he needed it to determine the value of $\frac{e}{m}$. Thomson measured the radius of curvature R of the electrons in a magnetic field.[1] In section (3.2) we will show that the radius of this circular trajectory is given by[3]

$$R = \frac{mv}{eB}. \tag{2.10}$$

Thus

$$\frac{m}{e} = \frac{BR}{v}. \tag{2.11}$$

Thomson had measured the values of B and R and obtained v through the ratio of the two fields, Eq. (2.9). He thus was able to determine a value for $\frac{e}{m}$ which he averaged to be 1.6×10^{-7} in c.g.s units.[1]

2.3 The capacitor model and sloped wells

In classical physics we deal with forces. Particles or objects in a field experience a force. We can use Newton's equations to find the motion of these particles as they travel through space. In quantum mechanics we don't deal with forces, we deal with potentials. As such we must model the environment as a potential well rather than through a force diagram. If we consider the Thomson experiment we have an electric field being generated by a parallel plate capacitor. Neglecting edge effects at the edge of the plates, this field is uniform throughout the capacitor itself. Recall the relation between field and potential[2]

$$\vec{E} = -\vec{\nabla}V. \tag{2.12}$$

We can solve for the potential by integrating. Constant fields correspond to linear potentials. We have a linear potential in the region inside the capacitor. The potential

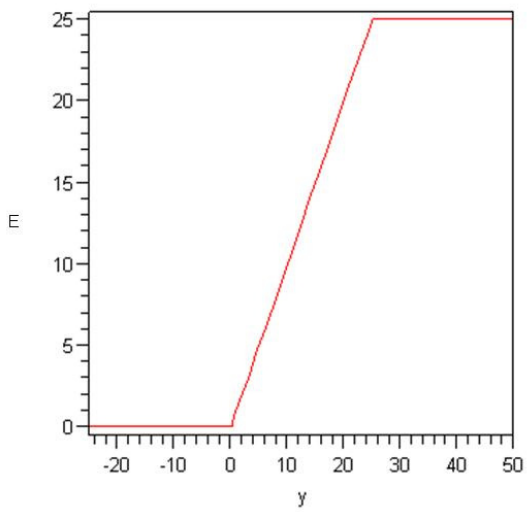

Figure 2.1 The potential energy of the parallel plate capacitor in Thomson's experiment

decreases in the direction in which the field is pointing. It is not the direction the beam is traveling in but is actually perpendicular to it.

What happens outside of the capacitor? Again, neglecting edge effects, the field outside is zero. The potential energy is zero in the region below the capacitor. Once we have traversed the field to the other side, we again have a constant potential but it is no longer zero. The potential on this side is some non-zero constant. This analysis of the potential allows us to create a model potential well for our capacitor. The capacitor can be constructed as shown in Figs. 2.1 and 2.2.

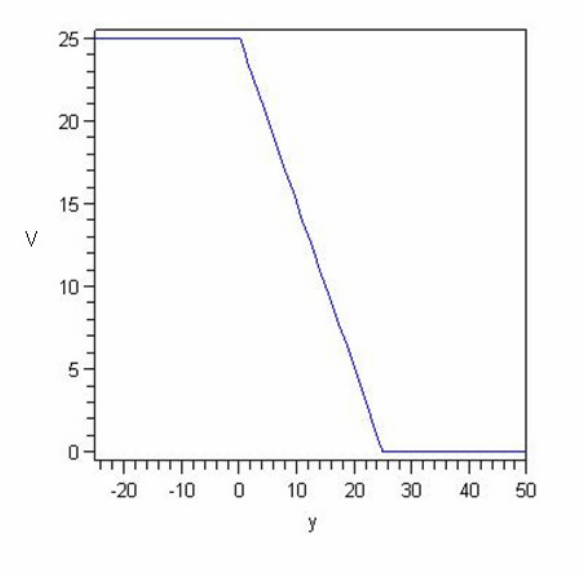

Figure 2.2 The potential of the parallel plate capacitor in Thomson's experiment

Chapter 3

Trajectories as Classical Solutions

In this chapter we obtain the trajectories of the particles based on Newtonian theory. It is important to have these trajectories as a guide for our quantum results. The electrons in this apparatus experience both an electric and a magnetic force. We model the system by considering the two fields separately, beginning with the electric field.

3.1 The electric field

To begin we select appropriate coordinates. For the case of the electric field, the trajectory will be parabolic rather than circular. I will choose to work in Cartesian coordinates because a polar coordinate system will not be of any particular advantage. The initial velocity v_0, E, and B are orthogonal in Thomson's setup. Let us consider the capacitor to be parallel to the xz plane. Neglecting edge effects we only have a force in the y direction. A real capacitor is finite and has skewed field lines around the edges. This creates forces in the xz plane. For our purposes, however, we treat the capacitor as ideal (see Fig. 3.1).

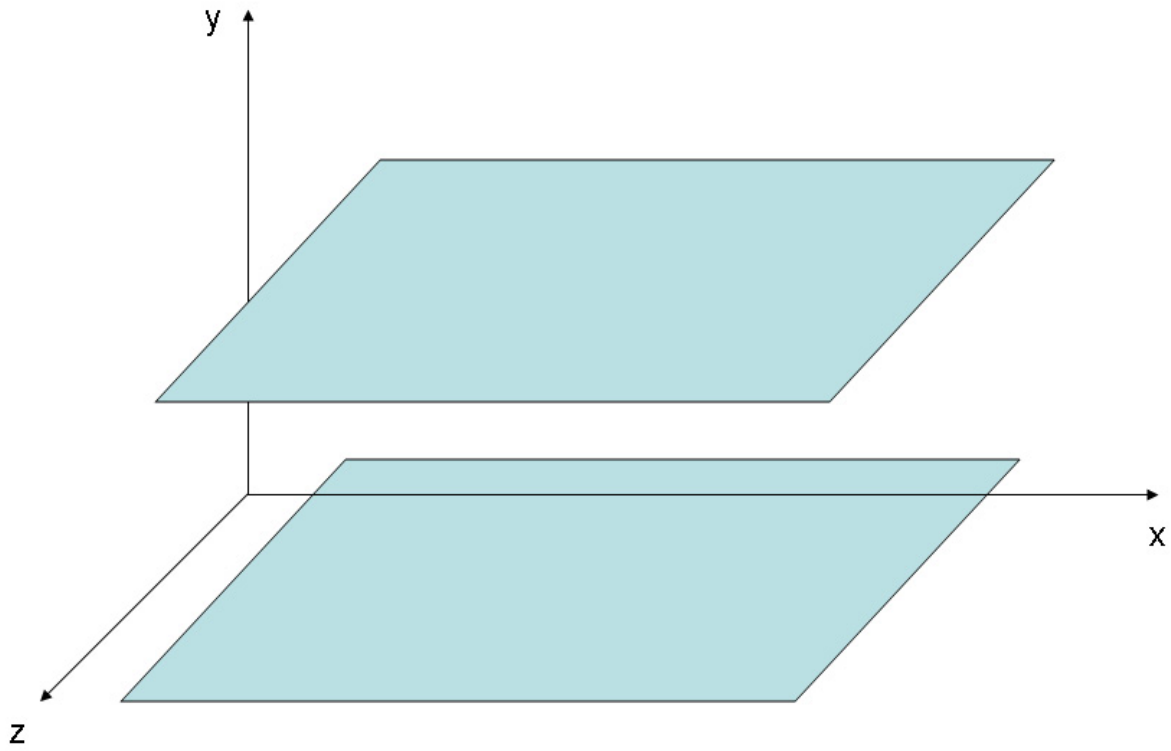

Figure 3.1 The coordinate setup for our model of the Thomson experiment

The electric force

$$\vec{F} = \vec{E}q \tag{3.1}$$

can be inserted into Newton's second law and expanded as three differential equations describing the force in each direction.

$$m\ddot{x} = 0 \tag{3.2}$$

$$m\ddot{y} = Eq \tag{3.3}$$

$$m\ddot{z} = 0 \tag{3.4}$$

Introducing the acceleration ω, where

$$\omega = \frac{Eq}{m} \tag{3.5}$$

we obtain

$$\ddot{x} = 0 \tag{3.6}$$

$$\ddot{y} = \omega \tag{3.7}$$

$$\ddot{z} = 0. \tag{3.8}$$

Our equation set contains second-order differential equations. The general solutions of these equations as functions of time are

$$x(t) = At + B \tag{3.9}$$

$$y(t) = \frac{1}{2}\omega t^2 + Ct + D \tag{3.10}$$

$$z(t) = Gt + H, \tag{3.11}$$

where A, B, C, D, G and H are constants of integration. To find particular solutions we specify boundary conditions for the functions and their first derivatives. Differentiating our general solutions gives

$$\dot{x} = A \tag{3.12}$$

$$\dot{y} = \omega t + C \tag{3.13}$$

$$\dot{z} = E. \tag{3.14}$$

By considering the initial conditions we find a physical meaning for the integration constants

$$x(0) = B \tag{3.15}$$

$$y(0) = D \tag{3.16}$$

$$z(0) = H \tag{3.17}$$

$$\dot{x_0} = \dot{x}(0) = A \tag{3.18}$$

$$\dot{y_0} = \dot{y}(0) = C \tag{3.19}$$

$$\dot{z_0} = \dot{z}(0) = G. \tag{3.20}$$

We can now rewrite our general solutions using the interpretation of the constants that we just discovered

$$x(t) = \dot{x_0}t + x_0 \tag{3.21}$$

$$y(t) = \frac{1}{2}\omega t^2 + \dot{y_0}t + y_0 \tag{3.22}$$

$$z(t) = \dot{z_0}t + z_0. \tag{3.23}$$

We have a quadratic relation perpendicular to the capacitor axis and linear relations along the capacitor. We can combine the two linear equations

$$t = \frac{z - z_0}{\dot{z_0}} \tag{3.24}$$

$$t = \frac{x - x_0}{\dot{x_0}} \tag{3.25}$$

to eliminate the time from our solution. The resulting equation describes the parabolic trajectory as it is projected onto the xz plane. The relation is linear because it is an overhead view of the parabola

$$\frac{z - z_0}{\dot{z_0}} = \frac{x - x_0}{\dot{x_0}}. \tag{3.26}$$

We can also eliminate t to create spatial representations of the trajectory in y as functions of either x or z. These equations describe the projection of the parabola on either the xy or the yz planes

$$y(x) = \frac{1}{2}\omega(\frac{x - x_0}{\dot{x_0}})^2 + \dot{y_0}(\frac{x - x_0}{\dot{x_0}}) + y_0 \tag{3.27}$$

$$y(z) = \frac{1}{2}\omega(\frac{z - z_0}{\dot{z_0}})^2 + \dot{y_0}(\frac{z - z_0}{\dot{z_0}}) + y_0. \tag{3.28}$$

If we allow the initial x and z coordinates to be zero we can find an equation for the parabola in terms of an r axis. This r axis lies in the xz plane and is the horizontal distance the particle has traveled

$$x_0 = 0 \qquad z_0 = 0$$

$$r = \sqrt{x^2 + z^2} = \sqrt{\dot{x_0}^2 t^2 + \dot{z_0}^2 t^2} = t\sqrt{\dot{x_0}^2 + \dot{z_0}^2} = tv_0. \tag{3.29}$$

From the Pythagorean relation we can solve for r in terms of x and z. We can then use our earlier values for t to relate t and r

$$t = \frac{r}{v_0} \tag{3.30}$$

$$y(r) = \frac{1}{2}\omega\left(\frac{r^2}{v_0^2}\right) + \dot{y_0}\left(\frac{r}{v_0}\right) + y_0. \tag{3.31}$$

Let us find an expression for the angle at which the beam is deflected after passing through the capacitor. The particle enters the capacitor at the origin. For simplicity the initial velocities in the y and z directions are also chosen to vanish. After making these assumptions we can simplify the equation for y in terms of x

$$y(x) = \frac{1}{2}\omega\frac{x^2}{\dot{x_0}^2}. \tag{3.32}$$

When the particle leaves the field, it follows a straight line that is tangent to its trajectory

$$y'(x) \equiv \frac{dy(x)}{dx} = \frac{\omega x}{\dot{x_0}^2}. \tag{3.33}$$

This derivative gives the slope of the parabola. We are interested in the slope when the particle exits the field, namely $x = L$

$$y'(L) = \frac{\omega L}{\dot{x_0}^2}. \tag{3.34}$$

We know that the slope of a line is equal to the tangent of the angle it forms with the x axis

$$y'(L) = \tan\theta. \tag{3.35}$$

If θ is small, as is the case in Thomson's experiment, we can approximate the tangent with its argument[3]

$$\theta \ll \frac{\pi}{2} \qquad \Longrightarrow \qquad \tan\theta \approx \theta. \tag{3.36}$$

The deflection angle can then be written as

$$\theta \approx \frac{EqL}{m\dot{x_0}^2}. \tag{3.37}$$

3.2 The magnetic field

Now we turn our attention to the magnetic field. Taking an approach similar to the electric field case, we begin by considering the forces acting on the particle. We again approximate our system to be ideal. We will consider a constant, uniform magnetic field acting in the positive z direction

$$B_x = 0 \qquad B_y = 0 \qquad B_z = B. \tag{3.38}$$

This means that the Lorentz force will be in the xy plane,

$$\vec{F} = q(\vec{v} \times \vec{B}). \tag{3.39}$$

We again use Newton's second law to model the force in each direction

$$m\ddot{x} = q(\vec{v} \times \vec{B})_x \tag{3.40}$$

$$m\ddot{y} = q(\vec{v} \times \vec{B})_y \tag{3.41}$$

$$m\ddot{z} = 0. \tag{3.42}$$

Working out the cross product we obtain the components in the force equation,

$$m\ddot{x} = \dot{y}Bq \tag{3.43}$$

$$m\ddot{y} = -B\dot{x}q \tag{3.44}$$

$$m\ddot{z} = 0. \tag{3.45}$$

Introducing

$$\omega = \frac{qB}{m} \tag{3.46}$$

the differential equations simplify further,

$$\ddot{x} = \omega\dot{y} \tag{3.47}$$

$$\ddot{y} = -\omega\dot{x} \tag{3.48}$$

$$\ddot{z} = 0. \tag{3.49}$$

It would be easier for us to solve this system if it were uncoupled, meaning each equation contained only one variable. We introduce variables for the first derivatives of x and y:

$$\chi = \dot{x} \tag{3.50}$$

$$\Upsilon = \dot{y} \tag{3.51}$$

as well as

$$\dot{\chi} = \ddot{x}, \tag{3.52}$$

$$\dot{\Upsilon} = \ddot{y}. \tag{3.53}$$

Combining Eqs. (3.47)-(3.51) we obtain

$$\dot{\chi} = \omega\Upsilon \qquad \dot{\Upsilon} = -\omega\chi \tag{3.54}$$

and finally

$$\ddot{\chi} = -\omega^2 \chi \tag{3.55}$$

$$\ddot{\Upsilon} = -\omega^2 \Upsilon. \tag{3.56}$$

We now have differential equations with constant coefficients that can be solved directly for χ and Υ as functions of t,

$$\chi(t) = A\cos\omega t + B\sin\omega t \tag{3.57}$$

$$\Upsilon(t) = C\cos\omega t + D\sin\omega t. \tag{3.58}$$

We must remember, however, that χ and Υ were the first derivatives of x and y. We are really interested in finding $x(t)$ and $y(t)$ so we must integrate Eqs. (3.57), (3.58), and (3.49)

$$x(t) = \frac{A}{\omega}\sin\omega t - \frac{B}{\omega}\cos\omega t + C_1 \tag{3.59}$$

$$y(t) = \frac{C}{\omega}\sin\omega t - \frac{D}{\omega}\cos\omega t + C_2 \tag{3.60}$$

$$z(t) = Gt + H. \tag{3.61}$$

We notice that the motion describes a periodic closed trajectory. Let us consider first the two constants, C_1 and C_2. With some foresight we can consider the average value of integrating the functions over one period. The average value is obtained by integrating over the period and dividing by its length

$$\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{A}{\omega}\sin\omega t - \frac{B}{\omega}\cos\omega t + C_1\right)dt = C_1 \tag{3.62}$$

$$\bar{y} = \frac{1}{2\pi}\int_0^{2\pi} y(t)dt = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{C}{\omega}\sin\omega t - \frac{D}{\omega}\cos\omega t + C_2\right)dt = C_2 \tag{3.63}$$

leading to

$$\bar{x} = C_1, \qquad \bar{y} = C_2. \tag{3.64}$$

Indeed we have found the average values of these functions. Let us try now to interpret the physical nature of these constants. Equations (3.59) and (3.60) are

parametric equations that suggest elliptical motion. Since the magnetic field is perpendicular to the velocity at all times, the speed is constant and the motion is circular. The magnetic field does no work.[2] When we consider an orbiting particle, the average position of that particle is the center of the circle. This is what we have solved for. We have found two coordinates (C_1,C_2) for the center of the circle that the electron describes in the plane orthogonal to the magnetic field.

We look for the physical significance of the integration constants A, B, C, D, G, and H that remain in our equation. This information comes from the initial conditions of the system. We will need the functions and their first derivatives. The motion is completely described by the following set of equations

$$x(t) = \frac{A}{\omega}\sin\omega t - \frac{B}{\omega}\cos\omega t + C_1 \tag{3.65}$$

$$y(t) = \frac{C}{\omega}\sin\omega t - \frac{D}{\omega}\cos\omega t + C_2 \tag{3.66}$$

$$z(t) = Gt + H \tag{3.67}$$

$$\dot{x}(t) = A\cos\omega t + B\sin\omega t \tag{3.68}$$

$$\dot{y}(t) = C\cos\omega t + D\sin\omega t \tag{3.69}$$

$$\dot{z}(t) = G \tag{3.70}$$

which at the initial time become another set

$$x_0 = -\frac{B}{\omega} + C_1 \tag{3.71}$$

$$y_0 = -\frac{D}{\omega} + C_2 \tag{3.72}$$

$$z_0 = H \tag{3.73}$$

$$\dot{x}_0 = A \tag{3.74}$$

$$\dot{y}_0 = C \tag{3.75}$$

$$\dot{z}_0 = G. \tag{3.76}$$

Solving the second set for the constants and substituting in the first set we obtain

$$x(t) = \frac{\dot{x}_0}{\omega}\sin\omega t + (x_0 - \bar{x})\cos\omega t + \bar{x} \tag{3.77}$$

$$y(t) = \frac{\dot{y}_0}{\omega}\sin\omega t + (y_0 - \bar{y})\cos\omega t + \bar{y} \tag{3.78}$$

$$z(t) = \dot{z}_0 t + z_0. \tag{3.79}$$

We have already mentioned that the parametric equations we are dealing with are suggestive of a circular orbit. As is the case of the electric field it will be important to have a detailed description of this orbit for comparing with the quantum results later on. Let us then turn our attention to the circle. We choose a coordinate system in a way that is consistent with the electric field from the Thomson experiment and that will allow us to simplify our results. We could come to expect a path that is similar to Fig. 3.2.

Since we are dealing with a closed orbit, we expect some kind of force acting toward the center. This force is required to keep the particle in its trajectory. Force is a vector that acts along the direction of the particle's acceleration. We know that the particle has an acceleration component acting toward the center of the circle. We refer back to the diagram to see if we can make any conjectures. We notice that by the choice of the coordinate system there are certain places along the path that are going to have zero acceleration along a given component. For example, a vertical arrow has no acceleration in the x direction.

With this in mind we refer back to our equations and evaluate them at different intervals. Combining what we have observed about the acceleration with the evaluated equations will allow the simplification to become evident.

Let's evaluate the x and y components of the acceleration on the axes. We can take the derivative of the velocity equations to find functions for the acceleration in

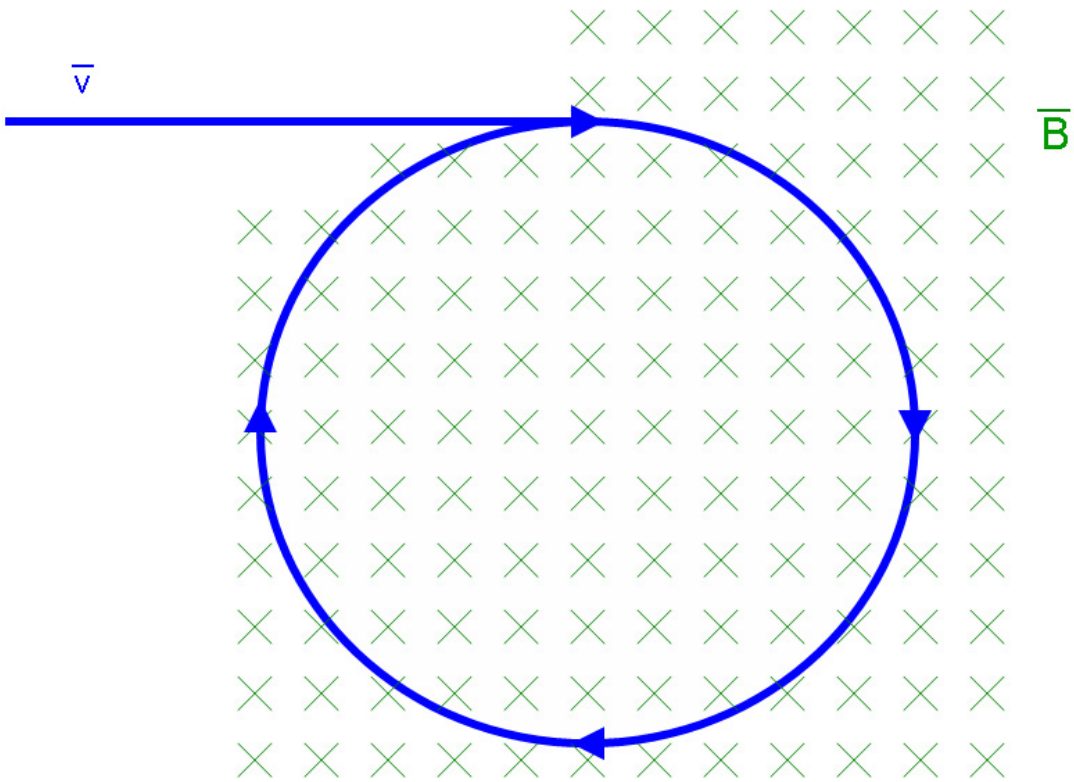

Figure 3.2 The trajectory of an electron in a constant magnetic field

each direction

$$\ddot{x}(t) = -\dot{x}_0\omega \sin\omega t + (\bar{x} - x_0)\omega^2 \cos\omega t \tag{3.80}$$

$$\ddot{y}(t) = -\dot{y}_0\omega \sin\omega t + (\bar{y} - y_0)\omega^2 \cos\omega t \tag{3.81}$$

$$\ddot{z}(t) = 0. \tag{3.82}$$

We now look at the different points along the circle. The argument of the trigonometric functions is ωt corresponding to an angular frequency multiplied by time. The value of the argument gives the angle in radians around the circle. For example $\omega t = \frac{3\pi}{2}$ leads to a location on the negative x axis. Choosing successively $t = 0, \frac{\pi}{2\omega}, \frac{\pi}{\omega}$ and $\frac{3\pi}{\omega}$ we obtain

$$\ddot{y}\left(\frac{\pi}{2\omega}\right) = 0 \qquad \Longrightarrow \qquad \dot{y}_0 = 0 \tag{3.83}$$

$$\ddot{x}\left(\frac{\pi}{2\omega}\right) = -\dot{x}_0\omega \tag{3.84}$$

$$\ddot{y}(0) = (\bar{y} - y_0)\omega^2 \tag{3.85}$$

$$\ddot{x}(0) = 0 \qquad \Longrightarrow \qquad \bar{x} = x_0 \tag{3.86}$$

$$\ddot{y}\left(\frac{\pi}{\omega}\right) = -(\bar{y} - y_0)\omega^2 \tag{3.87}$$

$$\ddot{x}\left(\frac{\pi}{\omega}\right) = 0 \qquad \Longrightarrow \qquad \bar{x} = x_0 \tag{3.88}$$

$$\ddot{y}\left(\frac{3\pi}{2\omega}\right) = 0 \qquad \Longrightarrow \qquad \dot{y}_0 = 0 \tag{3.89}$$

$$\ddot{x}\left(\frac{3\pi}{2\omega}\right) = \dot{x}_0\omega. \tag{3.90}$$

As a result of this process we simplify the equations. Using $B = C = 0$ in Eqs. (3.77) and (3.78) gives

$$x(t) = \frac{\dot{x}_0}{\omega} \sin\omega t + \bar{x} \tag{3.91}$$

$$y(t) = -(y_0 - \bar{y}) \cos\omega t + \bar{y}. \tag{3.92}$$

Since the motion is circular, the magnitude of the acceleration in the negative y direction at $t = 0$ will be equal in magnitude to that of the negative x direction at $t = \frac{\pi}{2}$. This will create yet another relationship between the constants

$$\ddot{y}(0) = \ddot{x}\left(\frac{\pi}{2\omega}\right) \qquad \Longrightarrow \qquad (\bar{y} - y_0)\omega^2 = -\dot{x}_0\omega \tag{3.93}$$

$$\ddot{y}\left(\frac{\pi}{\omega}\right) = \ddot{x}\left(\frac{3\pi}{2\omega}\right) \qquad \Longrightarrow \qquad (y_0 - \bar{y})\omega^2 = \dot{x}_0\omega \tag{3.94}$$

$$\omega(y_0 - \bar{y}) = \dot{x}_0. \tag{3.95}$$

We can now express both position equations in terms of only one unknown constant

$$x(t) = \frac{\dot{x}_0}{\omega}\sin\omega t + \bar{x} \tag{3.96}$$

$$y(t) = \frac{\dot{x}_0}{\omega}\cos\omega t + \bar{y}. \tag{3.97}$$

Through a little manipulation with some prior knowledge we alter Eqs. (3.96) and (3.97) further to obtain

$$x - \bar{x} = \frac{\dot{x}_0}{\omega}\sin\omega t \tag{3.98}$$

$$y - \bar{y} = \frac{\dot{x}_0}{\omega}\cos\omega t. \tag{3.99}$$

Squaring both sides,

$$(x - \bar{x})^2 = \frac{\dot{x}_0^2}{\omega^2}\sin^2\omega t \tag{3.100}$$

$$(y - \bar{y})^2 = \frac{\dot{x}_0^2}{\omega^2}\cos^2\omega t, \tag{3.101}$$

and adding the two equations together while invoking a trigonometric identity gives

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \frac{\dot{x}_0^2}{\omega^2}\sin^2\omega t + \frac{\dot{x}_0^2}{\omega^2}\cos^2\omega t \tag{3.102}$$

$$= \frac{\dot{x}_0^2}{\omega^2}(sin^2\omega t + cos^2\omega t) = \frac{\dot{x}_0^2}{\omega^2}. \tag{3.103}$$

We have succeeded in getting our equation into the form that we anticipated

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \frac{\dot{x_0}^2}{\omega^2}. \tag{3.104}$$

Equation (3.104) describes a circle centered at $(\bar{x}, \bar{y})$. The term on the right corresponds to the radius R squared

$$R = \frac{\dot{x_0}}{\omega} \tag{3.105}$$

or in terms of the physical parameters

$$R = \frac{m\dot{x}_0}{qB} \tag{3.106}$$

and

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \frac{m^2\dot{x}_0^2}{q^2B^2}. \tag{3.107}$$

The constant ω has a well-defined physical significance. We first determine the SI units

$$[\omega] = \frac{C}{Kg}\frac{Kg}{Cs} = \frac{1}{s} = 1Hz \tag{3.108}$$

corresponding to a frequency. Indeed it is a frequency, often called the cyclotron frequency, or the frequency at which the particle cycles through one revolution around the circle just found.[3]

Returning to the Thomson experiment, with the goal of comparing our results with Thomson's, what we are really concerned with is what Thomson saw when he did his experiment. The radius is a useful parameter but Thomson never saw a radius, he saw a deflection. Let us extract the deflection from our solutions.

The particle will follow a circular trajectory centered at some point. We aren't really concerned about where the circle is centered because we are only dealing with part of the curve. Thomson's electron will be leaving the field before it has completed a significant part of the circumference so t is small.[3] Using Eq. (3.104)

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \frac{\dot{x_0}^2}{\omega^2} \tag{3.109}$$

we can solve for the curve as a function of x. Selecting the positive root gives

$$y = \bar{y} + \sqrt{R^2 - (x - \bar{x})^2}. \tag{3.110}$$

We are dealing with a situation where the particle travels tangent to the curve of the circle. This would lead us to believe that we will be needing the derivative.

$$y' = \frac{-(x - \bar{x})}{\sqrt{R^2 - (x - \bar{x})^2}} \tag{3.111}$$

This expression gives us the slope of the curve at that point. Since the slope is equal to the tangent of the angle, we would have to take the inverse tangent of this expression in order to find θ. Thomson, however, was only concerned with small angles for the most part. As a result, we are probably safe to use the small angle approximation for tangents.

$$\theta \approx \tan\theta \tag{3.112}$$

This approximation will then give us a final result that we can use to find the angle. If we relabel $(x - x_0)$ as L then we can write

$$\theta \approx \frac{-L}{\sqrt{R^2 - L^2}}. \tag{3.113}$$

At this point we can make another approximation. L is small compared to R so we can eliminate the L^2 term and we are left with:

$$\theta \approx \frac{-L}{\sqrt{R^2}} = \frac{-L}{R}. \tag{3.114}$$

Recall that

$$R = \frac{mv}{qB}, \tag{3.115}$$

which results in our final expression for the deflection angle

$$\theta \approx -\frac{qBL}{mv} = -\frac{qBL}{m\dot{x}_0}. \tag{3.116}$$

3.3 The electromagnetic field

In order to remain consistent with the Thomson experiment we consider the classical description of the case in which both fields are acting on the electron. In section

3.1 we found that the force from the electric field acting on the electron could be described by Eq. (3.3). It is a differential equation describing the force in the y direction,

$$m\ddot{y} = Eq. \tag{3.117}$$

Using the right-hand rule confirms that when the electron first enters the magnetic field it experiences a force in the y direction. In section 3.2 we found that this force could be described by Eq. (3.44)

$$m\ddot{y} = -B\dot{x}q. \tag{3.118}$$

The net force acting on the electron in the y direction is then the sum of the two forces

$$m\ddot{y} = Eq - B\dot{x}q. \tag{3.119}$$

Thomson, however, arranged his setup so there would not be any deflection of the beam.[1] This means that the net force on the particle is zero and Eq. (3.119) becomes

$$0 = Eq - B\dot{x}q. \tag{3.120}$$

Which gives

$$Eq = B\dot{x}q \tag{3.121}$$

$$E = B\dot{x} \tag{3.122}$$

$$\dot{x} = \frac{E}{B}. \tag{3.123}$$

Substituting this result back into Eq. (3.119) gives

$$m\ddot{y} = Eq - Eq = 0, \tag{3.124}$$

as we established by assuming no deflection. We found in section 3.1 that the force from the electric field acts only in the y direction and since the velocity remains solely

along x in this case, there is no force from the magnetic field in the x or z directions. This means

$$m\ddot{x} = 0 \tag{3.125}$$

$$m\ddot{y} = 0 \tag{3.126}$$

$$m\ddot{z} = 0 \tag{3.127}$$

and

$$x(t) = \dot{x}_0 t + x_0 \tag{3.128}$$

$$y(t) = \dot{y}_0 t + y_0 \tag{3.129}$$

$$z(t) = \dot{z}_0 t + z_0. \tag{3.130}$$

Knowing that $\dot{y}_0 = 0$ and $\dot{z}_0 = 0$ gives

$$x(t) = \dot{x}_0 t + x_0 \tag{3.131}$$

$$y(t) = y_0 \tag{3.132}$$

$$z(t) = z_0 \tag{3.133}$$

which is the straight line trajectory we expected.

Chapter 4

Wave Packets as Quantum Solutions

When we consider the classical version of the Thomson experiment, we rely upon Newton's equations to obtain trajectories. Newton's equations deal with forces but the quantum treatment is interested in potentials. We use an equation that reflects the wave nature of our quantum solutions $\Psi(x)$. The foundation for the quantum description comes from the equation governing nonrelativistic wave mechanics, the Schrödinger equation. For time-independent potentials we can use the time-independent Schrödinger equation[4]

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} = (E - U(x))\Psi(x). \tag{4.1}$$

This equation is a second-order differential equation in one dimension. Here E stands for the total energy of the system. The term $U(x)$ represents the potential energy of the particle in its environment. The term in parentheses $(E - U(x))$ can therefore be considered as the kinetic energy of the particle. We solve for the wavefunction $\Psi(x)$.

4.1 Constant potentials

The term in the Schrödinger equation that governs the behavior of the wavefunction is the potential energy term $U(x)$. We consider some relevant potentials. In the simplest possible case $U(x) = 0$ and there is no external influence

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} = E\Psi(x). \tag{4.2}$$

Using $k = \sqrt{\frac{2mE}{\hbar^2}}$ we get

$$\frac{d^2\Psi(x)}{dx^2} = -k^2\Psi(x). \tag{4.3}$$

This Ordinary Differential Equation (ODE) says that the second derivative of a function is equal to itself multiplied by a negative constant. This is in very different circumstances the same equation as that encountered in Eq. (3.56). The solution to this equation consists of sines and cosines. We know from superposition that any linear combination of these solutions will also be a solution.[3] This gives us a generic form for the wavefunction in a zero-potential region

$$\Psi(x) = A\sin kx + B\cos kx. \tag{4.4}$$

It is a stationary solution and its time dependence has been removed. We can express it as a complex exponential or a cosine term with a phase shift

$$\Psi(x) = \left(\frac{A+B}{2}\right)e^{ikx} + \left(\frac{-A+B}{2}\right)e^{-ikx} \tag{4.5}$$

$$\Psi(x) = \tilde{B}\cos(kx+\phi). \tag{4.6}$$

Another simple case is a constant potential V_0 different from zero. Suppose also that the potential V_0 is greater than the total energy E of the particle. If this is the case, it changes the sign in front of $\Psi(x)$ in Eq. (4.1)

$$\frac{d^2\Psi(x)}{dx^2} = \kappa^2\Psi(x) \tag{4.7}$$

where $\kappa = \sqrt{\frac{2m(E-V_0)}{-\hbar^2}}$. Again, we have a simple differential equation but the solution has a different form now. Rather than trigonometric functions we obtain exponentials. The generic solution from superposition is

$$\Psi(x) = Ae^{\kappa x} + Be^{-\kappa x}. \tag{4.8}$$

These solutions apply to the two domains outside of the capacitor.

4.2 Linear potentials and Airy functions

If we refer back to our model for the capacitor from section 2.3 we notice that in the central region the potential is linear. We can set up the Schrödinger equation to solve for solutions in this region. For $U(x) = \alpha x$, a linear potential energy, we obtain

$$\frac{d^2}{dx^2}\Psi\left(x\right) = \left(\alpha\, x - E\right)\Psi\left(x\right) \tag{4.9}$$

where the constant slope α depends on the strength of the applied electric field. This equation is not as easy to solve because we no longer have a constant multiplying the wavefunction. Solving this equation using Maple we find

$$\Psi\left(x\right) = \_C1\ AiryAi\left(\frac{\alpha\, x - E}{\left(-\alpha\right)^{2/3}}\right) + \_C2\ AiryBi\left(\frac{\alpha\, x - E}{\left(-\alpha\right)^{2/3}}\right), \tag{4.10}$$

where C_1 and C_2 are integration constants. $AiryAi$ and $AiryBi$ are Maple's way of expressing a special class of functions called Airy functions. These functions are related to the more familiar Bessel functions and they come in two types, Ai and Bi.[5] A plot of $AiryAi$ is displayed in Fig. 4.1. We can make a few observations about the behavior of this function. Towards the right the function decays to zero once it crosses into the positive x values after reaching a final maximal peak. Towards the left the function oscillates at an increasing pace as it dies off. This oscillation makes sense physically if we consider what is happening to the electron. As the electron

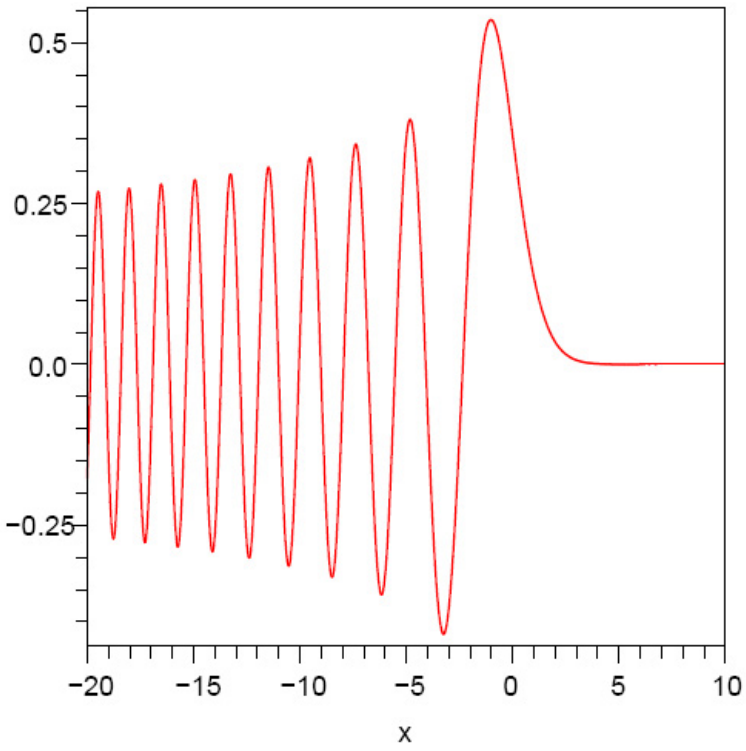

Figure 4.1 The Airy Ai Function

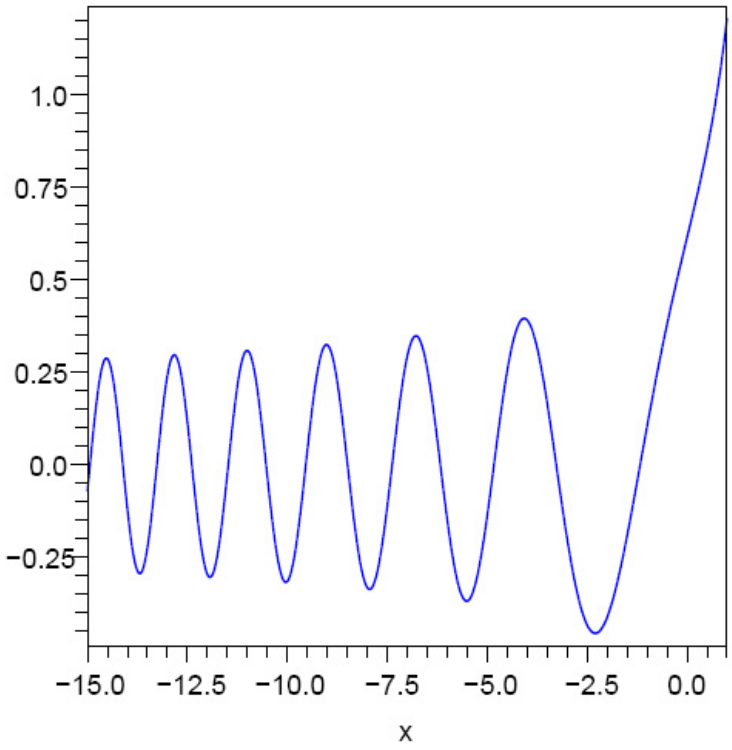

Figure 4.2 The Airy Bi Function

traverses this sloped well, it sees a decreasing potential energy and thus has more kinetic energy. As a result, the oscillations become increasingly rapid on the left and the amplitude decreases as the particle spends less time at any individual location.

We have a second solution as is expected for a second order differential equation, $AiryBi$. It is similar to the $AiryAi$ function but has a different asymptotic behavior on the right. A plot of this function is shown in Fig. 4.2. We see that $AiryBi$ blows up in the positive region and has the same oscillating behavior on the left. This unboundedness in the positive region will limit our use of $AiryBi$ later.

4.3 Airy wave packets

Having established a model for the potential energy of the capacitor we can now solve the Schrödinger equation to obtain wavefunctions. Sections 4.1 and 4.2 explored solutions to this equation in the types of potentials that compose our capacitor model. We can split our model up into three regions based on the type of potential that exists in these regions. We consider zero, linear, and constant potential regions. The solutions in each region are given by

$$\Psi_1(x) = A\sin kx + B\cos kx \tag{4.11}$$

$$\Psi_2(x) = \_C1\ AiryAi\left(\frac{\alpha\, x - E}{(-\alpha)^{2/3}}\right) + \_C2\ AiryBi\left(\frac{\alpha\, x - E}{(-\alpha)^{2/3}}\right) \tag{4.12}$$

$$\Psi_3(x) = Ae^{kx} + Be^{-kx} \tag{4.13}$$

as found in sections 4.1 and 4.2. We make two simplifications. The first simplification occurs in Ψ_2. The term $(-\alpha)^{\frac{2}{3}}$ in the denominator of the Airy function's argument can be imaginary. We are only concerned with real functions so we must choose α in a way that will allow us to retain real solutions. If we allow the slope to be unity then we can choose the real roots of $(-\alpha)^{\frac{2}{3}}$ and it will allow the factor to be equal to one. The second simplification comes in Ψ_3. We want our function to go to zero at positive infinity. The positive exponent blows up in the positive x direction. We can eliminate it from our solution. After making these simplifications and organizing our unknown constants we obtain (remember that $\alpha = 1$ in what follows)

$$\Psi_1(x) = A\sin kx + B\cos kx \tag{4.14}$$

$$\Psi_2(x) = C\,AiryAi\left(\frac{\alpha\, x - E}{\alpha}\right) + D\,AiryBi\left(\frac{\alpha\, x - E}{\alpha}\right) \tag{4.15}$$

$$\Psi_3(x) = Fe^{-kx}. \tag{4.16}$$

At this point we make a numeric simplification. It was mentioned in section 4.1 that $k = \sqrt{\frac{2mE}{\hbar^2}}$. We establish our system with natural units such that $\frac{2m}{\hbar^2} = 1$. This implies that $k = \sqrt{E}$.

We must now apply boundary conditions to our functions in order to obtain a solution for the entire system. The boundary condition on the left requires that the function and its derivative be continuous at the boundary between Ψ_1 and Ψ_2.[4] Using the functions with adjusted units gives

$$B = CAiryAi\left(-\frac{E}{\alpha}\right) + DAiryBi\left(-\frac{E}{\alpha}\right), \tag{4.17}$$

having evaluated Ψ_1 at $x = 0$ to obtain B on the left side of Eq. (4.17). We also use the condition that the derivatives of Ψ_1 and Ψ_2 must be equal at $x = 0$. Taking the derivative of these functions and evaluating them at $x = 0$ gives

$$A\sqrt{E} = CAiryAi\left(1, -\frac{E}{\alpha}\right) + DAiryBi\left(1, -\frac{E}{\alpha}\right) \tag{4.18}$$

where the number 1 in the argument of the $AiryAi$ and $AiryBi$ functions, $\left(1, -\frac{E}{\alpha}\right)$, refers to the first derivatives of $AiryAi\left(-\frac{E}{\alpha}\right)$ and $AiryBi\left(-\frac{E}{\alpha}\right)$ respectively. We divide one equation by the other and get

$$\frac{B}{A\sqrt{E}} = \frac{\left(CAiryAi\left(-\frac{E}{\alpha}\right) + DAiryBi\left(-\frac{E}{\alpha}\right)\right)}{\left(CAiryAi\left(1, -\frac{E}{\alpha}\right) + DAiryBi\left(1, -\frac{E}{\alpha}\right)\right)}. \tag{4.19}$$

We also have a boundary condition on the right at $x = d$ where d is the length of the capacitor. The symbol L has been used for the same quantity. We have already taken care of the end behavior by choosing the exponential term that decays rather than blowing up, so we are left with conditions at $x = d$. We have the same conditions at $x = d$ for Ψ_2 and Ψ_3 as we do at $x = 0$ for Ψ_1 and Ψ_2. The functions and their derivatives must be equal when evaluated at the boundary point. Setting the functions equal to each other gives

$$CAiryAi\left(\frac{\alpha\, d - E}{\alpha}\right) + DAiryBi\left(\frac{\alpha\, d - E}{\alpha}\right) = Fe^{-\sqrt{\alpha\, d - E}d}. \tag{4.20}$$

Setting their derivatives equal gives

$$CAiryAi\left(1,\frac{\alpha\,d-\,E}{\alpha}\right)+DAiryBi\left(1,\frac{\alpha\,d-\,E}{\alpha}\right)=-\,F\sqrt{\alpha\,d-\,E}e^{-\sqrt{\alpha\,d-E}d}. \quad (4.21)$$

Dividing the two equations produces

$$\frac{\left(CAiryAi\left(\frac{\alpha\,d-E}{\alpha}\right)+DAiryBi\left(\frac{\alpha\,d-E}{\alpha}\right)\right)}{\left(CAiryAi\left(1,\frac{\alpha\,d-E}{\alpha}\right)+DAiryBi\left(1,\frac{\alpha\,d-E}{\alpha}\right)\right)}=-\,\frac{1}{\sqrt{\alpha\,d-\,E}}. \quad (4.22)$$

We now have two equations Eq. (4.19) and Eq. (4.22) and four unknowns A, C, D, and F. We reduce this system to two unknowns by simplifying down to a ratio of constants. In Eq. (4.19) we already have a ratio of $\frac{B}{A}$. We can obtain a ratio for $\frac{C}{D}$. The way we accomplished this in Maple was to set $D = 1$ and solve Eq. (4.22) for C. This result was really equal to the ratio of $\frac{C}{D}$. We did the same with Eq. (4.19) by setting $B = 1$ and solving for A, obtaining the ratio for $\frac{A}{B}$. We set $D = 1$ and $C = \frac{C}{D}$ and solve Eq. (4.20) for $\frac{F}{D}$. We set the same conditions and solve Eq. (4.17) for $\frac{B}{D}$. We take the expressions for each of these combinations of constants and substitute them into Ψ_1, Ψ_2 and Ψ_3 as follows,

$$\Psi_1(x)=\frac{B}{D}\left(\frac{A}{B}\sin kx+\cos kx\right)=A\sin kx+B\cos kx \quad (4.23)$$

$$\Psi_2\,(x)=\frac{C}{D}AiryAi\left(\frac{\alpha\,x-E}{\alpha}\right)+AiryBi\left(\frac{\alpha\,x-E}{\alpha}\right) \quad (4.24)$$

$$=CAiryAi\left(\frac{\alpha\,x-E}{\alpha}\right)+DAiryBi\left(\frac{\alpha\,x-E}{\alpha}\right) \quad (4.25)$$

$$\Psi_3(x)=\frac{F}{D}(e^{-kx})=Fe^{-kx}. \quad (4.26)$$

This method works because we set $D = 1$ and we found expressions for $\frac{B}{D}$, $\frac{A}{B}$, $\frac{C}{D}$ and $\frac{F}{D}$.

We now have the correct boundary conditions applied to our wavefunctions in each region. We construct a piecewise function in Maple consisting of each of these solutions in their respective domain. This piecewise solution is a function of x and

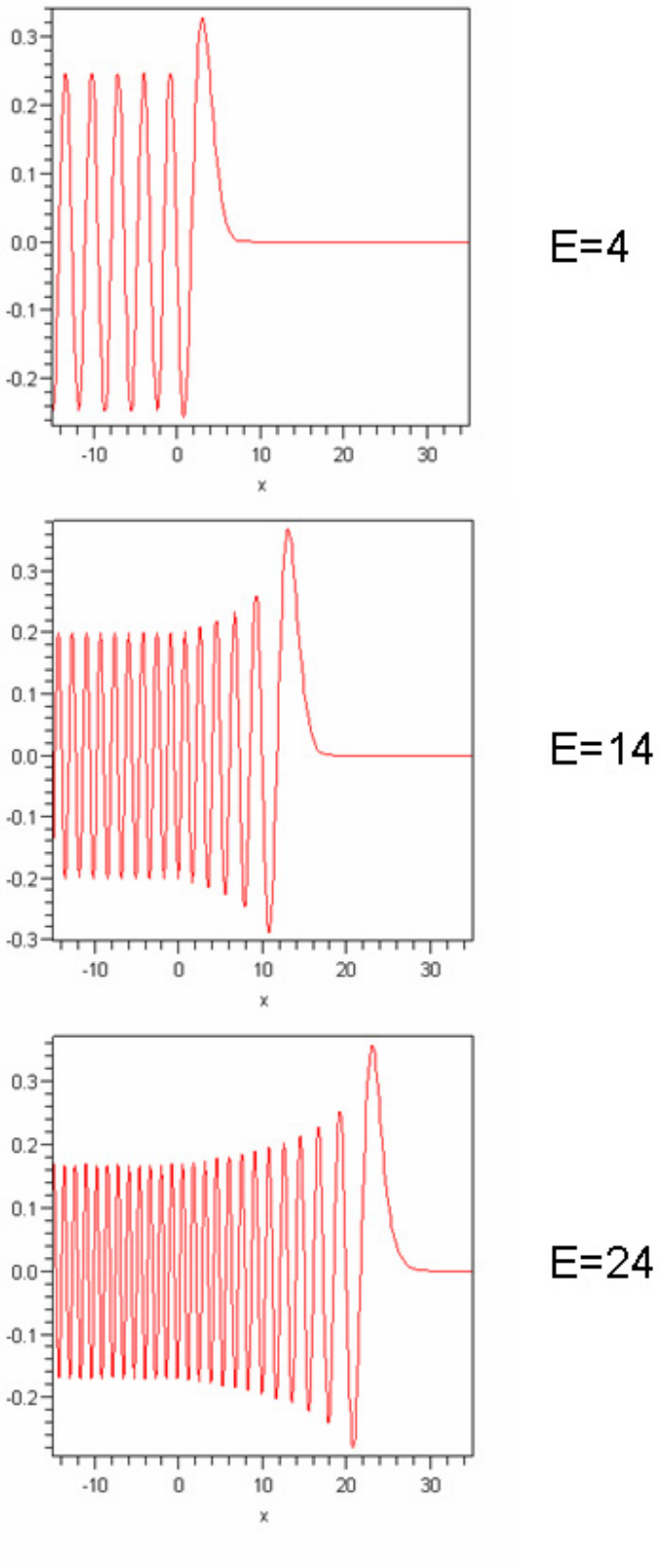

Figure 4.3 The wavefunction solution plotted for 3 different values of the energy

also a function of E, the energy of the particle. Figure 4.3 displays plots of this piecewise function for different values of the energy E. The wavefunction behaves as expected in each of the three regions. The behavior is also affected by the value of the energy.

We have found the wavefunction that is the solution to our model of the capacitor in the Thomson experiment. We must now use this wavefunction to construct a localized wave packet. We construct our wave packet as a δ function. The wavefunction is multiplied by a weight factor and integrated,[4]

$$\delta(x-a) = \int_0^\infty \Psi_E(x)\Psi_E(a)dE. \tag{4.27}$$

Maple is not able to analytically integrate the complicated combination of Airy functions that exists in the expression for our wavefunction. Therefore we must approximate the integral with a finite sum. We also want to localize our wave packet around a specific point. The wavefunction squared is going to correspond to the probability density for the cathode ray. We localize the beam in the center of the capacitor which, according to our model in Fig. 2.1, is equivalent to $x = 12.5$. Thus we have

$$\delta(x-12.5) = \sum_{E=0}^{n} \Psi_E(x)\Psi_E(12.5). \tag{4.28}$$

We substitute this value of x into Ψ_E in our Maple code and sum over a range of energies. This finite sum may actually have benefits because of the Heisenberg uncertainty. The less localized our wave packet is, the less dispersive it will be.

4.4 Normalization

Once we had completed the necessary steps for constructing our wave packet, there remained a plaguing error that prevented us from obtaining a localized peak. We had summed the Airy terms and constructed a wave packet but it was not localized as

we had expected it to be. We noticed in the plots of our wavefunction that changing the value of the energy was also changing the amplitude. We made a table of values corresponding to the different energies and the amplitudes they corresponded to. There did not seem to be a pattern and some of the amplitudes were several orders of magnitude larger than others. This was an interesting result and it was important because it helped us to realize that even though our wave packet was normalized in the end, we had neglected to normalize the wavefunctions from which the packet was constructed. The relation Eq. (4.27) holds for normalized Ψ_E.

We returned to the beginning of our Maple code where we had first composed our piecewise wavefunction from the individual wavefunctions in each region. A term was added to the code which summed the squares of $\Psi(x)_1$, $\Psi(x)_2$ and $\Psi(x)_3$ over their respective regions. Again we had to approximate the integral with a sum because of Maple's inability to analytically integrate the expressions. This approximation accounts for the reason that some of the curves in the plots don't appear to enclose an area of exactly one. Summing each wavefunction over a sufficiently large region gives

$$n_{1E}^2 = \sum_{x=-25}^{0} \Psi_E(x)_1^2 \tag{4.29}$$

$$n_{2E}^2 = \sum_{x=0}^{25} \Psi_E(x)_2^2 \tag{4.30}$$

$$n_{3E}^2 = \sum_{x=25}^{35} \Psi_E(x)_3^2. \tag{4.31}$$

We then obtain the normalization constant

$$N_E = \frac{1}{\sqrt{n_{1E}^2 + n_{2E}^2 + n_{3E}^2}}. \tag{4.32}$$

We multiplied our wavefunction by this normalization constant and constructed the approximate δ function again

$$\delta(x - 12.5)_N = \sum_{E=0}^{25} N_E \Psi_E(x) \Psi_E(12.5) \tag{4.33}$$

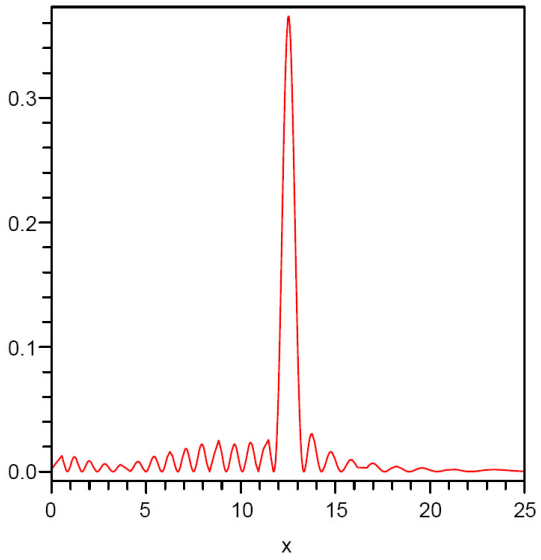

Figure 4.4 The approximate δ function representing the transverse probability amplitude of the electron beam in the Thomson experiment

Figure 4.4 shows a plot of this approximate δ function representing the probability amplitude for the beam of electrons in the Thomson experiment. This plot is made for a sum containing 23 terms. A more detailed explanation of how to further localize these functions by increasing the number of terms is given in Appendix C. In what follows we leave out the index E from the expressions for Ψ and N. It is understood that all wavefunctions have been properly normalized.

Chapter 5

Deflection from Wave Packet Evolution

Now that we have constructed a wave packet we follow its behavior in the potential region as time progresses. We include the time dependence by multiplying the stationary states by the factor $e^{-i\omega t}$ in the expression for the δ function[4]

$$\delta(x - x_c, t)_N = \sum_{E=0}^{n} N\Psi(x)\Psi(x_c)e^{-i\omega t} \tag{5.1}$$

where $\omega = \frac{E}{\hbar}$ and x_c is the center of the capacitor. In our setup $x_c = 12.5$. Based on the results of the Thomson experiment and our classical derivations we can make some assumptions about the wave packet evolution. We should obtain a packet that moves to the left at an accelerating rate. This corresponds to the probability amplitude of the electron as it is accelerated through the electric field.

5.1 Dispersion

Once we have obtained an expression for the time-dependent wave packet we animate it in Maple to see how it behaves. The surprising results illustrated in Fig. 5.1

show that the peak starts out localized at $t = 0$ as constructed. As time progresses, however, the packet rapidly disperses and we lose any information about the motion of the packet. This is seen in Figs. 5.2 and 5.3. Figure 5.4 is an overlay of times $t = 0$, $t = 0.3$, $t = 0.6$, and $t = 1$.

Dispersion is a quantum effect that happens to wave packets as time progresses. We expect to see some dispersion in our packet as it moves through time but it is quite surprising to see it dominating the motion of the peak. We investigate this dispersion thoroughly and attempt several techniques to understand its origins.

5.2 Dispersion-controlling techniques

5.2.1 Expectation values

One of the techniques is to look at the expectation value for the wave packet as it moves through time. To compute the expectation value we use its general definition:[4]

$$< x >= \frac{\int_{-\infty}^{\infty} x|\psi(x)|^2 dx}{\int_{-\infty}^{\infty} |\psi(x)|^2 dx}. \tag{5.2}$$

Again we approximate the integral in Maple using a sum,

$$< x >= \frac{\sum_{x=0}^{30} x|\psi(x)|^2 dx}{\sum_{x=0}^{30} |\psi(x)|^2 dx}. \tag{5.3}$$

The Maple code for this operation is given below

```
exp:=sum(subs(t=0,x*shift),x=0..30)/sum(subs(t=0,shift),x=0..30);.
```

The Maple function shift stands for the solution $\psi(x)$ or $\delta(x-x_c)_N$. We construct an array in Maple that takes the expectation value and pairs it with the corresponding value of t that it was created from. This creates a scatter plot of expectation values plotted against time. We want to see how the expectation value changes relative to

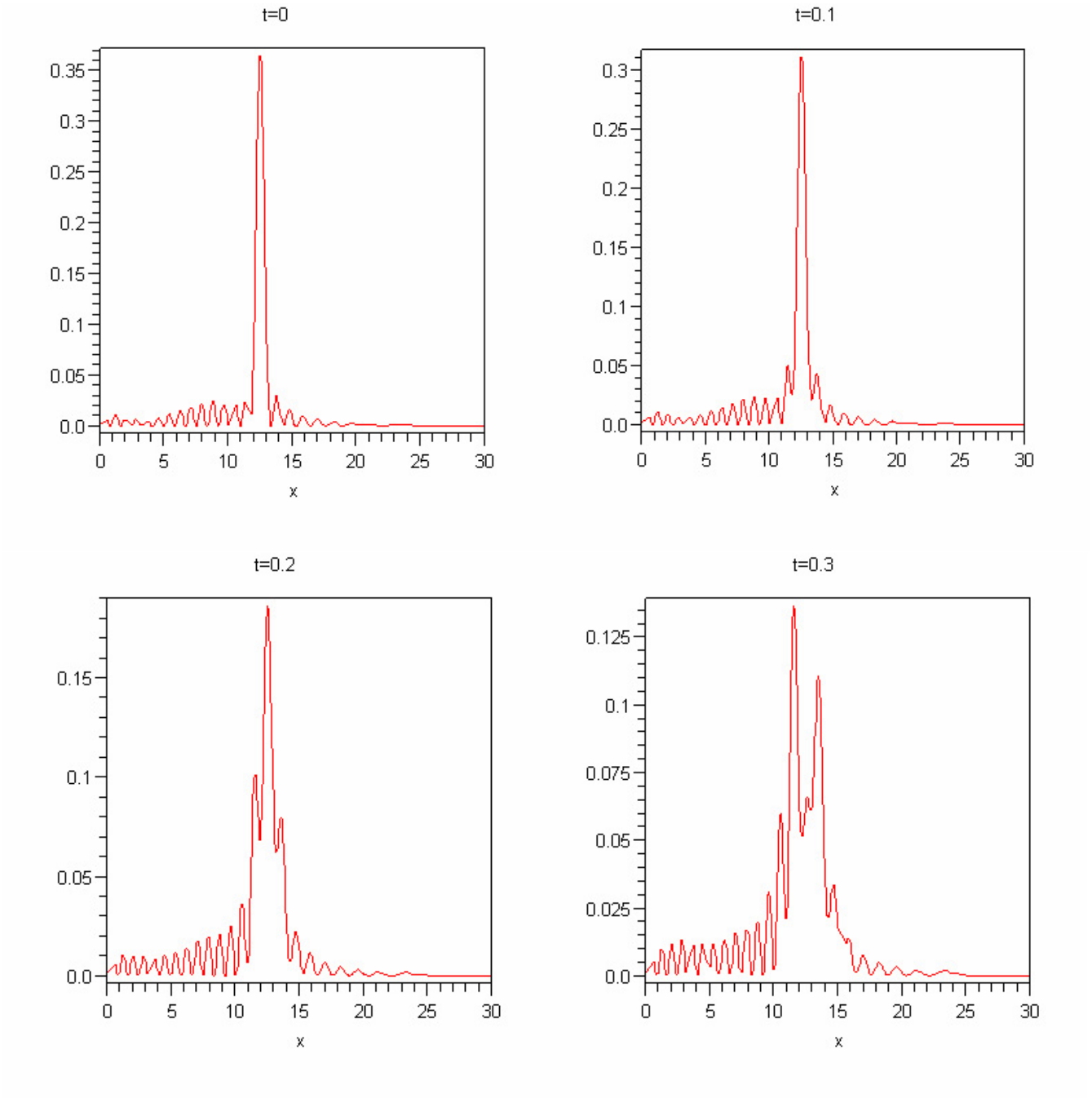

Figure 5.1 Time evolution of the wave packet for $0 \leq t \leq 0.3$

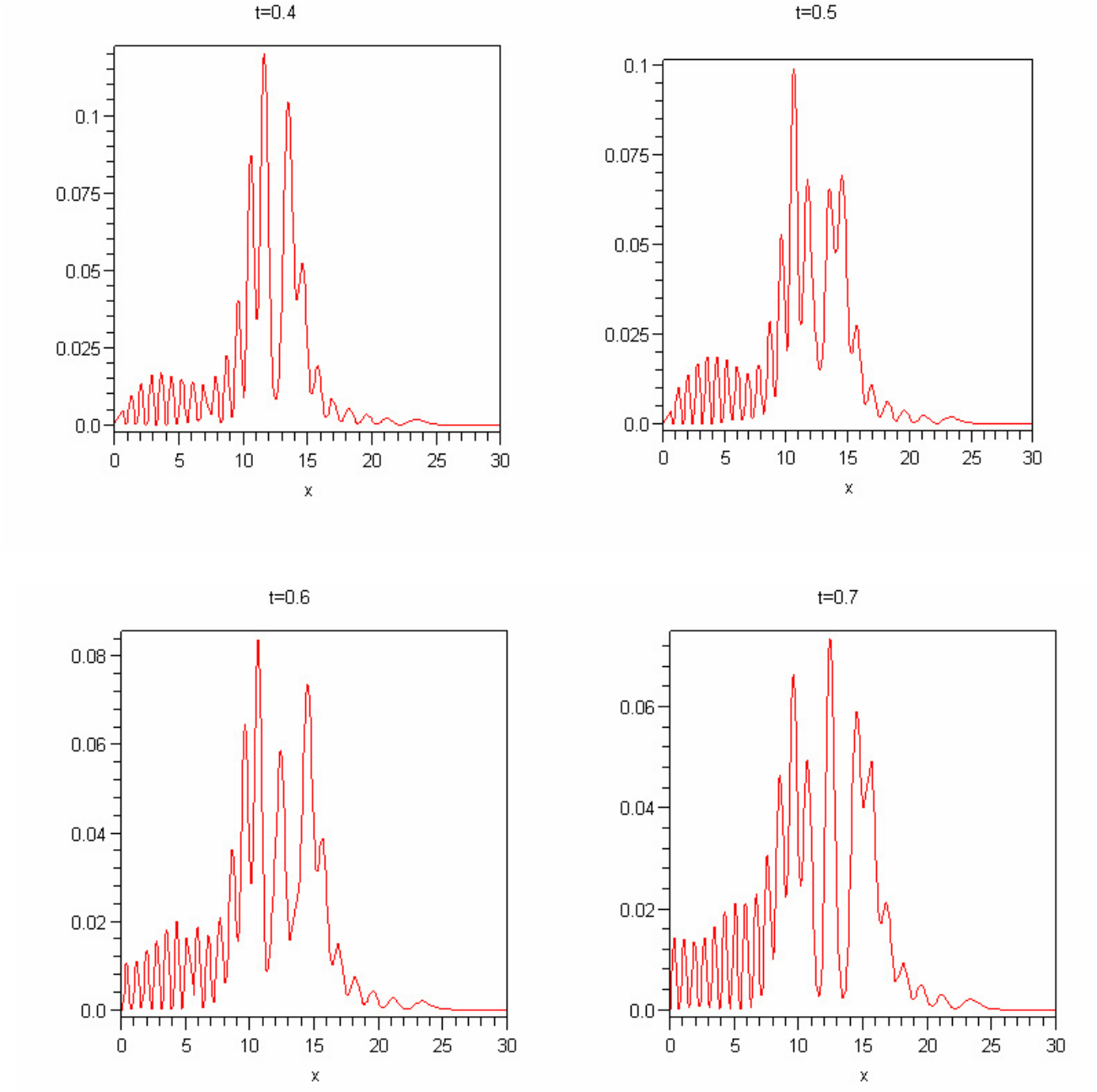

Figure 5.2 Time evolution of the wave packet for $0.4 \leq t \leq 0.7$

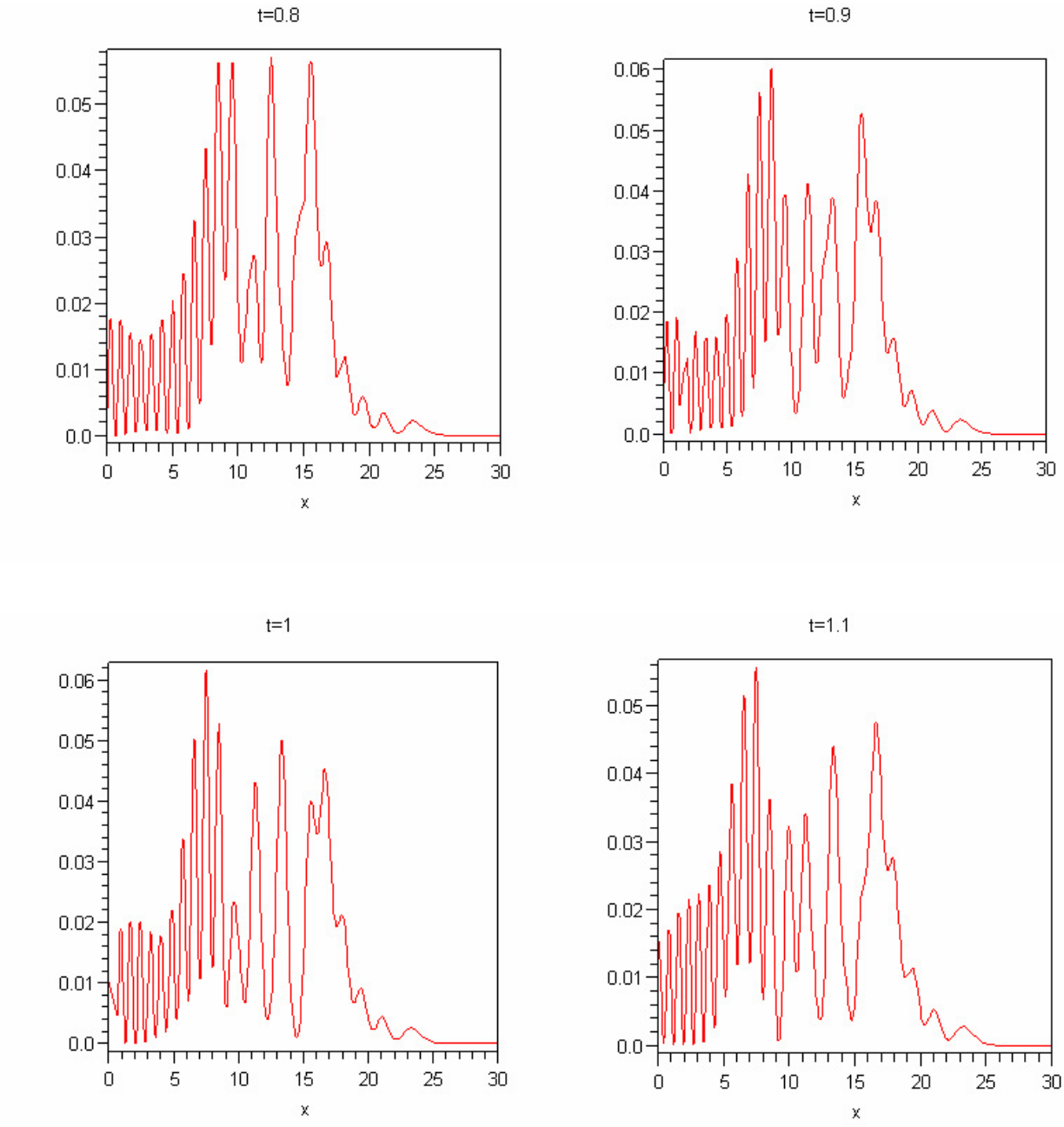

Figure 5.3 Time evolution of the wave packet for $0.8 \leq t \leq 1.1$

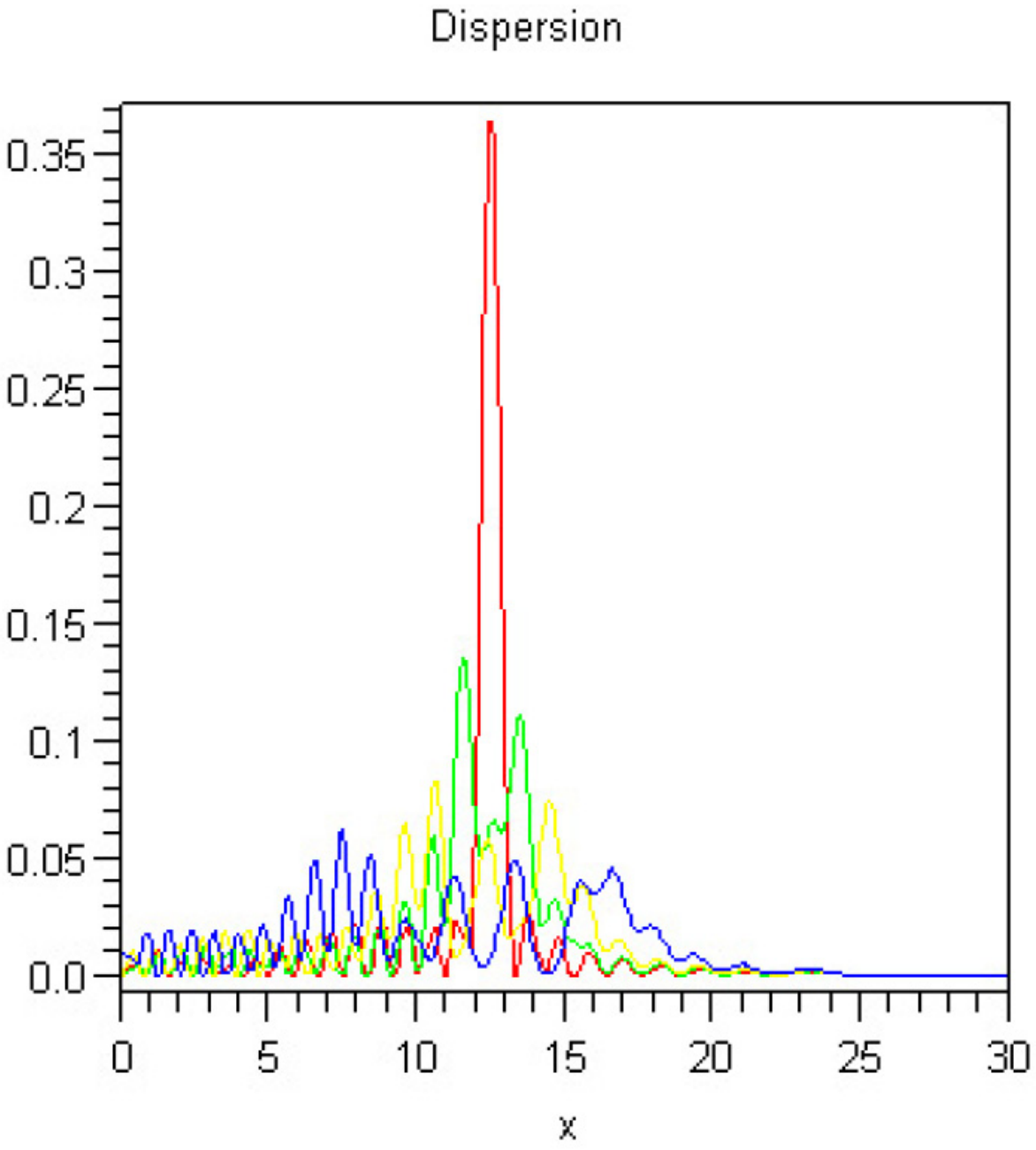

Figure 5.4 An overlay of four plots ($t = 0$, $t = 0.3$, $t = 0.6$ and $t = 1$) showing the dispersion of the wave packet (color in electronic version)

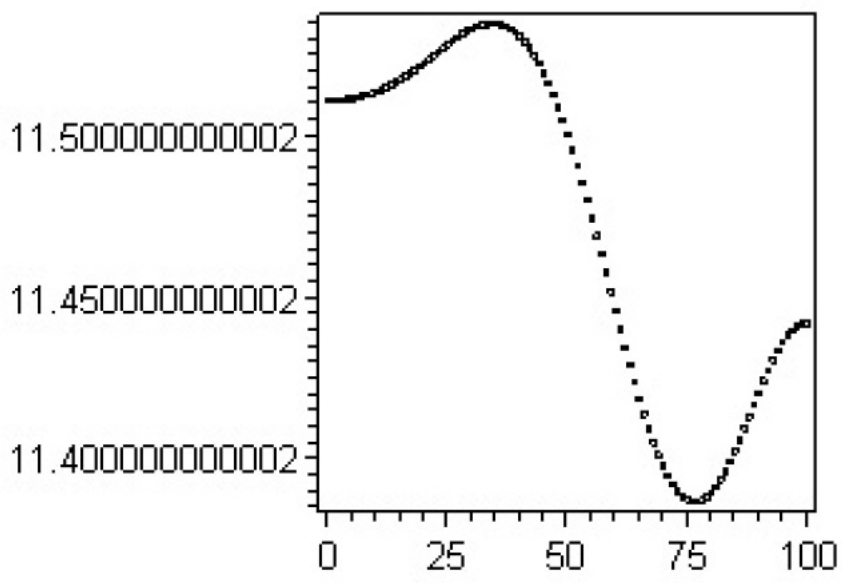

Figure 5.5 The expectation value $< x >$ plotted against time for a wave packet consisting of 23 terms

the spreading of the wave packet. It is interesting to note that at time $t = 0$, the expectation value is closer to $x = 11.5$ than to $x = 12.5$ which is where the beam was localized. This is a result of the small amplitude oscillations on either side of the peak. Figure 5.1 shows a plot of the localized beam at $t = 0$. It can be seen from this figure that the area under the oscillations on the left of the peak is larger than the area on the right. This shifts the expectation value of the peak to the left because of the integrals in the expression for the expectation value.

The importance of the way in which the wave packet was constructed becomes apparent in the motion of the expectation value. Depending on which terms are taken to create the sum, the wave packet will disperse differently. We would expect a parabola that opened downward for a plot of the expectation value. This would indicate that the electron is accelerating to the left. If we construct the scatter plot for the wave packet composed of 23 terms we obtain Fig. 5.5. This expectation value

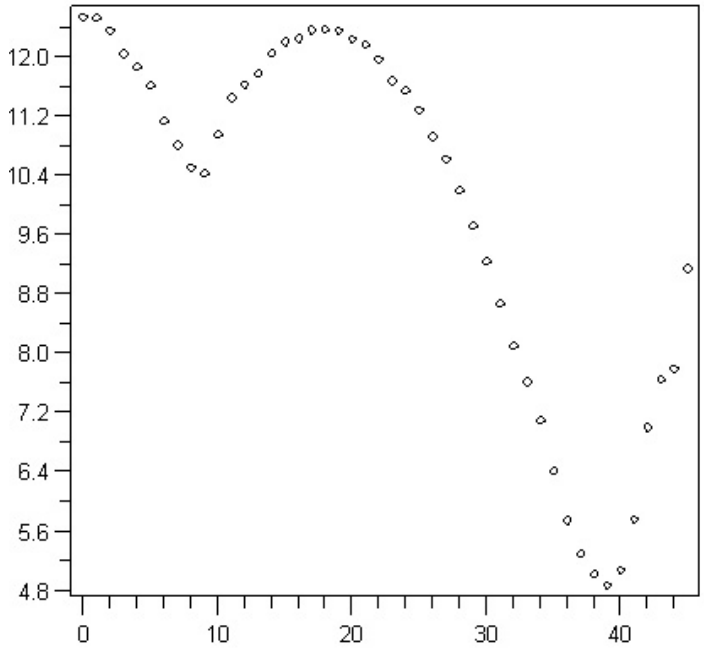

Figure 5.6 The expectation value plotted against time for a wave packet consisting of 248 terms

oscillates back and forth without moving a significant distance in either direction. If we refine our grid so that we take 248 terms rather than 23 we get a much more localized peak. If we perform the same analysis on its expectation value we find Fig. 5.6. This plot still does not look like what is expected. However, it does move a significant distance in the desired direction. It is particularly interesting to notice the initial few seconds of the plot. The initial motion of this expectation value is a parabola that opens downward as expected. It could be suggested that the peak is trying to move to the left as we had expected but that its motion is being washed out by dispersion. Significant dispersion happens only one time unit into the evolution. It is possible that, once the peak has been defined using enough terms, the expectation value begins moving as expected, only to be biased out by the noise created by the dispersion of the packet.

We tried several other techniques to understand the nature of the spreading including an analysis using sine waves instead of Airy functions. We did measurements of the time required for significant dispersion and compared those times for different parameters. Section 5.4 will discuss our findings.

While conducting the investigation into the nature of the dispersion, we were able to learn about the behavior of the wave packet. We tried changing the slope of the potential region, changing the relative dimensions of the capacitor and other alterations to our model. We also tested several ways of selecting which energy values to take when we were creating our sum.

5.2.2 Energy level selection

We take more terms to create a finer grid in the sum in Eq. (4.33). We do this to improve the approximation made in expressing the integral as a sum. Taking more terms leads to more localized wave packets initially but does not remove the dispersion in time.

Another method consists of changing the selection of the terms. Rather than spacing them evenly over the entire range we select them by hand. We do this to avoid coincidences from the existing periodicity in the frequency of the energy. Otherwise it might lead to resonances that would be misleading and falsely represent the data over the entire range. Again this method does affect the packets but does not remove the dispersion.

As mentioned in the discussion of expectation values, the way in which the packet is constructed affects its behavior. Taking more terms creates a more localized peak. Because of the Heisenberg uncertainty relation, a more localized wave packet will disperse more quickly.[4] Our packet disperses much faster when it is composed of more terms. However, it does lead to a more accurate expectation value plot.

Taking different combinations of values and trying to stagger the energy terms affects the packet and the way the packet disperses. Each method seems to have its own unique effect leading to slightly different dispersion in each case.

5.3 Robinett's analytic solution

To avoid the difficulties of the numerical analysis of the wave packet we follow a description of this problem found in Robinett.[6] An analytic solution can be found in momentum space and then Fourier transformed back into position representation. Rather than modeling the wave packet with δ functions, Gaussian distributions model the probability amplitude for the electron. We will discuss this selection in detail in the current and following sections.

The discussion below follows Robinett.[6] He begins with the time-dependant Schrödinger equation in momentum space,

$$\frac{p^2}{2m}\phi(p,t) - F\left[i\hbar\frac{\partial}{\partial p}\right]\phi(p,t) = i\hbar\frac{\partial\phi(p,t)}{\partial t} \tag{5.4}$$

or

$$i\hbar\left(F\frac{\partial\phi(p,t)}{\partial p} + \frac{\partial\phi(p,t)}{\partial t}\right) = \frac{p^2}{2m}\phi(p,t) \tag{5.5}$$

where F plays the role of our slope, $-F = \alpha$. Assume a solution of the form $\phi(p,t) = \Phi(p - Ft)\tilde{\phi}(p)$. With this form, Eq. (5.5) reduces to

$$\frac{\partial\tilde{\phi}(p)}{\partial p} = -\frac{ip^2}{2m\hbar F}\tilde{\phi}(p) \tag{5.6}$$

with the solution

$$\tilde{\phi}(p) = e^{\frac{-ip^3}{6mF\hbar}}. \tag{5.7}$$

The general solution can then be written as

$$\phi(p,t) = \Phi(p - Ft)e^{\frac{-ip^3}{6mF\hbar}} \tag{5.8}$$

or, using the arbitrariness of $\Phi(p)$, as

$$\phi(p,t) = \phi_0(p - Ft)e^{\frac{i((p-Ft)^3 - p^3)}{6mF\hbar}} \tag{5.9}$$

where $\phi_0(p)$ is some initial momentum distribution since $\phi(p,0) = \phi_0(p)$. It is interesting to note that if the initial momentum distribution is characterized by $< p >_0 = p_0$, then using a change of variables gives

$$< p >_t = \int_{-\infty}^{\infty} p|\phi(p,t)|^2 dp \tag{5.10}$$

$$= \int_{-\infty}^{\infty} p|\phi_0(p - Ft)|^2 dp. \tag{5.11}$$

Using $q = p - Ft$ gives

$$\int_{-\infty}^{\infty} q|\phi_0(q)|^2 dq + Ft\int_{-\infty}^{\infty} |\phi_0(q)|^2 dq \tag{5.12}$$

$$= < p >_0 + Ft. \tag{5.13}$$

Thus

$$< p >_t = < p >_0 + Ft \tag{5.14}$$

which is a result comparable to Eq. (D.5). This equation states that the average momentum value increases linearly with time, consistent with the classical result for a constant force, $F = \frac{dp}{dt}$. The momentum distribution translates uniformly to the right with no change in shape since

$$|\phi(p,t)|^2 = |\phi_0(p - Ft)|^2. \tag{5.15}$$

The corresponding position space wavefunction can then be obtained through a Fourier transform

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi_0(p - Ft)e^{\frac{i((p-Ft)^3 - p^3)}{6mF\hbar}} e^{\frac{ipx}{\hbar}} dp. \tag{5.16}$$

Since the p^3 terms cancel in the exponent this transform can be done analytically for a Gaussian momentum distribution characterized by a parameter γ. In that case, we have

$$\phi_0(p) = \sqrt{\frac{\gamma}{\sqrt{\pi}}} e^{\frac{-\gamma^2 p^2}{2}} \tag{5.17}$$

so that

$$\psi(x,t) = \frac{1}{\sqrt{\gamma\hbar\sqrt{\pi}(1+\frac{it}{t_0})}} e^{\frac{iFt}{\hbar}(\frac{x-Ft^2}{6m})} e^{-\frac{(x-\frac{Ft^2}{2m})^2}{2\hbar^2\gamma^2(1+\frac{it}{t_0})}} \tag{5.18}$$

where the spreading time is defined by $t_0 \equiv m\hbar\gamma^2$. The corresponding probability density is then

$$|\psi(x,t)|^2 = \frac{1}{\beta_t\sqrt{\pi}} e^{-\frac{(x-\frac{Ft^2}{2m})^2}{\beta_t^2}} \tag{5.19}$$

where $\beta_t = \hbar\gamma\sqrt{1+\frac{t^2}{t_0^2}}$. This analytic solution has produced a Gaussian distribution as the description of the probability amplitude. We modify Eq. (5.19) slightly by recalling that our setup requires the packet to move to the left. This introduces an additional negative sign in the exponent. We also added a term in the denominator, σ to control the width of the peak and shifted the peak into the center of the capacitor at $x = 12.5$. The Maple code for this modified Gaussian function is given below,

```
f:=(1/sigma)*1/(betat*sqrt(Pi))*exp(((-(-(x-12.5)-t^2)^2)
   /((sigma^2)*betat^2)));
```

where

```
betat:=sqrt(1+0.0001*t^2):.
```

Here we have set $t_0 = 10000$ so that the peak would spread slowly. We have also included the natural units and the choice of our slope equal to one. This gives

$$f = \frac{1}{\sigma\beta_t\sqrt{\pi}} e^{-\frac{\left(-x+12.5-t^2\right)^2}{\sigma^2{\beta_t}^2}}. \tag{5.20}$$

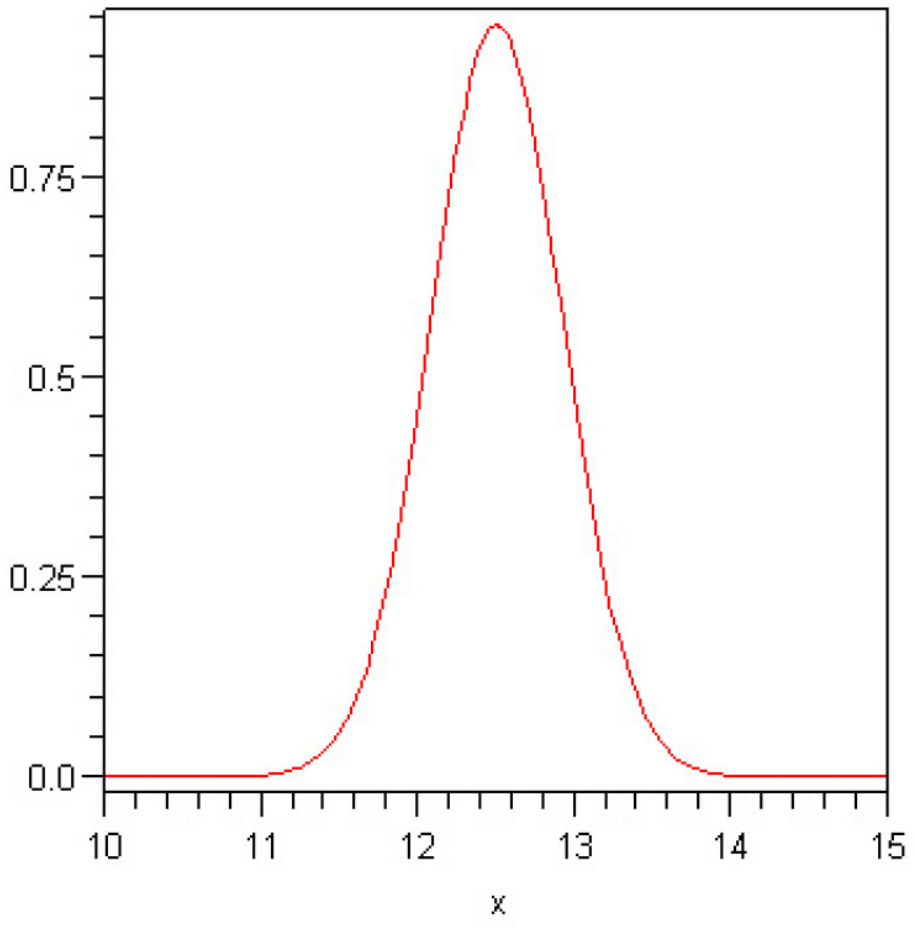

Figure 5.7 The Gaussian function representing the probability amplitude from the analytic solution

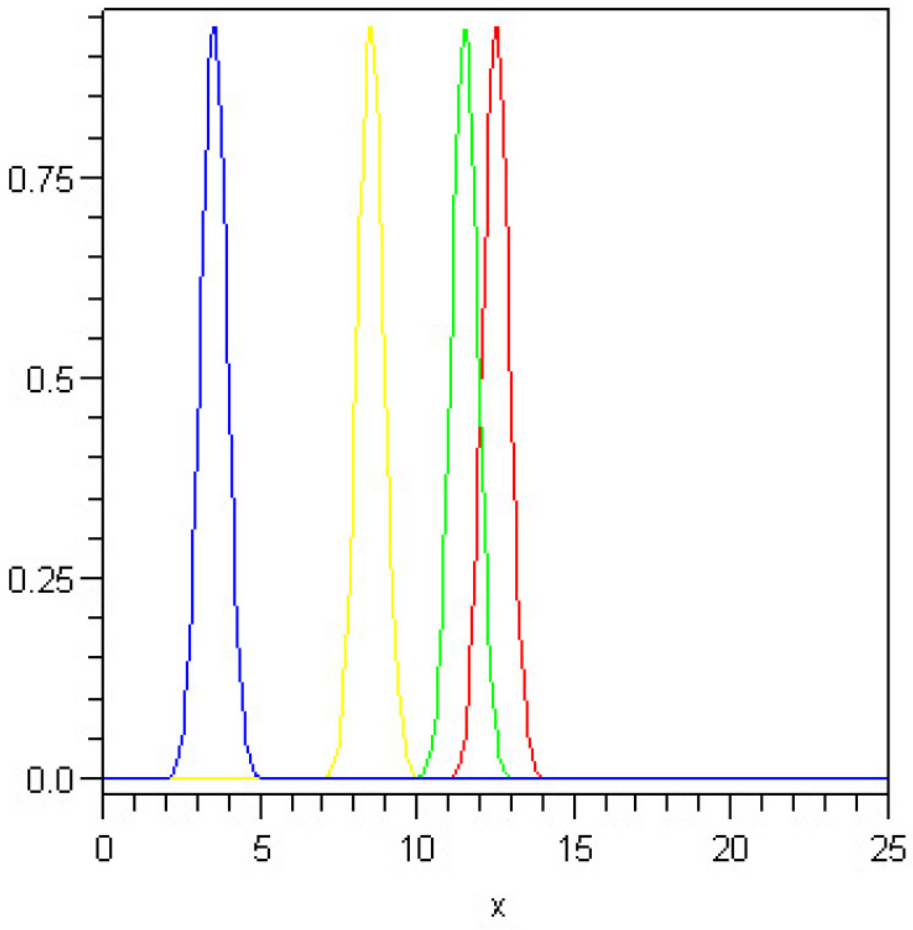

Figure 5.8 The Gaussian function at times $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (color in electronic version)

A plot of this Gaussian function for $\sigma = 0.6$ at time $t = 0$ is given in Fig. 5.7. As time progresses, this Gaussian function accelerates to the left. It spreads out very slowly because of our choice for t_0. A plot of this Gaussian function for times $t = 0$, $t = 1$, $t = 2$, and $t = 3$ is shown in Fig. 5.8.

5.4 Dispersion times comparison

One of the interesting features that comes from the analytic solutions is the ability to compare them with the numerical solutions. We are particularly interested in the dispersion time of each packet. This is the amount of time required before significant dispersion or spreading is observed in either case. In the case of the δ function, significant dispersion occurs after only a fraction of a time unit. We are interested in comparing the dispersion of this δ packet with the motion of the Gaussian packet. We are also interested in any similarities that exist between the motion of the two peaks. A plot of the two peaks is shown in Fig. 5.9 for $t = 0$, $t = 0.1$, $t = 0.2$, and $t = 0.3$.

We know that the Gaussian function is accelerating but it starts from rest and thus moves very slowly during the first few fractions of a time unit. The δ function disperses before the Gaussian has moved a significant distance. We had suspected that the dispersion we were seeing was related to the numeric issues of the problem rather than any actual specific quantum phenomenon. This comparison helped to support our beliefs about the nature of the dispersion.

5.5 Discussion

When initially considering the model for the wave packet we did consider the idea of using a Gaussian distribution. We decided that either method should work, because of the relationship showing that δ functions can be written as the limit of infinitely narrow Gaussians,[2]

$$\delta(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\sqrt{\pi\epsilon}} e^{\frac{-x^2}{4\epsilon}}. \tag{5.21}$$

Thus our approach of using an approximate δ function seems to be mathematically sound.

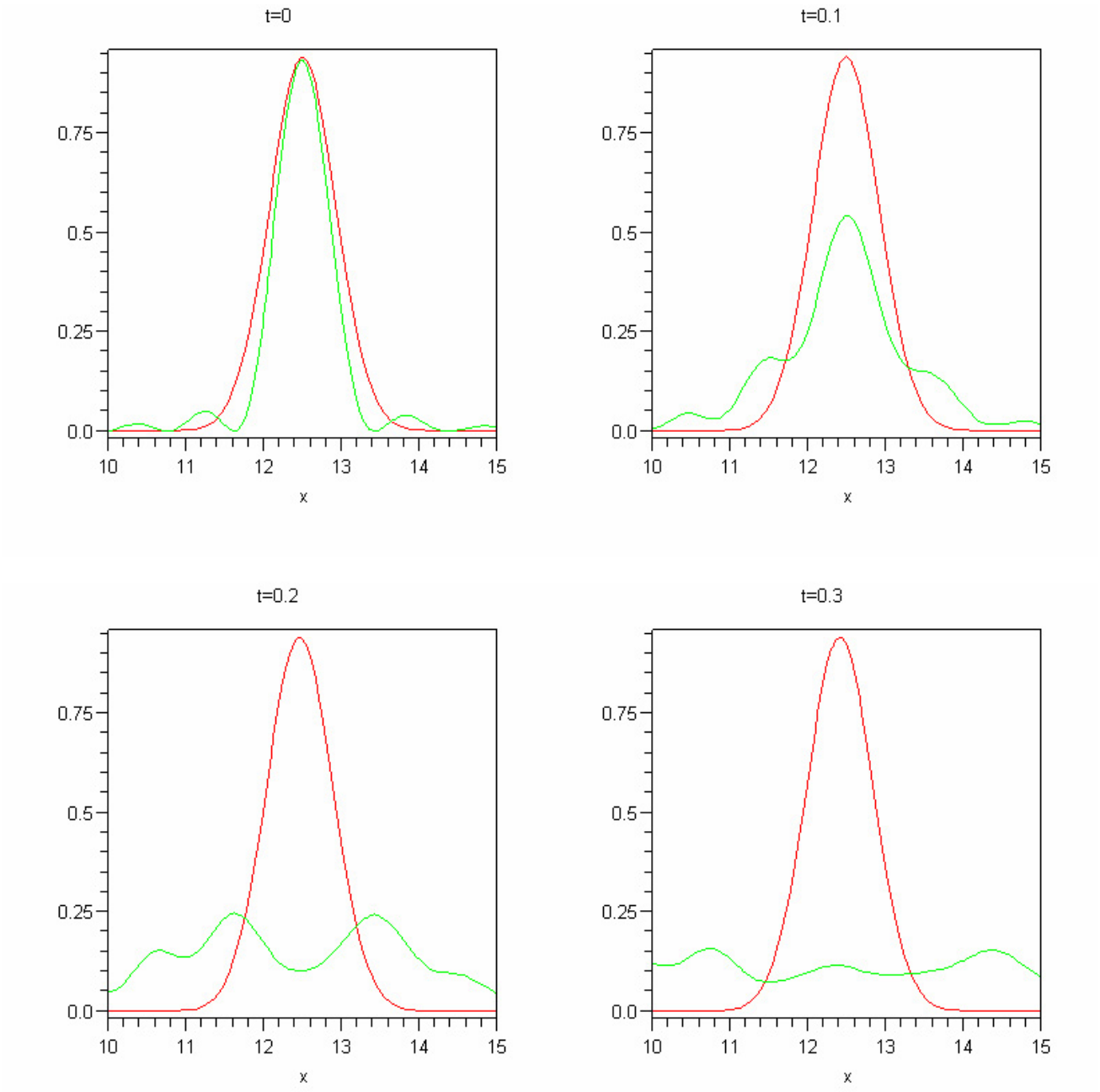

Figure 5.9 The Gaussian and δ plotted for $t = 0$, $t = 0.1$, $t = 0.2$, and $t = 0.3$

It is interesting to note that while we were studying the dispersion of our wave packet we encountered an article addressing a similar problem.[7] We ultimately found the solution in Robinett's text[6] which, as previously mentioned, was derived using a Gaussian packet in momentum space. In Churchill's paper[7] we found a description of our problem with a statement about the nature of its difficulties. Churchill states that modeling this problem from a numeric approach as we did was "extremely difficult and numerically intensive."[7]

The combination of Robinett's Gaussian solution in momentum space and the numeric description by Churchill leads us to a few conclusions about the nature of our project.[6,7] From a mathematical standpoint, we are unaware of any mistakes that were made in our derivations and expressions. We believe that we have correctly modeled the wave packet.

The first conclusion we can make comes from the analytic solution. Since the derivation was carried out in momentum space with a Gaussian function, we can assume certain special properties about the wavefunctions. The first observation is that the Gaussian distribution seems to be a more stable solution for this particular problem. Airy functions are difficult to work with because of their complexity and the math seems to be simpler with a Gaussian function.

Another observation we can make comes from comparing the dispersion times for the two peaks. Since the δ function disperses so quickly when compared to the Gaussian, we are lead to believe that the dispersion is not a purely quantum effect. The numeric setup we are dealing with in Maple seems to have some complications. It is possible that we made too many approximations or that Maple is unable to obtain a stable solution. Modeling the system numerically does not work as well as the analytic setup.

Chapter 6

Conclusion

In this project we sought to find a quantum mechanical description of Thomson's determination of the charge-to-mass ratio $\frac{e}{m}$ for the electron. After modeling the system and finding stationary solutions in the capacitor we combined them to construct approximate δ function packets. We allowed these wave packets to evolve in time and found the packets dispersing before we could observe their deflection. This complication prevented us from comparing the quantum results to the classical results with the accuracy that we had expected.

We also found that the problem can be solved analytically in momentum space if the peak is modeled with a Gaussian function. The apparent lack of dispersion in the Gaussian packet would indicate that it is a better model for the probability amplitude than the approximate δ function. The Gaussian and the δ function can be related mathematically through the limiting relationship between the two functions. This leads us to believe that there are problems with modeling the probability amplitude with a δ function that are not inherent in the functions themselves.

This discrepancy was only one of the reasons that we had to question our numeric approach using Maple. We also compared dispersion times between the two peaks,

one being a Gaussian and the other a δ function. The dispersion in the δ function happened far more quickly than it should have for the system involved. It also dispersed long before any quantum mechanical dispersion or physical deflection was observed in the Gaussian distribution. This leads us to believe that the dispersion we are seeing is related to the numeric approach using Maple as opposed to any actual physical quantum dispersion.

We can thus form some conclusions about our numeric approach. Although Maple was an exceptionally useful aid during this research project, it struggled towards the end with the time evolution. The primary reason for this was the analytic manipulation of Airy functions. Maple is not able to perform important operations such as integration when dealing with complicated expressions involving Airy functions. There were several integrals we had to approximate with finite sums for this reason. It is possible that our level of accuracy was just not adequate for the calculations involved. It is also possible that Maple was not accurate enough to describe the time evolution graphically. Maple was helpful but may not have been the best choice for this project.

We also discovered an interesting property of Airy functions. The analytic derivation was carried out in momentum space to take advantage of the fact that Gaussian wave packets of Airy functions can be integrated analytically.

In conclusion, we were able to model the Thomson experiment with the analytic solution. The numeric approach in modeling the peak with an approximate δ function was not as successful as we had hoped for. We could not avoid the occurrence of dispersion in the peak. Our recommendation for any future study of the problem is to better understand the nature of this dispersion. This might be accomplished by systematically studying dispersion in different potentials. Also a systematic study of wave packets of different shapes should give insights into the dispersion issue.

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[7]J.N. Churchill, Am. J. Phys. **46**(5), 537 (1978).

Appendix A

Another Approach to the E-B Relationship

It is interesting to note that there is a second way to obtain a mathematical link between the fields from our original definition. We know that we are considering the case where the forces are balanced and we have already used this to find a common ratio. We can also consider that if the forces are balanced, there will be no deflection of the ray and θ will be zero. We can set our two angle functions equal and opposite to each other:

$$\frac{EqL}{mv^2} = \frac{qBL}{mv}. \tag{A.1}$$

We can cancel some variables

$$\frac{E}{v} = B \tag{A.2}$$

and we obtain

$$E = Bv \tag{A.3}$$

$$v = \frac{E}{B} \tag{A.4}$$

which is the same relationship found in Eq. (3.123) using another method involving forces.

Appendix B

Maple Code for the Construction of the Wave Packet

Below is the Maple code that was used to create the Airy wave packet with time evolution included. The expression named shift at the end of the code is the expression that is plotted for various times in Chapter 5 when dispersion is discussed.

```
> restart;with(plots):
> S1:=diff(diff(psi(x),x),x)=((-E))*psi(x):
> dsolve(S1,psi(x)):
> psi1:=A*sin(sqrt(E)*x)+B*cos((sqrt(E))*x):
> S2:=diff(diff(psi(x),x),x)=((alpha*x-E))*psi(x):
> dsolve(S2,psi(x)):
> psi2:=C*AiryAi(-(-1)*(alpha*x-E)/alpha)+D*AiryBi(-(-1)*(alpha*x-E)/alpha):
> S3:=diff(diff(psi(x),x),x)=(alpha*d-E)*psi(x):
> dsolve(S3,psi(x)):
> psi3:=F*exp(-(alpha*d-E)^(1/2)*x):
> dpsi1:=diff(psi1,x):
```

```
> dpsi2:=diff(psi2,x):
> dpsi3:=diff(psi3,x):
> LBC:=evalf(subs(x=0,psi1)=subs(x=0,psi2)):
> LBCP:=evalf(subs(x=0,dpsi1)=subs(x=0,dpsi2)):
> LBCR:=evalf(subs(x=0,psi1))/evalf(subs(x=0,dpsi1))
=evalf(subs(x=0,psi2))/evalf(subs(x=0,dpsi2)):
> RBC:=evalf(subs(x=d,psi2)=subs(x=d,psi3)):
> RBCP:=evalf(subs(x=d,dpsi2)=subs(x=d,dpsi3)):
> RBCR:=evalf(subs(x=d,psi2))/evalf(subs(x=d,dpsi2))
=evalf(subs(x=d,psi3))/evalf(subs(x=d,dpsi3)):
> CoD:=solve(subs(D=1,RBCR),C):
> AoB:=solve(subs(B=1,C=CoD,D=1,LBCR),A):
> FoD:=solve(subs(D=1,C=CoD,RBC),F):
> BoD:=subs(D=1,C=CoD,rhs(LBC)):
> psi1:=BoD*(subs(A=AoB,B=1,psi1)):
> psi2:=subs(C=CoD,D=1,psi2):
> psi3:=FoD*(subs(F=1,psi3)):
> psifda1:=subs([alpha=1,d=25],psi1):
> psifda2:=subs([alpha=1,d=25],psi2):
> psifda3:=subs([alpha=1,d=25],psi3):
> n1:=sum(psifda1*psifda1,x=-25..0):
> n2:=sum(psifda2*psifda2,x=0..25):
> n3:=sum(psifda3*psifda3,x=25..35):
> N:=sqrt(1/(n1+n2+n3)):
> P1:=N*psifda1:
> P2:=N*psifda2:
```

```
> P3:=N*psifda3:

> G:=piecewise(x<0 and x>-25,P1,x=-25,P1,x=0,P2,x>0 and
x<25,P2,x=25,P2,x>25 and x<35,P3,x=35,P3):

> F:=subs(x=12.5,G):

> Delta:=F*G:

> piece1:=sum(subs(E=E/1,Delta)*sin((2*E/1)*t),E=0.1..24.9):

> piece2:=sum(subs(E=E/1,Delta)*cos((2*E/1)*t),E=0.1..24.9):

> psixt:=piece2+I*piece1:

> conpsixt:=piece2-I*piece1:

> shift:=psixt*conpsixt:
```

Appendix C

Further Localization of the Probability Amplitude

The sum command in Maple is designed to add the terms of a series that differ by integer increments. Chapter 4 describes the need for the integrals in our calculations to be replaced by finite sums. We summed the terms in Maple using the sum command but were then limited by the size of our interval. We used a change of variables method in which the variables we were summing over were replaced by a smaller fraction of the same variable. For example, summing E from 1 to 25,

$$\sum_{E=1}^{25} E \tag{C.1}$$

gives 25 terms from $E = 1$ to $E = 25$. This includes $E = 1, 2, 3$...etc. If we replace E with another variable G such that $G = \frac{E}{10}$, then summing G from $G = 10$ to $G = 250$ gives 241 terms in E from $E = 1$ to $E = 25$

$$\sum_{G=10}^{250} G \Longrightarrow \sum_{E=1}^{25} E. \tag{C.2}$$

This includes $E = 1, 1.1, 1.2, 1.3$...*etc.* We are summing over the same region but we now have a much finer grid. The Maple code for this substitution is

```
>Sum:=sum(subs(E=E/10,Delta),E=10..250);
```

where Delta is a function of the energy E. Adding more terms creates a more localized peak in our probability distribution. This inherently carries with it certain quantum effects such as dispersion that come from uncertainty.

Figure C.1 shows plots of similar wave packets that were constructed using sine waves instead of Airy functions. They were constructed in the same way but represent a different situation such as a particle in a rigid box with a flat potential. They are plotted for different numbers of terms in the sum and show the effect on the localization of the packet. In the plots, N is the number of terms in the sum used to create the packet. Figure C.2 displays the four plots overlayed.

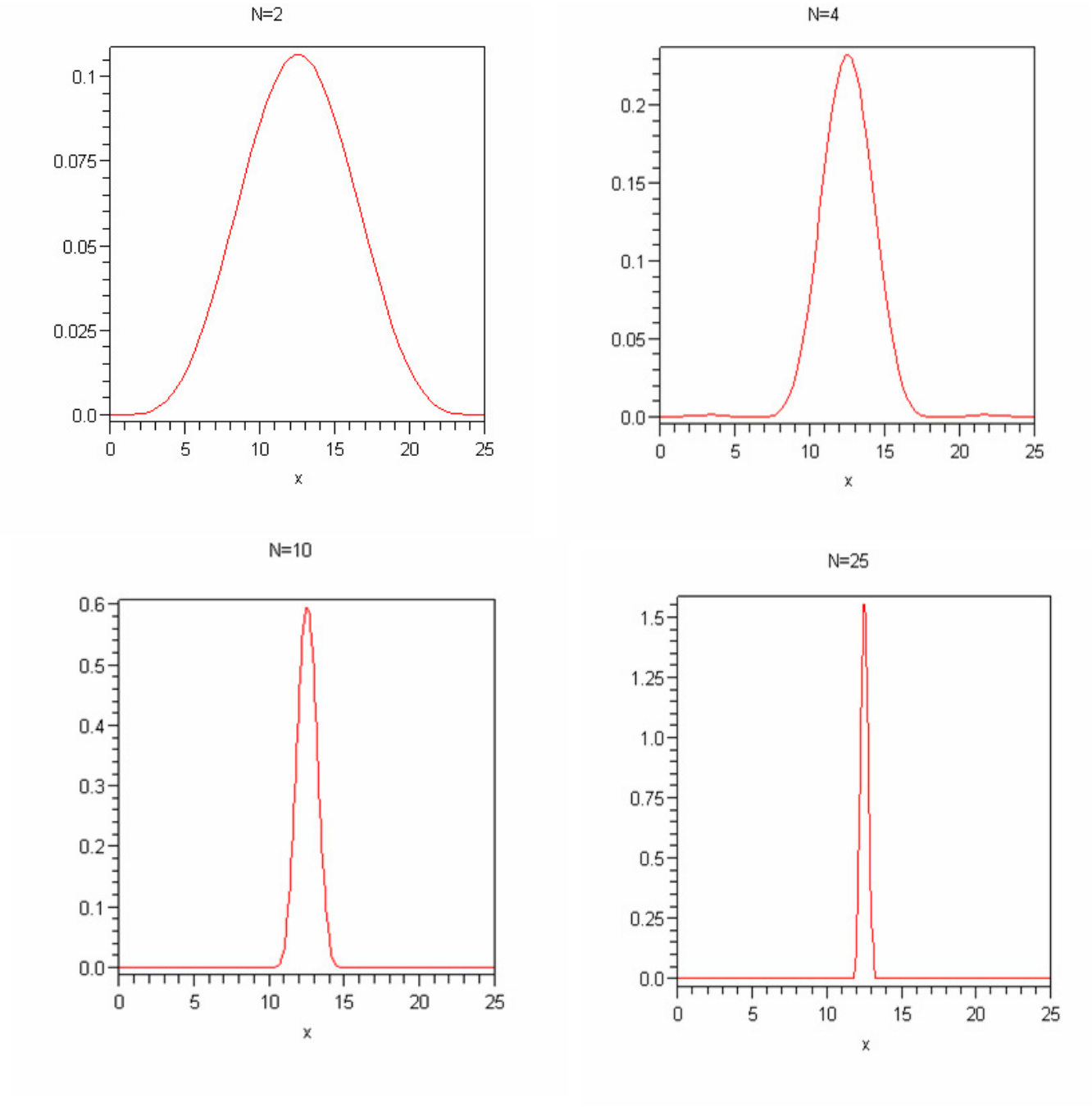

Figure C.1 The approximate δ function constructed from sine waves plotted for various values of terms in the sum

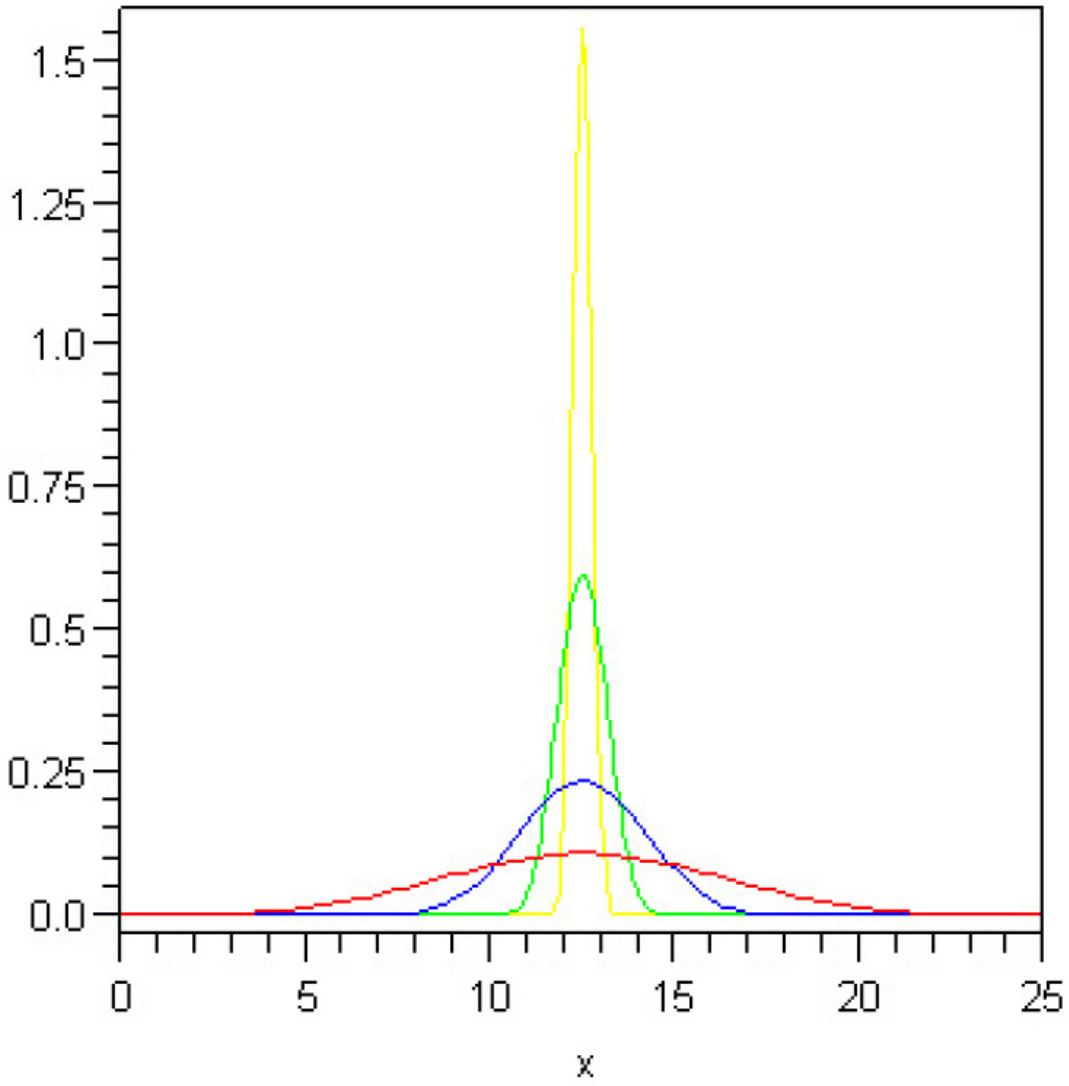

Figure C.2 An overlay of the four plots from Fig. C.1. The curves with decreasing halfwidths correspond to $N = 2, 4, 10, 25$

Appendix D

Quantum Mechanical Justification Based on the Heisenberg Picture

In this project we took an approach to the Thomson experiment that was based on the quantum mechanical description of the system involved. We sought to recover the same results that we found in the classical case. The Thomson experiment is most frequently treated as a classical system. Viewing it from a purely quantum mechanical standpoint is not nearly as common. It makes sense to solve this system using classical physics because everything works nicely. The claim of quantum theory, however, is that it is a better description of how the physical world operates than classical mechanics. We should thus be able to do anything with quantum mechanics that we can do with classical physics. In this section I give some justification for the treatment of the Thomson experiment, which is traditionally classical, as a quantum mechanical system. This justification is based on the Heisenberg representation.

When working with constant state vectors, we are using the Heisenberg picture. The time dependence of the operators is given by the Heisenberg equation.[4] This

equation states that for any operator A, its time evolution is given by

$$\dot{A} = \frac{1}{i\hbar}[A, H]. \tag{D.1}$$

The Hamiltonian for an electron in the electric field of the Thomson experiment is[3]

$$H = \frac{P^2}{2m} + \alpha x \tag{D.2}$$

where α is the slope of the potential energy in the capacitor model of section 2.3. Since α is positive the electron is expected to accelerate towards the left. When we substitute the Hamiltonian into the Heisenberg equation and use $[x, P] = i\hbar$ we get[4]

$$\dot{P} = \frac{1}{i\hbar}[P, H] = \frac{1}{i\hbar}[P, \alpha x] = -\frac{\alpha}{i\hbar} i\hbar = -\alpha. \tag{D.3}$$

Thus

$$\dot{P} = -\alpha \tag{D.4}$$

which is the constant force we expect from the electric field. Integrating to find $P(t)$ gives

$$P(t) = P_0 - \alpha t. \tag{D.5}$$

We can do the same thing for the position operator. Applying the Heisenberg equation gives

$$\dot{x} = \frac{1}{i\hbar}[x, \frac{P^2}{2m} + \alpha x] = \frac{2i\hbar}{i\hbar 2m} P = \frac{P}{m}. \tag{D.6}$$

Therefore

$$\ddot{x} = \frac{1}{i\hbar}[\dot{x}, H] = \frac{1}{i\hbar}[\frac{P}{m}, \frac{P^2}{2m} + \alpha x] = -\frac{\alpha}{m}. \tag{D.7}$$

We can integrate to solve for $x(t)$,

$$\ddot{x} = -\frac{\alpha}{m} \tag{D.8}$$

$$\dot{x} = \frac{P_0}{m} - \frac{\alpha}{m} t \tag{D.9}$$

$$x(t) = x_0 + \frac{P_0}{m}t - \frac{\alpha}{2m}t^2. \tag{D.10}$$

The evolution of the operators can be compared to Eq. (3.22). It is showing constant acceleration. From Eq.(D.10) we can derive the evolution of the expectation value of the position.

$$< x(t) >=< x_0 > + \frac{< P_0 >}{m}t + \frac{< \alpha >}{2m}t^2. \tag{D.11}$$

We can also derive an expression for the spreading of the wave packet. The uncertainty in x is given by[4]

$$\Delta x^2 =< x^2 > - < x >^2 . \tag{D.12}$$

We have just found an expression for x given by

$$x = x_0 + \frac{P_0}{m}t - \frac{\alpha}{2m}t^2 \tag{D.13}$$

and thus

$$x^2 = x_0^2 + \left(\frac{P_0}{m}\right)^2 t^2 + \frac{\alpha^2 t^4}{4m^2} + \frac{x_0 P_0 t}{m} + \frac{P_0 x_0 t}{m} - \frac{x_0 \alpha t^2}{m} - \frac{\alpha P_0 t^3}{m^2}. \tag{D.14}$$

Therefore

$$< x^2 >=< x_0^2 > + < P_0^2 > \frac{t^2}{m^2} + \frac{\alpha^2 t^4}{4m^2} + \frac{< x_0 P_0 >}{m}t + \frac{< P_0 x_0 >}{m}t - \frac{\alpha t^2}{m} < x_0 > - \frac{\alpha t^3}{m^2} < P_0 > . \tag{D.15}$$

The expectation value of x is given by

$$< x >=< x_0 > + < P_0 > \frac{t}{m} - \frac{\alpha t^2}{2m} \tag{D.16}$$

and thus

$$< x >^2 =< x_0 >^2 + \frac{< P_0 >^2 t^2}{m^2} + \frac{\alpha^2 t^4}{4m^2} + 2 < x_0 >< P_0 > \frac{t}{m} - \frac{< x_0 > \alpha t^2}{m} - \frac{\alpha t^3 < P_0 >}{m^2}. \tag{D.17}$$

The spreading is then given by

$$\Delta x^2 =< x^2 > - < x >^2 \tag{D.18}$$

$$=< x_0^2 > - < x_0 >^2 +(< P_0^2 > - < P_0 >^2)\frac{t^2}{m^2}+\frac{2t}{m}\left(\frac{< P_0x_0 + x_0P_0 >}{2}- < x_0 >< P_0 >\right). \tag{D.19}$$

Thus

$$\Delta x^2 = \Delta x_0^2 + \Delta P_0^2\frac{t^2}{m^2} + \frac{2t}{m}\left(\frac{< P_0x_0 + x_0P_0 >}{2}- < x_0 >< P_0 >\right). \tag{D.20}$$

It is interesting to note that α is not contained in the expression. This means that the spreading is independent of the slope of the potential. It is also interesting to note that the uncertainty in x is identical to that of the free particle. We can use the same method for the momentum. We begin with

$$P(t) = P_0 - \alpha t \tag{D.21}$$

and

$$< P >=< P_0 > -\alpha t. \tag{D.22}$$

We then proceed to find the values we need for the uncertainty. We find the square of the momentum

$$P^2 = P_0^2 + \alpha^2t^2 - 2\alpha P_0t. \tag{D.23}$$

Thus

$$< P >^2=< P_0 >^2 +\alpha^2t^2 - 2\alpha < P_0 > t \tag{D.24}$$

and

$$< P^2 >=< P_0^2 > +\alpha^2t^2 - 2\alpha < P_0 > t. \tag{D.25}$$

We can then find the uncertainty,

$$\Delta P^2 =< P^2 > - < P >^2= \Delta P_0^2 \tag{D.26}$$

$$\Delta P^2(t) = \Delta P_0^2. \tag{D.27}$$

If we minimize the uncertainty between these two quantities by setting their product equal to $\hbar$,

$$\Delta x \Delta P = \frac{\hbar}{2} \tag{D.28}$$

then we can assign values as

$$\Delta x_0 = \sqrt{\frac{\hbar}{2}} \tag{D.29}$$

and

$$\Delta P_0 = \sqrt{\frac{\hbar}{2}}. \tag{D.30}$$

This results in the following expression for the square of the uncertainty in x,

$$\Delta x^2 = \frac{\hbar}{2}\left(1 + \frac{t^2}{m^2}\right) + \frac{2t}{m}\left(\frac{< P_0 x_0 + x_0 P_0 >}{2} - < x_0 >< P_0 >\right). \tag{D.31}$$

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