Space-Time-Resolved Simulation of Photon Vacuum Polarization and Bhabha Scattering

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ABSTRACT

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We create a limited particle content model in one spatial dimension to study the interaction between single particle photon states, and electron-positron pair states. We use this model to simulate the phenomena of vacuum polarization in a propagating photon, and Bhabha Scattering. This approach supplements the S-matrix approach to Quantum Field Theory, which provides an effective approach for predicting the results of scattering experiments but gives little physical insight into the dynamics of the field during interactions. We find divergences arise in attempting to calculate the eigenvalues of particles states with interaction in out limited content model. This leads to areas requiring further study of further study, namely, how to implement renormalization within space-time-resolved frameworks.

ACKNOWLEDGMENTS

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Chapter 1

Introduction

1.1 Motivation

Quantum Field Theory (QFT) is the study of relativistic quantum mechanics. It combines two of the most effective theories for describing the physical world: quantum mechanics and special relativity. Quantum Mechanics describes the fundamental uncertainty inherent in measuring very small particles and is fundamentally probabilistic. Special relativity describes objects moving at velocities near the speed of light, and encompasses the inherent relative nature of space and time. QFT describes fundamental particles as excitations of quantum fields, and the fundamental forces as interactions between these fields. The first historically developed Quantum Field Theory was Quantum Electrodynamics (QED), and it is one of the most precise physical theories ever. For example, the calculation of the anomalous magnetic moment of the electron, which matches experiment to a remarkable 11 digits of precision [1].

The normal approach to solving problems in QFT is the S-matrix scattering approach. This approach maps initial particle states at times long before a scattering event to states at times long after the interaction. This approach has produced extremely accurate predictions of the outcomes of

scattering experiments. However, it provides little insight into the dynamics of the particles during such an interaction. During the interactions the dynamics can be complicated because arbitrarily high numbers of particles can be created and destroyed as a part of the interaction. Because of quantum uncertainty, it is not even clear that the intermediate states of the field look like what we would conventionally describe as particles.

Space-time-resolved QFT is an attempt to gain insight on the intermediate dynamics of quantum states. It involves taking quite a few simplifying assumptions, such as limiting interactions to include only a fixed number of particles, and limiting to one spatial dimension. Once these simplifications are made, numerical simulation of the reduced model is possible with more modest computational resources than are required for the full description. The space-time-resolved approach was pioneered by the research group of Grobe et al. They created several simple models to visualize the time evolution of quantum field interactions. For example, in 2010 Wagner et al. from this group looked at a simple model of Compton scattering using simple Boson-Fermion model with a Yukawa potential [2]. In 2011, they were able to show using this space-time-resolved approach how the force between tow Fermions develpoed from the exchange of virtual bosons [3].

In Scott Glasgow's research group, we have also simulated quantum field dynamics using reduced space-time-resolved models. In 2016, Glasgow et al. were able to simulate the process of a bare photon dressing itself using a simplified model [4]. In 2020, Glasgow and Ware were able to use this model to simulate Compton Scattering [5]. This thesis follows a very similar path to this previous work. We develop a reduced particle content model to study the interaction between the electron-positron, and photon states of the field. This allows us to simulate vacuum polarization of a photon, and scattering of positrons and electrons.

1.2   Background

The state of quantum particles are described using normalized vectors in a complex Hilbert space. We represent one of these vectors, or "states", as a "ket" such as $|\psi\rangle$. The corresponding "bra", $\langle\psi|$, represents the associated covector in the dual space [6]. Physically observable quantities are represented by Hermitian operators, such as Ω, and observation of these quantities yields one of the eigenvalues ω of this operator, with probability given by

$$P(\Omega = \omega) = |\langle\omega|\psi\rangle|^2, \tag{1.1}$$

where $|\omega\rangle$ is the eigenvector associated to the eigenvalue ω. In the Schrödinger picture of quantum mechanics, the time evolution of the quantum state is then described by the abstract Schrödinger equation given by

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle. \tag{1.2}$$

In order to accommodate the creation and annihilation of particles we must use a space in with arbitrary particle content. This gives the state space have a more complicated structure called a "Fock Space" which is defined as the direct sum of all possible particle number spaces, i.e. we have

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n. \tag{1.3}$$

Where Hilbert spaces containing multiple identical particles are given by the tensor product of single particle spaces written as

$$\mathscr{H}_n = \mathscr{H} \bigotimes \mathscr{H} \bigotimes ... \bigotimes \mathscr{H}. \tag{1.4}$$

To keep track of the number of particles we use creation and annihilation operators, which are analogous to the ladder operators of the quantum harmonic oscillator. For bosons we have commutation relations, or anticommutation relations for fermions, given by

$$[a(k), a^\dagger(k')]_\pm = \delta(k - k'). \tag{1.5}$$

This guarantees the proper Dirac orthonomality of states, and thus we can view any multi particle states as excitations of the zero particle vacuum state $|\rangle$ i.e we have that a state of N particles each of species type n_i with momentum $\mathbf{k}_i$ is given by

$$|\psi(\mathbf{k}_1, n_1; \mathbf{k}_2, n_2; ...; \mathbf{k}_N, n_N)\rangle \propto a^\dagger_{n_1}(\mathbf{k}_1) a^\dagger_{n_2}(\mathbf{k}_2) ... a^\dagger_{n_N}(\mathbf{k}_N) |\rangle \tag{1.6}$$

The set of all kets of the form (1.6), for all possible particle content, give us a natural a basis for the state space. The use of creation and annihilation operators provide a convenient way to keep track of the particle content of our space. They are also very important because any operator on the Fock space can be written in terms of creation and annihilation operators [7]. As such, the Hamiltonian operator which we will study below is naturally given in terms of these creation and annihilation operators.

The Hamiltonian of an interacting field will contain terms which lead to couplings between subspaces of the Fock space with different particle number. It is for this reason that the the full dynamics of the quantum field are extremely complex. The abstract Schrödinger equation (1.2) will then become an infinite set of coupled differential equations in an increasing number of independant variables.

1.3   Outline of Approach

In chapter 2 we review the full QED Hamiltonian in the Lorenz gauge. Starting with this Hamiltonian we derive a reduced Hamiltonian that models interactions of the field with limited particle content in one spatial dimension. We start by nondimensionalizing the model, finding that the Hamiltonian depends on only one essential parameter a dimensionless coupling constant defined in terms of

the fine structure content. After doing this we limit our model to one spatial dimension which allows for easier numerical computation. We then project our Hamiltonian onto a subspace of states with limited particle content. Namely states with a single free photon, and states with a bare electron-positron pair.

In chapter 3 we study the behavior of this simplified model. We first solve the time independant Schrödinger equation for the energy of the self interacting photon states. We find that divergences arise in attempting to calculate these eigenvalues within our reduced model. Though we do not yet understand how renormalize our model to fully remove these divergences, we are able to regulate them using a finite cutoff and a small photon "mass". This allows us to numerically simulate the space-time-resolved dynamics of our model. We simulate the vacuum polarization of a propagating photon, and Bhabha scattering of a electron and positron. In the conclusion, we discuss next steps and further work that could build off of this thesis.

Chapter 2

Methods

2.1 QED Hamiltonian

In principle, the dynamics of a quantum system can be completely determined by the full QED Hamiltonian. However, the full dynamics of the quantum field remain intractable even numerically when the complete description is retained. To study these dynamics with a practical amount of resources we must make two important simplifying assumptions. We limit our state space to only states of interest, and we limit our model to one spatial dimension. These approximations make numerical simulation tractable. The standard scattering approach to QFT makes heavy use of perturbation theory. Our method is partially perturbative in that by limiting the particle content of the interaction we are only considering lower order interactions. However, the notable difference between the traditional approach is that after making this approximation we simulate the full dynamics of the wave function rather than just scattering amplitudes.

In the Lorenz gauge the full QED Hamiltonian is given by [8]

$$H = H_D + H_R + H_I.$$ (2.1)

The first term is the Dirac term which describes the kinetic energy of matter particles, and is given

by

$$H_D = \int d^3k \sqrt{(mc^2)^2 + (\hbar c \, \|\mathbf{k}\|)^2} \left[c_\uparrow^\dagger(\mathbf{k}) c_\uparrow(\mathbf{k}) + c_\downarrow^\dagger(\mathbf{k}) c_\downarrow(\mathbf{k}) + d_\uparrow^\dagger(\mathbf{k}) d_\uparrow(\mathbf{k}) + d_\downarrow^\dagger(\mathbf{k}) d_\downarrow(\mathbf{k}) \right]. \quad (2.2)$$

Where m is the mass of an electron, c is the speed of light, and $\hbar$ is the reduced planck's constant.

The second term is the radiation term which describes the kinetic energy of photons. Here the sum

over ε represents the possible vector components of the different photon states:

$$\int d^3k \, \|\hbar c \mathbf{k}\| \sum_{x \in \{s, l, \varepsilon_1, \varepsilon_2\}} a_x^\dagger(\mathbf{k}) a_x(\mathbf{k}). \quad (2.3)$$

Where $s, l, \varepsilon_1, \varepsilon_2$ represent scalar, longitudinal, and the two directions of polarization respectively.

Scalar and longitudinal photons are included in the Lorenz gauge and give rise to the static Coulomb

force. The third term in (2.1) is the interaction term which describes the coupling between the

electromagnetic field and matter particles. It is given by a linear combination of triples of creation

and annihilation operators. As concisely as possible it is given by

$$H_I = - \int d^3r \mathbf{J}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) + c \int d^3r \rho(\mathbf{r}) A_0(\mathbf{r}), \quad (2.4)$$

where the vector potential is defined in terms of photon creation and annihilation operators by

$$A_\mu(\mathbf{r}) = \int d^3k \sqrt{\frac{\hbar}{2\varepsilon_0 c \, \|\mathbf{k}\| \, (2\pi)^3}} \left[a_\mu(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} + \bar{a}_\mu(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right]. \quad (2.5)$$

Here we have define the barred operators as $\bar{a}_i(\mathbf{k}) = a_i^\dagger(\mathbf{k})$ for $i \in \{1, 2, 3\}$ and $\bar{a}_0(\mathbf{k}) = -a_0^\dagger(\mathbf{k})$.

Our current model does not include any scalar or longitudinal photon states, and so the distinction

between barred and adjoint operators will not matter for our model.

The current and charge density operators present in the interaction Hamiltonian are given by

$$\mathbf{J}(\mathbf{r}) = qc\Psi^\dagger(\mathbf{r})\vec{\alpha}\Psi(\mathbf{r}) \text{ and } \rho(\mathbf{r}) = q\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}). \quad (2.6)$$

Note in the above equation the presence of $\vec{\alpha}$ and β defined in terms of the well known Dirac matrices. The current and charge density operators are given in terms of the Dirac field given by

$$\Psi(\mathbf{r}) = \int d^3k e^{i\mathbf{k}\cdot\mathbf{r}} U(\mathbf{k})\chi(\mathbf{k}) \tag{2.7}$$

where

$$\chi(\mathbf{k}) = \begin{bmatrix} c_\uparrow(\mathbf{k}) \\ c_\downarrow(\mathbf{k}) \\ d_\uparrow^\dagger(\mathbf{k}) \\ d_\downarrow^\dagger(\mathbf{k}) \end{bmatrix} \tag{2.8}$$

and

$$U(\mathbf{k}) = I_{4\times 4}\cos(\theta_k/2) - \beta\vec{\alpha}\cdot\hat{\mathbf{k}}\sin(\theta_k/2). \tag{2.9}$$

Here θ_k is given by $\arctan(\hbar\|\mathbf{k}\|/mc)$.

2.2   Reduction to One Spatial Dimension

Before we simplify our model to one spatial dimension, we rewrite it in term of dimensionless quantities. We nondimensionalize it by choosing characteristic scalings given by

$$E_c = mc^2, \ k_c = \frac{mc}{\hbar}, \ \text{and} \ t_c = \frac{\hbar}{mc^2}. \tag{2.10}$$

Now, we define dimensionless variables,

$$\mathscr{E} = E/E_c, \kappa = k/k_c, \tau = t/t_c \tag{2.11}$$

Also, we redefine our creation and annihilation operators in terms of these quantities as,

$$\tilde{a}(\kappa) = (k_c)^{3/2} a(k_c\kappa). \tag{2.12}$$

After significant simplification and rewriting the Hamiltonian in terms of dimensionless quantities, we obtain a dimensionless hamiltonian $\mathscr{H} = H/E_c$ with terms given by

$$\mathscr{H}_D = \int d^3\kappa \sqrt{1 + \|\kappa\|^2} \left[\tilde{c}_\uparrow^\dagger(\kappa)\tilde{c}_\uparrow(\kappa) + \tilde{c}_\downarrow^\dagger(\kappa)\tilde{c}_\downarrow(\kappa) + \tilde{d}_\uparrow^\dagger(\kappa)\tilde{d}_\uparrow(\kappa) + \tilde{d}_\downarrow^\dagger(\kappa)\tilde{d}_\downarrow(\kappa) \right], \qquad (2.13)$$

$$\mathscr{H}_R = \int d^3\kappa \, \|\kappa\| \sum_{x \in \{s,l,\varepsilon_1,\varepsilon_2\}} \tilde{a}_x^\dagger(\kappa)\tilde{a}_x(\kappa), \qquad (2.14)$$

and

$$\mathscr{H}_I = -\lambda \int d^3\kappa_1 \int d^3\kappa_2 \int d^3\kappa_\gamma \frac{1}{\sqrt{\|\kappa_\gamma\|}} \sum_{x \in \{s,l,\varepsilon_1,\varepsilon_2\}} \tilde{\chi}^\dagger(\kappa_1) U^\dagger(\kappa_1) \left[\vec{\alpha} \cdot \hat{x}(\kappa_\gamma) \right] U(\kappa_2)\chi(\kappa_2)$$
$$\cdot \left[\tilde{a}_x \delta(-\kappa_1 + \kappa_2 + \kappa_\gamma) + \bar{\tilde{a}}(\kappa_\gamma)\delta(-\kappa_1 + \kappa_2 + \kappa_\gamma) \right]. \quad (2.15)$$

Here we have extended $\vec{\alpha}$ to a four vector with $\alpha_0 = I_{4\times 4}$. λ is a dimensionless coupling parameter related to the fine structure constant by $\lambda = \frac{\sqrt{\alpha}}{2\pi}$. Now that we have obtained the dimensionless form of the Hamiltonian we will revert back to the old notation, e.g. $\mathbf{k}$ rather than κ, but all quantities will be assumed to be dimensionless.

Now that we have non-dimensionalized the model we can put it in one spatial dimension easily without causing any complications due to units. We do this by choosing all momenta to act in the z-direction, i.e. $\mathbf{k} = k\hat{z}$, which reduces all vector quantities to scalars. This is equivalent to quantizing the model in a long narrow box [5], with the caveat that the coupling constant becomes dependant upon the dimensions of said box. We will thus view the coupling constant λ as a variable parameter which we can adjust in our model.

The one-dimensional version of the Dirac and radiation terms is given by

$$H_D = \int dk \sqrt{1 + k^2} \left[c_\uparrow^\dagger(k)c_\uparrow(k) + c_\downarrow^\dagger(k)c_\downarrow(k) + d_\uparrow^\dagger(k)d_\uparrow(k) + d_\downarrow^\dagger(k)d_\downarrow(k) \right], \qquad (2.16)$$

and

$$H_R = \int dk |k| \sum_{x \in \{s,l,\varepsilon_1,\varepsilon_2\}} a_x^\dagger(k) a_x(k). \tag{2.17}$$

Finding the interaction term requires simplifying the dot product with Dirac matrices in (2.15), and thus depends on our choice of polarization vectors. Choosing the momenta in the z direction gives us the definition of the polarization basis vectors given by

$$\hat{l} = \hat{k} = \text{sign}(k)\hat{z}, \tag{2.18}$$

$$\hat{\varepsilon}_1 = \hat{x}, \text{ and} \tag{2.19}$$

$$\hat{\varepsilon}_2 = \text{sign}(k)\hat{y}. \tag{2.20}$$

Using these definitions to simplify the Dirac matrices gives us an interaction Hamiltonian with 32 terms but only two that interact with our states of interest. We choose the states of interest to be a free photon state with positive helicity, and an electron-positron pair state with total spin 1. What we end up with is

$$H_I = -\lambda \int dk_1 \int dk_2 \int dk_\gamma \frac{1}{\sqrt{|k_\gamma|}} \Big[g(k_1,k_2) c_\uparrow^\dagger(k_1) d_\downarrow^\dagger(k_2) a_+(k_\gamma) \delta(-k_1 + k_2 + k_\gamma)$$

$$+ g(k_1,k_2) c_\uparrow(k_2) d_\downarrow(k_1) a_+^\dagger(k_\gamma) \delta(-k_1 + k_2 - k_\gamma) + \dots \text{ other terms} \Big]. \tag{2.21}$$

The annihilation operator for a positively circularly polarized photon is given by $a_+(k_\gamma) = \frac{1}{\sqrt{2}}\big[a_1(k_\gamma) - i \cdot \text{sign}(k_\gamma) a_2(k_\gamma)\big]$, with the corresponding creation operator given by the Hermitian conjugate. Note that in the current convention $d_\downarrow^\dagger(k)$ represents the creation operator of a positron with spin $+\hbar/2$ and dimensionless momentum $-k$, thus these terms in the Hamiltonian can be seen to conserve angular momentum. Knowing this we can identify the delta functions in (2.21) as ensuring conservation of total momentum. The coupling function g gives the strength of the coupling between these states and is given by,

$$g(k_1, k_2) = \sqrt{\left(1 + \frac{1 + k_1 \cdot k_2}{E_0(k_1)E_0(k_2)}\right)}, \tag{2.22}$$

and

$$E_0(k) = \sqrt{1 + k^2}. \tag{2.23}$$

2.3 Limited Particle Content Projection

We limit the particle interaction of the model by looking at the action of the Hamiltonian on a projected subspace of the full state space. The basis for our subspace of interest is given by $\{|k_\gamma\rangle, |k_-, k_+\rangle\}$, where our basis kets are

$$|k_\gamma\rangle = a_+^\dagger(k_\gamma)|\rangle, \text{ and } |k_-, k_+\rangle = c_\uparrow^\dagger(k_-)d_\downarrow^\dagger(k_+)|\rangle. \tag{2.24}$$

To study the dynamics in this space we must understand what the effect of the hamiltonian is projected on our subspace of states of interest. We use a projection operator given by

$$P = \int dk_\gamma |k_\gamma\rangle \langle k_\gamma| + \int dk_- \int dk_+ |k_-, k_+\rangle \langle k_-, k_+|. \tag{2.25}$$

By applying the projector to both sides of the Hamiltonian we get a reduced hamiltonian which acts on the projected subspace of interest. The projected radiation term in this hamiltonian is given by

$$PH_R P = \int dk_\gamma |k_\gamma| |k_\gamma\rangle \langle k_\gamma|. \tag{2.26}$$

Applying the projector to the Dirac term gives us

$$PH_D P = \int dk_- \int dk_+ \left[\sqrt{1 + k_-^2} + \sqrt{1 + k_+^2}\right] |k_-, k_+\rangle \langle k_-, k_+|. \tag{2.27}$$

The Dirac and Radiation terms represent diagonal terms in the hamiltonian, and correspond to the dynamics of particles in the absence of interaction. They dictate how non interacting particles naturally propagate through space.

Next, we apply the projector to the interaction term in the Hamiltonian. The interaction Hamiltonian maps photon states to pair states and vice versa. It represents the interaction between the quantized EM field and the Dirac field. The projected interaction Hamiltonian is

$$PH_I P = -\lambda \int dk_\gamma \int dk_- \int dk_+ \frac{g(k_-, k_+)}{\sqrt{|k_\gamma|}} \left[|k_-, k_+\rangle \langle k_\gamma| \delta(-k_- + k_+ + k_\gamma) \right.$$

$$\left. + |k_\gamma\rangle \langle k_-, k_+| \delta(-k_- + k_+ + k_\gamma) \right]. \quad (2.28)$$

Chapter 3

Results

Now that we have applied these approximations we can numerically simulate the physical principles predicted by this model.

3.1 Self Interacting Photon Energy Calculation

The presence of the interaction changes the behavior of particle states. A single "free" photon state is no longer a stable particle. Instead a physical photon state includes interaction dependant contributions from electron-positron states. This phenomena is sometimes described as a probability of a photon creating an electron-positron pair and then this pair annihilating and recreating a photon which continues to travel through space. These dynamics are responsible for the physical result of vacuum polarization. We can plug the Hamiltonian into the time independent Schrödinger equation,

$$H \ket{\psi} = E \ket{\psi},\tag{3.1}$$

to find the energies of a perturbed photon. Doing so with an arbitrary polarized photon state vector leads to a transcendental equation for the energy, $E(k_\gamma)$, given as

$$E(k_\gamma) = |k_\gamma| - \frac{\lambda^2}{|k_\gamma|} \int d\xi \frac{g(\xi + k_\gamma, \xi)^2}{\sqrt{(\xi + k_\gamma)^2 + 1} + \sqrt{\xi^2 + 1} - E(k_\gamma)} \qquad (3.2)$$

Unfortunately, upon examination it becomes clear that the numerical value of the energy predicted by (3.2) is not well defined. The integral above is over all values of ξ. However, this integral does not converge; the integrand is $\mathcal{O}(1/\xi)$ for large ξ and thus the integral is logarithmic divergent. This sort of divergence is often called an "Ultraviolet" divergence because it occurs in the high momentum and short wavelength range of our model. This equation also predicts that the energy diverges as photon momentum goes to zero because of the $\frac{1}{|k_\gamma|}$ in the second term. Analysis of (3.2) show that this makes $E(k_\gamma) = \mathcal{O}(\frac{1}{\sqrt{k_\gamma}})$ for small values of k_γ. Divergences of this kind are often called "Infrared" divergences because they happen in the low momentum and long wavelength portion of the momentum domain.

Discovering divergences within quantum field theoretic problems is not unexpected. There is a whole structure of theories built around taming these divergences. The procedures for removing these divergences are called regularization and renormalization. Regularization typically involves introducing some sort of finite bounds or "cutoffs" on the range of our theory to cause terms to be finite. Renormalization, which usually occurs after regularization, involves redefining the parameters of the theory, such as charge and mass, to be dependant upon the cutoffs in such a way that we can re-extend the range to all values [7]. However, the current theory of renormalization has only been applied previously within the standard scattering approach to QFT. It is unclear how to apply these methods within space-time-resolved models for QFT. How to apply renormalization methods within space-time-resolved models for QFT remains an open question, and represents an area of further study.

While it is still unclear how to extend the full renormalization procedure to our space-time-resolved model, we can regulate the ultraviolet divergence by introducing a momentum cutoff on the bounds of the integral term. In Fig. 3.1 we can see the infrared divergence which will still occur

in the low momentum limit of the energy. We can regulate this divergence by introducing a photon "mass", which will remove the singularity in the energy at $k_\gamma = 0$. Applying both these changes to the energy transcendental equation we arrive at a regularized equation for the energy,

$$E(k_\gamma) = \sqrt{\varepsilon^2 + k_\gamma^2} - \frac{\lambda^2}{\sqrt{\varepsilon^2 + k_\gamma^2}} \int_{-\Lambda}^{\Lambda} d\xi \frac{g(\xi + k_\gamma, \xi)^2}{\sqrt{(\xi + k_\gamma)^2 + 1} + \sqrt{\xi^2 + 1} - E(k_\gamma)}, \tag{3.3}$$

where the photon mass is $m_\gamma = \varepsilon \cdot m_{e_-}$, and Λ is the momentum cutoff of the integral. In Fig. 3.1 we can see the behavior of the eigenvalues of the photon states of the interacting hamiltonian. As in [4], we find that the energy of the free particles states is perturbed negatively by the presence the interaction. In the case of the massless photon this perturbation approaches infinity as k_γ goes to zero.
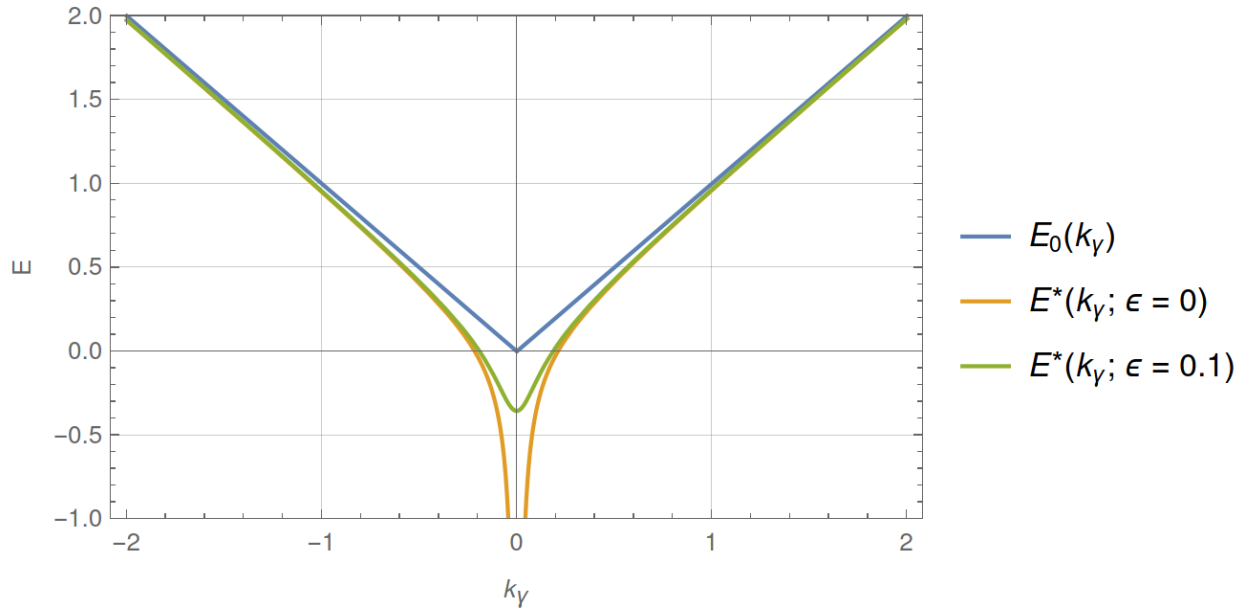
Figure 3.1 E^* is the Energy of the interacting photon eigenstates with coupling constant set to $\lambda = 0.1$. The free photon energy is given by $E_0 = |k_\gamma|$. k_γ is given in units of $mc/\hbar$. The momentum cutoff here is $\Lambda = 20mc/\hbar$

3.2   Vacuum Polarization of a Propogationg Photon

Although it is unclear how to renormalize this model, we can regulated the divergences using the methods described above. Removing these divergences makes our model well defined, which allows us to simulate the dynamics predicted by this simplified model. Following the results of previous work, we use a small coupling constant $\lambda = 0.4$ [4, 5]. Because we expect our model to not be accurate near the low momentum range of photon states, we use initial conditions of a photon distribution with momenta well above zero. We define our Hamiltonian in such a way that it ignores interactions with unphysical zero momentum photons.

To simulate the dynamics we must solve the abstract Schrödinger equation, which is given in dimensionless form by

$$i\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle.\tag{3.4}$$

Given an orthonormal set of eigenstates $\{\left|\psi_n\right\rangle\}$ and corresponding eigenvalues $\{E_n\}$, the general solution, for an initial condition $\left|\psi_0\right\rangle$, is given by

$$\left|\psi(t)\right\rangle = \sum_n e^{-iE_n t}\left\langle\psi_n|\psi_0\right\rangle\left|\psi_n\right\rangle.\tag{3.5}$$

We find these eigenstates by discretizing our model in momentum space, which allows us to represent our Hamiltonian operator as a matrix. We can then use numerical methods to find the eigenvalues and eigenvectors of this matrix, and use these to simulate the solution using (3.5). We choose an initial condition as a normalized superposition of bare photon states given by

$$\left|\psi_0\right\rangle \propto \int dk_\gamma e^{-(k_\gamma - l)^2/w^2}\left|k_\gamma\right\rangle.\tag{3.6}$$

The time dynamics are evaluated using (3.5) which gives us $\left|\psi(t)\right\rangle$. This is naturally represented in

the momentum basis. Applying Fourier transforms we find our position space wave functions,

$$\psi_\gamma(x,t) = \frac{1}{\sqrt{2\pi}} \int dk_- e^{-ik_\gamma \cdot x} \langle k_\gamma | \psi(t) \rangle, \text{ and} \tag{3.7}$$

$$\psi(x_-, x_+, t) = \frac{1}{2\pi} \int dk_- \int dk_+ e^{-ik_- \cdot x_- - ik_+ \cdot x_+} \langle k_-, k_+ | \psi(t) \rangle. \tag{3.8}$$

We can then take the absolute value squared to get the photon distribution and correlated electron-positron distribution. These are given by $P_\gamma(x,t) = |\psi_\gamma(x,t)|^2$, and $P(x_-, x_+, t) = |\psi(x_-, x_+, t)|^2$. We can find the probability distribution for finding a bare electron, and the probability distribution for finding a bare positron which are

$$P_{e_-,*}(x,t) = \int dx_+ P(x, x_+), \text{ and } P_{*,e_+}(x,t) = \int dx_- P(x_-, x). \tag{3.9}$$

Fig. 3.2 plots the time evolution of these results using a discretization of 120 momentum points, with cutoff $\Lambda = 6$, for initial conditions (3.6) with $w = 0.6$, and $l = 2$. We see that an initial free photon distribution quickly creates an electron-positron distribution which moves with it as it propagates through space. Permanent pair creation from a single photon is impossible due to relativistic constraints [9]. Therefore, the electron-postron distribution should be regarded as virtual particle states due to the vacuum polarization of the photon. These dynamics are often heuristically described as a photon creating a short lived electron-positron pair which then annihilate. In Fig. 3.2 we can see the virtual electron and positron distribution overlap each other exactly. This should be expected because the model is symmetric under interchange of positrons and electrons.

3.3   Electron- Positron Scattering

We can also use an initial condition of electron-positron states. We start with a distribution containing only bare electron-positron states which are spatially separated with opposite momentum. Explicitly this is a normalized state given by
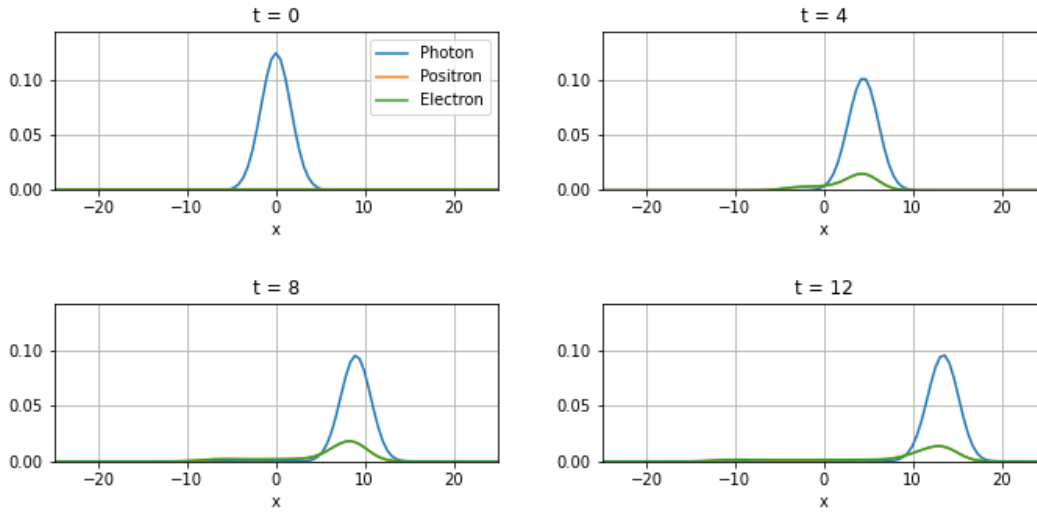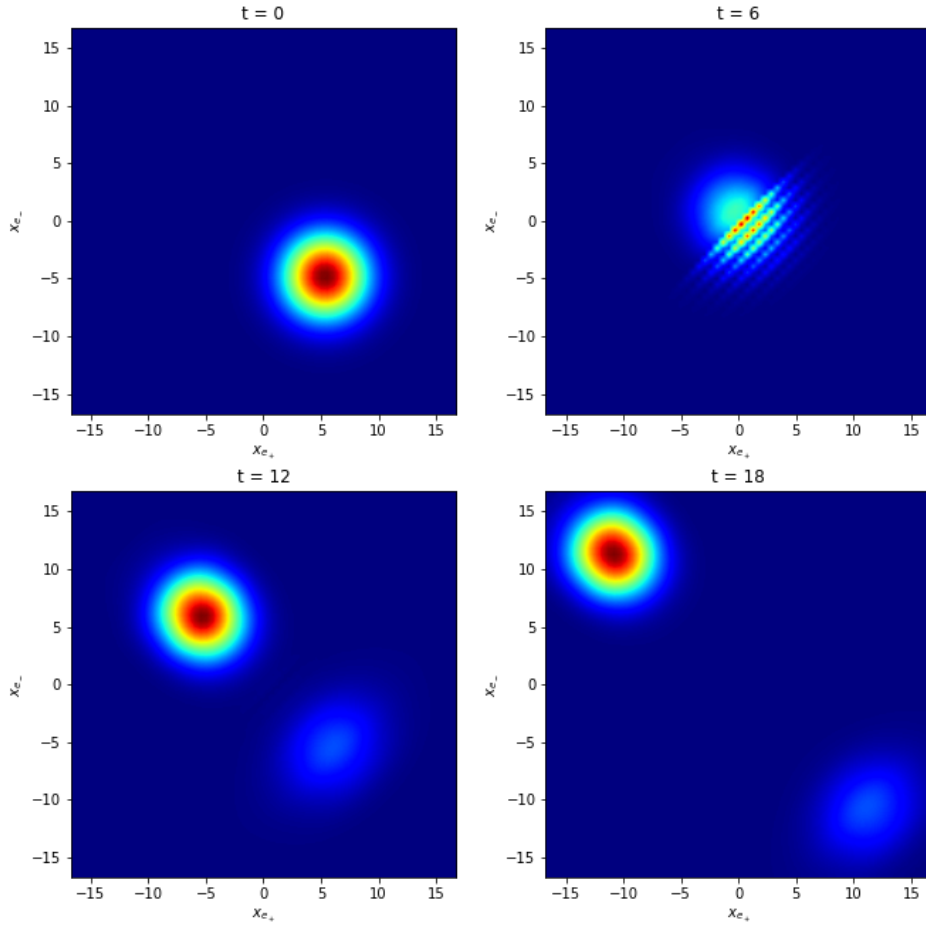
Figure 3.2 Dynamics of vacuum polarization of a propagating photon. Here time,t, is measured in units of $\frac{\hbar}{mc^2}$, and position, x, is measured in units of $\frac{mc}{\hbar}$.

$$|\psi_0\rangle \propto \int dk_- \int dk_+ \exp\left(-\frac{(k_- - l)^2}{w^2} - i(k_- - l)x_0 - \frac{(k_+ + l)^2}{w^2} + i(k_+ + l)x_0\right)|k_-, k_+\rangle. \quad (3.10)$$

We use the same methods as in the previous section to find the time evolution of this state. In Fig. 3.3 we plot $P(x_{e_-}, x_{,e_+})$ for various values of time. Here we have used the same numerical parameters as before with initial condition parameters of $l = 2, w = 0.4$, and $x_0 = 5$. Now, while these results contain the full information of the dynamics they may be hard to interpret visually. We can use (3.9) to find separate distributions for the electron and positron. While this allows us to visualize them separately, it may obscure some correlation between electron and positron position. Fig. 3.4 plots the dynamics of the separate positron, electron, and photon distributions with parameters $l = 2, w = 0.6$, and $x_0 = 5$. From these two plots we can see that the electron and positron are most likely to directly pass through each other without interacting, but there is some probability of scattering.

Electron-positron scattering is known as "Bhabha scattering". It generally involves two pro-

Figure 3.3 Dynamics of correlated electron-positron distribution, $P(x_{e_-}, x_{,e_+})$. Here time,t, is measured in units of $\frac{\hbar}{mc^2}$, and position, x, is measured in units of $\frac{mc}{\hbar}$.

cesses: an annihilation process (s-channel) where the electron-positron pair annihilate into a photon which then recreates a new pair, and a scattering process (t-channel) in which one particle emits a photon which is absorbed by the other [10]. Due to the states which are included in our model, it is only capable of modeling the s-channel dynamics. Upon initial inspection it may seem strange that the electron and positron scatter in opposite directions. Having opposite charges one might
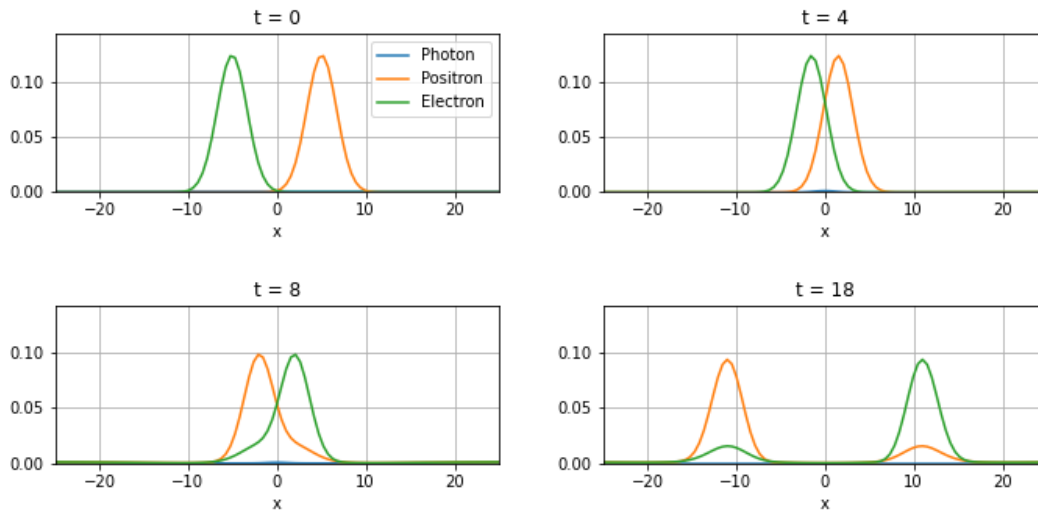
Figure 3.4 Dynamics of electron-positron scattering. Here time,t, is measured in units of $\frac{\hbar}{mc^2}$, and position, x, is measured in units of $\frac{mc}{\hbar}$.

think they should attract each other. Our results makes sense however, if one considers the fact

that our model does not include longitudinal or scalar photons thus does not include any Couloumb

interaction. Thus the largest interaction between electron and positron in this model should be as

opposing magnetic dipoles. Examining the dynamics in Fig. 3.4 we observe the curious result that

the interaction is accompanied by only a very small photon distribution despite the fact that the

interaction is mediated by the photon state.

3.4   Conclusion

In this thesis we have have undertaken a similar work to that done by Glasgow et. al in [4, 5]. We

have developed a simple model in one spatial and one time dimension for studying the interaction

between electron-positron pair states, and photon states. This model allows us to describe simplified

dynamics of Quantum Field interactions. We use this model to visualize the vacuum polarization

of a propagating photon. We find that an initial free photon distribution naturally creates a virtual

electron-positron distribution which moves with the photon. We are also able to use this model to visualize the process of Bhabha scattering. We find that initially spatially separated electron and positron distributions interact and create a probability of scattering. This simulation is analagous to the work done by Glasgow and Ware in modeling Compton scattering [5].

Unlike the model in [4, 5], we have found that our approximations give rise to divergences in the calculation of the energy of an interacting photon. This leads us to consider the question of how to implement renormalization methods within space-time-resolved models, which represents an area requiring further study. Numerical frameworks for renormalization in space-time-resolved frameworks have been proposed [11, 12], but the cases that have been addressed all involve mass renormalization. Renormalization of the vacuum polarization of a photon typically involves a field normalization [7].

More work is needed in analyzing and optimizing the numerical parameters of the model. The dynamics simulated here represent a first attempt at numerical simulation of these results. Further work could include simulating the model with a greater resolution in the discretization, and an analysis of the effect of the numerical parameters on the results of the simulation. The ultimate goal would be to develop methods to simulate more complex particle interaction such as permanent pair creation, or photon-photon scattering. This has not been done yet because doing so requires a larger state space, and thus greatly increases the computational resources required. This model represents an important step towards simulating the dynamics of these more interesting particle interactions.

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