A SEMI-CLASSICAL STUDY OF ELECTRON BLURRING IN THE LONGITUDINAL STERN-GERLACH MAGNET

by

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Abstract

We study the feasibility of polarizing electron beams using the longitudinal Stern-Gerlach effect. After a brief historical motivation we review a semi-classical analysis for electron dynamics in the presence of an axial current ring. We derive the complete set of differential equations for the trajectories and variances in this cylindrical geometry and point out differences with expressions found in the literature. We solve a subset of the equations numerically for a particular choice of initial conditions and we provide numerical simulations of the effect. We conclude by comparing the trajectory variance with the spin separation.

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I would like to thank Dr. Van Huele for never giving up on me. Thank you so very much. I would also like to thank Allison Jarstad, for inspiring me to finish what I had begun and to continually improve myself. You truly are a great friend.

Contents

1 Introduction

The truth is rarely appreciated, and perhaps that is why I feel a little trepidation in saying that I undertook this project originally with the intent to solely fulfill a requirement for graduation. However, the not so gentle nudging at times of Dr. Van Huele and the ability to utilize the limited mathematical knowledge that I have accumulated sufficiently excited me to take on a purely theoretical project. Throughout my academic career though, I had yet to undertake such an ambitious project as a senior thesis where the research and learning environments were entirely unstructured and self-imposed. In order to step-up and not become a victim of these circumstances I surrounded myself with caring people who inspired me to continue. People who eventually became my why's.

Within this paper you will find the historicity of electron beam polarization: where the idea was first generated, discussed, dashed, and once again slightly pried back open to the realm of possibility as discussed in the Background section. The development of a method for analyzing a mostly classical system with some quantum correction will be found in the Semi-Classical Method section. Its application to a specific apparatus, the Longitudinal Stern-Gerlach, will be covered in section four, Current Ring Dynamics. The equations of motion describing the electron beam in the Longitudinal Stern-Gerlach is discussed in section five, Dynamical Equations. The numerical solutions and the chosen method of solution for the equations of motion are found in the Numerical Solutions section, and in the Conclusion the question concerning the possibility of polarizing an electron beam will be presented based upon the results.

2 Background

2.1 Electron Polarization in Early Quantum Theory

Throughout the formative years of quantum theory, several of its discoverers debated at length the possiblity of polarizing an electron beam. Stern and Gerlach had polarized a beam of neutral silver atoms using what has come to be known as a transverse Stern-Gerlach magnet and it would seem natural to begin our study of electron beam polarization with an apparatus that has already proven capable of beam polarization. However, the transverse

Stern-Gerlach magnet is unable to decouple the splitting of the beam of free electrons from the blurring induced by the Lorentz force acting on a beam of finite width [3, 4, 5]. In order to circumvent the Lorentz force blurring Brillouin suggested a new experimental geometry, the longitudinal Stern-Gerlach where the magnetic inhomogeneity that generates the splitting is in the direction of propagation thereby minimizing the blurring force [6]. In response to this suggestion, the outspoken Pauli said, "it is impossible to observe the spin of the electron, separated fully from its orbital momentum, by means of experiments based on the concept of classical particle trajectories"[1]. In short, Pauli found that the longitudinal Stern-Gerlach would not polarize an electron beam. So the question remained dormant for years until Dehmelt did what Pauli thought unthinkable. Dehmelt isolated electrons of a given spin in a modified Penning trap (continuous Stern-Gerlach magnet) and measured their magnetic moment $\vec{\mu}$. Since the measurement of $\vec{\mu}$ is so closely related to the measurement of spin splitting, Dehmelt reopened the question of electron beam polarizability.

2.2 Brillouin's Proposal

With renewed potential we address Brillouin's proposal. The longitudinal Stern-Gerlach experimental setup is characterized by electrons with a specific energy being passed through an inhomogenous magnetic field at some angle to the principal direction of propagation ($\hat{z}$). The kinetic energy of the particle in the $\hat{z}$ direction depends solely upon the insertion angle, and the potential energy depends on the spin projection along $\hat{z}$. Electrons with spin parallel to the field require a different minimum insertion angle than those with spin antiparallel to it if they are to reach the detector. The difference in distance traveled associated with a given insertion angle effectively serves to split an electron beam by spin.

2.3 Pauli's Refutation

To understand both the strengths and limitations of the longitudinal Stern-Gerlach we now follow reference [9] in the following argument given by Pauli against its potential efficacy given at the Sixth Solvay Conference in 1930. We begin with a finite width beam of electrons moving in the positive $\hat{z}$ direction, anti-parallel to the primary magnetic field direction. If $\frac{\partial B_z}{\partial z} > 0$, then electrons with spin parallel to $\hat{z}$ and velocity v_z stop and reverse direction within a time t given by $mv_z = \mu_B \frac{\partial B_z}{\partial z} t$, where m is the mass of the electron. The number of electrons detected beyond $v_z t$ is half that which would be counted where spin properties are excluded. Now suppose that the magnetic field B is everywhere parallel to the xz plane, so by Maxwell's equations we know that $\frac{\partial B_x}{\partial x} = -\frac{\partial B_z}{\partial z}$. If the field at $x = 0$ is exactly along $\hat{z}$, then at a distance Δx from the z axis the magnetic field component $B_x = \frac{\partial B_x}{\partial x} \Delta x = -\frac{\partial B_z}{\partial z} \Delta x$. This field causes the velocity in the z direction to reverse sign in the Larmor precession time. Therefore, forces must act over a time much less than the Larmor precession time, i.e., $t \ll \frac{h}{\mu_B B_x}$, or equivalently $\mu_B(\frac{\partial B_z}{\partial z}) t \Delta x \ll h$, which reduces to $mv_z \Delta x \ll h$. Because of the wave nature of electrons the last condition cannot be fulfilled during the complete interaction time because the de Broglie wavelength λ is just $\frac{h}{mv_z}$ and beam widths $\Delta x \ll \lambda$ are not possible. However, if you tried to make one, then $\Delta v_x > \frac{h}{m\Delta x}$ by the uncertainty principle requires that $\Delta v_x \gg v_z$ and the outcome cannot be predicted by classical mechanics.

2.4 Problems with Pauli's Refutation

This constitutes the whole of Pauli's argument; however, there exists a point of questionable reasoning forwarded by Pauli concerning the actual classical trajectories. It is certain that an electron slightly displaced from the z-axis will experience a force that starts to rotate velocity towards the y axis. The change in the direction of motion is modified by the small induced v_y which causes a Lorentz force due to B_z. The resulting trajectory is a helical spiral about $\hat{z}$, with only one direction of motion along $\hat{z}$. Since the precession does not affect the result, the forces are not required to act over a time much less than the Larmor precession time. The uncertainty principle no longer inhibits the dynamics of the system and a time can be chosen that satisfies the initial conditions.

Therefore the essential difficulty in applying the transverse Stern-Gerlach geometry to the polarization of a finite electron beam is that Lorentz forces blur the beam, and in order to circumvent the blurring longitudinal geometry is used. To analyze this experimental situation, we cannot resort to the analytically simple mixed dynamical picture where the electron trajectory is treated classically and the spin quantum mechanically, nor can we treat the problem entirely quantum mechanically due to the analytical difficulty; so a method which exhibits partially the advantages of both dynamical pictures is employed which incorporates both classical and quantum mechanics more fully.

3 Semi-Classical Method

This method was originally developed by Sundaram and Milonni [2] for maser theory. More recently though H. Batelaan [7] has employed this method to describe electron polarization by the longitudinal Stern-Gerlach, the subject of this paper; we develop this analysis more fully and in the process we correct the form of some of the equations given in [7]. The purpose of this section is to restate the development of the method and correct the errors previously made. We begin by writing the position and momentum operators as the sum of their large classical expectation value and a small quantum correction,

$$\hat{x} = \langle x \rangle + \delta x, \qquad \hat{p}_x = \langle p_x \rangle + \delta p_x, \tag{1}$$

with the commutation relation $[\hat{x}, \hat{p}_x] = [\delta x, \delta p_x] = \imath\hbar$. In order to describe the time variance of these operators it is necessary to treat them quantum mechanically. Within quantum mechanics there are two equivalent pictures of describing time dependence. The first is the Schroedinger picture where the states, $\psi(x,t)$, are time dependent and the physical variables have no time dependence. In this picture the time dependence is described by

$$\imath\hbar \frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t). \tag{2}$$

The second is the Heisenberg picture. Within this framework the states, $\psi(x)$ are time independent and the physical variables have time dependence, the exact opposite of the Schroedinger picture. The time dependence for Heisenberg is described by

$$\imath\hbar \frac{d}{dt}A = [A, H] + \imath\hbar \frac{\partial}{\partial t}A \tag{3}$$

which leads to

$$\imath\hbar \frac{d}{dt}\langle A \rangle = \langle [A, H] \rangle + \imath\hbar \langle \frac{\partial}{\partial t}A \rangle. \tag{4}$$

Both pictures are equivalent. We will employ the Heisenberg picture to construct dynamical equations for the expectation values and for variances. This will correspond to trajectories with a point-like center and a finite extension similar to the motion of wave packets. Since our physical variables are explicitly time independent we will not need the last term in Eq. (4).

$$\imath\hbar\frac{d}{dt}\langle A\rangle = \langle[A,H]\rangle. \tag{5}$$

First, we develop a general function $F(x,p_x)$ into its Taylor series about the expectation values

$$\begin{aligned} F(x,p_x) \;=\;& F| + \left(\frac{\partial F}{\partial x}\bigg|\delta x + \frac{\partial F}{\partial p_x}\bigg|\delta p_x\right) + \\ & \frac{1}{2!}\left(\frac{\partial^2 F}{\partial x^2}\bigg|\delta x^2 + \frac{\partial^2 F}{\partial x\partial p_x}\bigg|(\delta x\delta p_x + \delta p_x\delta x) + \frac{\partial^2 F}{\partial p_x^2}\bigg|\delta p_x^2\right) + \\ & \frac{1}{3!}\left(\frac{\partial^3 F}{\partial x^3}\bigg|\delta x^3 + \frac{\partial^3 F}{\partial x^2\partial p_x}\bigg|\left(\delta x^2\delta p_x + \delta x\delta p_x\delta x + \delta p_x\delta x^2\right) + \right.\\ & \left.\frac{\partial^3 F}{\partial x\partial p_x^2}\bigg|\left(\delta x\delta p_x^2 + \delta p_x\delta x\delta p_x + \delta p_x^2\delta x\right) + \frac{\partial^3 F}{\partial p_x^3}\bigg|\delta p_x^3\right) + \cdots \end{aligned} \tag{6}$$

Where $F| = F(x,p_x)|_{x=\langle x\rangle,p_x=\langle p_x\rangle}$. We do the same for a different function $G(x,p_x)$ and we then evaluate their commutator where,

$$\begin{aligned} [F(x,p_x),G(x,p_x)] \;=\;& \imath\hbar\left\{F|,G|\right\} + \imath\hbar\left(\left\{\frac{\partial F}{\partial x}\bigg|,G|\right\} + \left\{F|,\frac{\partial G}{\partial x}\bigg|\right\}\right)\delta x \\ & + \imath\hbar\left(\left\{\frac{\partial F}{\partial p_x}\bigg|,G|\right\} + \left\{F|,\frac{\partial G}{\partial p_x}\bigg|\right\}\right)\delta p_x + \\ & \frac{\imath\hbar}{2!}\left(\left\{\frac{\partial^2 F}{\partial x^2}\bigg|,G|\right\} + 2\left\{\frac{\partial F}{\partial x}\bigg|,\frac{\partial G}{\partial x}\bigg|\right\} + \right.\\ & \left.\left\{F|,\frac{\partial^2 G}{\partial x^2}\bigg|\right\}\right)\delta x^2 + \frac{\imath\hbar}{2!}\left(\left\{\frac{\partial^2 F}{\partial x\partial p_x}\bigg|,G|\right\} + \right.\\ & \left\{\frac{\partial F}{\partial x}\bigg|,\frac{\partial G}{\partial p_x}\bigg|\right\} + \left\{\frac{\partial F}{\partial p_x}\bigg|,\frac{\partial G}{\partial x}\bigg|\right\} + \\ & \left.\left\{F|,\frac{\partial^2 G}{\partial x\partial p_x}\bigg|\right\}\right)(\delta x\delta p_x + \delta p_x\delta x) + \end{aligned}$$

$$\frac{\imath\hbar}{2!}\left(\left\{\frac{\partial^2 F}{\partial p_x^2}\bigg|, G|\right\} + 2\left\{\frac{\partial F}{\partial p_x}\bigg|, \frac{\partial G}{\partial p_x}\bigg|\right\} + \left\{F|, \frac{\partial^2 G}{\partial p_x^2}\bigg|\right\}\right)\delta p_x^2 + \cdots \tag{7}$$

Where $\{F|, G|\} = \left(\frac{\partial F}{\partial x}\Big|\frac{\partial G}{\partial p_x}\Big|\right) - \left(\frac{\partial F}{\partial p_x}\Big|\frac{\partial G}{\partial x}\Big|\right)$ represents the Poisson bracket. We now evaluate the expectation value of the commutator and get

$$\begin{aligned}\langle[F(x,p_x), G(x,p_x)]\rangle \;=\;& \imath\hbar\{F|, G|\} + \frac{\imath\hbar}{2!}\left(\left\{\frac{\partial^2 F}{\partial x^2}\bigg|, G|\right\} + 2\left\{\frac{\partial F}{\partial x}\bigg|, \frac{\partial G}{\partial x}\bigg|\right\} + \right.\\ & \left.\left\{F|, \frac{\partial^2 G}{\partial x^2}\bigg|\right\}\right)\langle\delta x^2\rangle + \frac{\imath\hbar}{2!}\left(\left\{\frac{\partial^2 F}{\partial x\partial p_x}\bigg|, G|\right\} + \right.\\ & \left\{\frac{\partial F}{\partial x}\bigg|, \frac{\partial G}{\partial p_x}\bigg|\right\} + \left\{\frac{\partial F}{\partial p_x}\bigg|, \frac{\partial G}{\partial x}\bigg|\right\} + \\ & \left.\left\{F|, \frac{\partial^2 G}{\partial x\partial p_x}\bigg|\right\}\right)(\langle\delta x\delta p_x\rangle + \langle\delta p_x\delta x\rangle) + \\ & \frac{\imath\hbar}{2!}\left(\left\{\frac{\partial^2 F}{\partial p_x^2}\bigg|, G|\right\} + 2\left\{\frac{\partial F}{\partial p_x}\bigg|, \frac{\partial G}{\partial p_x}\bigg|\right\} + \right.\\ & \left.\left\{F|, \frac{\partial^2 G}{\partial p_x^2}\bigg|\right\}\right)\langle\delta p_x^2\rangle + \cdots\end{aligned} \tag{8}$$

It is difficult to be able to discern where (8) differs from that provided by Batelaan given that it is not fully enumerated in the paper [7].

4 Current Ring Dynamics

In order to develop the Hamiltonian for Brillouin's thought experiment we need to find a physical apparatus which fulfills the longitudinal Stern-Gerlach criterion. Fortunately, a very simple apparatus exits which satisfies all the requirements, it is a circular current loop. The Hamiltonian in cylindrical coordinates for this apparatus is

$$H = \frac{(p_z - A_z)^2}{2m} + \frac{(p_\rho - A_\rho)^2}{2m} + \frac{(p_\phi - A_\phi)^2}{2m\rho^2}, \tag{9}$$

where A_z, A_ρ, and A_ϕ are the vector potentials in cylindrical coordinates. The vector potential is given [8] by $A_z = A_\rho = 0$ and

$$A_\phi(r,\theta) = \frac{4Ia}{\sqrt{a^2 + r^2 + 2ar\sin\theta}} \left[\frac{(2-k^2)K(k) - 2E(k)}{k^2} \right], \qquad (10)$$

where $r^2 = \rho^2 + z^2$ and the argument k of the elliptic integrals $K(k)$ and $E(k)$ is given by

$$k^2 = \frac{4ar\sin\theta}{a^2 + r^2 + 2ar\sin\theta}. \qquad (11)$$

When either $a \gg r$, $a \ll r$, or $\theta \ll 1$, then

$$\frac{(2-k^2)K(k) - 2E(k)}{k^2} \cong \frac{\pi k^2}{16}. \qquad (12)$$

Within the approximation, A_ϕ is now

$$A_\phi(\rho,\theta,z) \approx \frac{\pi I a^2}{c} \frac{\rho}{(a^2 + \rho^2 + z^2 + 2a\rho)^{\frac{3}{2}}}. \qquad (13)$$

Substituting the values for $\mathbf{A}$ into (9) we get

$$H = \frac{p_z^2}{2m} + \frac{p_\rho^2}{2m} + \frac{p_\phi^2}{2m\rho^2} - \omega(z)p_\phi + \frac{1}{2}m\omega(z)^2\rho^2 \qquad (14)$$

where $\omega(z) = \frac{1}{2}\frac{qB(z)}{m}$ and $B(z) = (\frac{a}{\sqrt{a^2+z^2}})^3$. The fourth term in eq. (14) is twice as large and our fifth term is four times as large as the corresponding terms in the results presented in [7]. The third term also differs from [7] where our p_ϕ^2 is replaced by $p_\phi^2 + 1/4$.

5 Dynamical Equations

We now apply Eq. (4) to the dynamical variables that we want to evaluate in order to find the trajectories. Starting with z and p_z, we discover that the geometry of the field environment forces us to look at an increasing number of expectation values of all the terms that appear in the right hand side of the equations that we developed. In order to close the set of equations we truncated the set at the third order in the small variances. We include a term $o(3)$ in the equations below to represent the truncation, corresponding to expectation values of the third order and higher (i.e. $\langle \delta p_z^3 \rangle$). Our system of equations reads now

$$\frac{d\langle z\rangle}{dt} = \frac{\langle p_z\rangle}{m}, \tag{15}$$

$$\begin{aligned}\frac{d\langle p_z\rangle}{dt} &= -\left.\frac{\partial H}{\partial z}\right| - \frac{1}{2}\left.\frac{\partial^3 H}{\partial z^3}\right|\langle \delta z^2\rangle - \left.\frac{\partial^3 H}{\partial z^2 \partial p_\phi}\right|\langle \delta z \delta p_\phi\rangle \\ &\quad - \frac{1}{2}\left.\frac{\partial^3 H}{\partial z \partial \rho^2}\right|\langle \delta \rho^2\rangle - \left.\frac{\partial^3 H}{\partial z^2 \partial \rho}\right|\langle \delta z \delta \rho\rangle + o(3),\end{aligned} \tag{16}$$

$$\frac{d\langle \delta z^2\rangle}{dt} = 2\frac{\langle \delta z \delta p_z\rangle}{m}, \tag{17}$$

$$\frac{d\langle \delta z \delta p_z\rangle}{dt} = \frac{\langle \delta p_z^2\rangle}{m} - A\langle \delta z^2\rangle - B\langle \delta z \delta p_\phi\rangle - C\langle \delta z \delta \rho\rangle + o(3), \tag{18}$$

$$\frac{d\langle \delta p_z^2\rangle}{dt} = -2A\langle \delta z \delta p_z\rangle - 2B\langle \delta p_z \delta p_\phi\rangle - 2C\langle \delta p_z \delta \rho\rangle + o(3), \tag{19}$$

$$\frac{d\langle \delta z \delta p_\phi\rangle}{dt} = \frac{\langle \delta p_z \delta p_\phi\rangle}{m}, \tag{20}$$

$$\frac{d\langle \delta z \delta \rho\rangle}{dt} = \frac{\langle \delta z \delta p_\rho\rangle}{m} + \frac{\langle \delta \rho \delta p_z\rangle}{m}, \tag{21}$$

$$\frac{d\langle \delta p_z \delta p_\phi\rangle}{dt} = -A\langle \delta z \delta p_\phi\rangle - B\langle \delta p_\phi^2\rangle - C\langle \delta \rho \delta p_\phi\rangle + o(3), \tag{22}$$

$$\frac{d\langle \delta \rho \delta p_z\rangle}{dt} = \frac{\langle p_z \delta p_\rho\rangle}{m} - A\langle \delta z \delta \rho\rangle - B\langle \delta \rho \delta p_\phi\rangle - C\langle \delta \rho^2\rangle + o(3), \tag{23}$$

$$\frac{d\langle \delta z \delta p_\rho\rangle}{dt} = \frac{\langle \delta p_z \delta p_\rho\rangle}{m} - C\langle \delta z^2\rangle - D\langle \delta z \delta \rho\rangle - E\langle \delta z \delta p_\phi\rangle + o(3), \tag{24}$$

$$\frac{d\langle \delta p_\phi^2\rangle}{dt} = 0, \tag{25}$$

$$\frac{d\langle \delta \rho \delta p_\phi\rangle}{dt} = \frac{\langle \delta p_\rho \delta p_\phi\rangle}{m}, \tag{26}$$

$$\frac{d\langle \delta \rho^2\rangle}{dt} = \frac{\langle \delta \rho \delta p_\rho\rangle + \langle \delta p_\rho \delta \rho\rangle}{m}, \tag{27}$$

$$\begin{aligned}\frac{d\langle \delta p_z \delta p_\rho\rangle}{dt} &= -A\langle \delta z \delta p_\rho\rangle - B\langle \delta p_\rho \delta p_\phi\rangle - C\langle \delta \rho \delta p_\rho\rangle - C\langle \delta z \delta p_z\rangle \\ &\quad -D\langle \delta \rho \delta p_z\rangle - E\langle \delta p_z \delta p_\phi\rangle + o(3),\end{aligned} \tag{28}$$

$$\frac{d\langle\delta\rho\delta p_\rho\rangle}{dt} = \frac{\langle\delta p_\rho^2\rangle}{m} - C\langle\delta z\delta\rho\rangle - D\langle\delta\rho^2\rangle - E\langle\delta\rho\delta p_\phi\rangle + o(3), \tag{29}$$

$$\frac{d\langle\delta p_\rho\delta p_\phi\rangle}{dt} = -C\langle\delta z\delta p_\phi\rangle - D\langle\delta\rho\delta p_\phi\rangle - E\langle\delta p_\phi^2\rangle + o(3), \tag{30}$$

$$\frac{d\langle\delta p_\rho^2\rangle}{dt} = -2C\langle\delta z\delta p_\rho\rangle - 2D\langle\delta\rho\delta p_\rho\rangle - 2E\langle\delta p_\phi\delta p_\rho\rangle + o(3), \tag{31}$$

$$\frac{d\langle\phi\rangle}{dt} = \frac{\langle p_\phi\rangle}{m\langle\rho^2\rangle} - \langle\omega(z)\rangle, \tag{32}$$

$$\frac{d\langle p_\phi\rangle}{dt} = 0, \tag{33}$$

$$\frac{d\langle\rho\rangle}{dt} = \frac{\langle p_\rho\rangle}{m}, \tag{34}$$

$$\frac{d\langle p_\rho\rangle}{dt} = \frac{\langle p_\phi^2\rangle}{m\langle\rho^3\rangle} - m\langle\omega(z)^2\rho\rangle, \tag{35}$$

with

$$A = \frac{\partial^2 H}{\partial z^2}, B = \frac{\partial^2 H}{\partial z\partial p_\phi}, C = \frac{\partial^2 H}{\partial z\partial\rho}, D = \frac{\partial^2 H}{\partial\rho^2}, E = \frac{\partial^2 H}{\partial p_\phi\partial\rho}. \tag{36}$$

Those equations whose form has been corrected are (24), (29), and (31). The last term in all three instances was omitted from the original. These three terms are equally important as their preceding terms in each of their respective equations because they have the same order of magnitude as each other term with an evaluated derivative as a coefficient. Also in the definition of C our $\partial\rho$ replaces the ∂p_ρ and our ∂p_ϕ replaces the $\partial\phi$ from [7].

6 Numerical Solutions

6.1 Spin Separation

Most important to the comparison of quantum diffusion and spin separation is the interesting subset of differential equations of motion which describe the longitudinal motion along $\hat{z}$ and the widths of the probability distributions for the beam. The subset is comprised of equations $(15)-(31)$ and they form a complete set (when truncated to the third order correction). Within the solution of the time evolution of the beam two variables are very important for they describe the beam dispersion. These values are $\langle \delta z^2 \rangle$ which describes the longitudinal width of the packet and $\langle \delta \rho^2 \rangle$ which is a measure of the transverse beam width. Splitting will be detectable if splitting, given by

$$\Delta z_{spin} = \int \int 2a_z dt' dt = \int \int \frac{2\mu_B}{m} \frac{\partial B_z}{\partial z} dt' dt, \tag{37}$$

versus dispersion remains reasonable.

6.2 Matlab

This application was chosen for its numerical analysis tools and familiarity. Recently, I had taken a lab class (Physics 430) where I had learned how to solve similar systems with relative ease in Matlab. The method taught and used is that two M-files are needed, one for the equations, rhs17.m, and one for the initial conditions of the system and the differential solver,spin.m. Both rhs17.m and spin.m are available in Appendix B. In solving the system we held the initial conditions of the apparatus geometry, magnetic field strength, and small quantum correctional uncertainties constant while allowing the beams macroscopic properties, initial position and momentum, to vary.

7 Analysis

7.1 A Typical Result

When the system is evaluated a figure appears, which describes graphically the spin separation versus the beam diffusion. If the spin separation lines exit the beam diffusion box, that result indicates the ability to quantitatively measure the spin of a free electron. However, if the spin separation lines were not to exit the beam diffusion box that would correspond to not having sufficient resolution to separate the diffusion from the separation. Under the given initial conditions, it appears that the spin separation is unable to overcome the large beam diffusion.

7.2 Possible Improvements

The method appears to be effective at approaching systems that are largely classical with a slight quantum correction. The method is appropriate for systems that are mostly classical. Also a careful determination of optimal initial conditions needs to be done in order to explore the parameter space to find out where spin polarization might be an experimental possibility. As we now have it, the mass of the electron, m_e, the Bohr magneton, μ_B, and the charge of the electron, e, are all unitary. Whereas, the beam width (ρ) is five, the current rings radius (R) is 100, and the initial strength of the magnetic field (B_0) is ten. This seems inappropriate and can be improved.

8 Conclusion

Although left for history and nearly forgotten, the longitudinal Stern-Gerlach magnet appears in simulation not to satisfy what Brillouin had hoped; however, a further study with expanded initial conditions should give us a more complete view of the validity of Brillouin's proposal.

References

[1] W. Pauli, *Collected Scientific Papers*, edited by R. Kronig and V. F. Weisskopf (Wiley, New York, 1964), Vol. 2, pp. 544-552.

[2] B. Sundaram and P. W. Milonni, "Chaos and low-order corrections to classical mechanics or geometrical optics," Phys. Rev. E **51**, 1971-1982 (1995).

[3] N. F. Mott, "The scattering of fast electrons by atomic nuclei," Proc. R. Soc. London, Ser. A **124**, 425-442 (1929).

[4] W. Pauli, *Magnetism* (Gauthier-Villars, Brussels, 1930), pp. 183-186, 217-226, 275-280.

[5] B. M. Garraway, and S. Stenholm, "Observing the spin of a free electron," Phys. Rev. A **60**, 63-79 (1999).

[6] L. Brillouin, "Is it possible to test by a direct experiment the hypothesis of the spinning electron?," Proc. Natl. Acad. Sci. U.S.A. **14**, 755-763 (1928).

[7] H. Batelaan, "Electrons, Stern-Gerlach, and quantum mechanical propagation," Am. J. Phys. **70**, 325-331(2002).

[8] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York, 1969), p. 270.

[9] H. Batelaan, T.J. Gay, and J.J. Schwendiman, "Stern-Gerlach Effect for Electron Beams," Phys. Rev. Lett. **79**, 4517-4521(1997).

A Appendices

A.1 3-Dimensional Taylor Expansion for a General Function

$$
\begin{aligned}
F(x_i,p_i) &= F| + \left(\sum_i \frac{\partial F}{\partial x_i}\bigg|\delta x_i + \sum_i \frac{\partial F}{\partial p_i}\bigg|\delta p_i\right) + \frac{1}{2!}\left(\sum_{i,j}\frac{\partial^2 F}{\partial x_i\partial x_j}\bigg|\delta x_i\delta x_j\right.\\
&\left.+\sum_{i,j}\frac{\partial^2 F}{\partial x_i\partial p_j}\bigg|\delta x_i\delta p_j + \sum_{i,j}\frac{\partial^2 F}{\partial p_i\partial x_j}\bigg|\delta p_i\delta x_j + \sum_{i,j}\frac{\partial^2 F}{\partial p_i\partial p_j}\bigg|\delta p_i\delta p_j\right)\\
&+\frac{1}{3!}\left(\sum_{i,j,k}\frac{\partial^3 F}{\partial x_i\partial x_j\partial x_k}\bigg|\delta x_i\delta x_j\delta x_k + \sum_{i,j,k}\frac{\partial^3 F}{\partial x_i\partial x_j\partial p_k}\bigg|\delta x_i\delta x_j\delta p_k\right.\\
&+\sum_{i,j,k}\frac{\partial^3 F}{\partial x_i\partial p_j\partial x_k}\bigg|\delta x_i\delta p_j\delta x_k + \sum_{i,j,k}\frac{\partial^3 F}{\partial p_i\partial x_j\partial x_k}\bigg|\delta p_i\delta x_j\delta x_k\\
&+\sum_{i,j,k}\frac{\partial^3 F}{\partial x_i\partial p_j\partial p_k}\bigg|\delta x_i\delta p_j\delta p_k + \sum_{i,j,k}\frac{\partial^3 F}{\partial p_i\partial x_j\partial p_k}\bigg|\delta p_i\delta x_j\delta p_k\\
&\left.+\sum_{i,j,k}\frac{\partial^3 F}{\partial p_i\partial p_j\partial x_k}\bigg|\delta p_i\delta p_j\delta x_k + \sum_{i,j,k}\frac{\partial^3 F}{\partial p_i\partial p_j\partial p_k}\bigg|\delta p_i\delta p_j\delta p_k\right)\\
&+\cdots. \qquad (38)
\end{aligned}
$$

A.2 Commutation Relation of Two 3-Dimensional General Functions

$$
\begin{aligned}
[F(x_i,p_i),G(x_i,p_i)] &= \imath\hbar\sum_i\{F|,G|\} + \imath\hbar\sum_i\left(\left\{\frac{\partial F}{\partial x_i}\bigg|,G|\right\}\delta x_i\right.\\
&+\left\{F|,\frac{\partial G}{\partial x_i}\bigg|\right\}\delta x_i + \left\{\frac{\partial F}{\partial p_i}\bigg|,G|\right\}\delta p_i\\
&\left.+\left\{F|,\frac{\partial G}{\partial p_i}\bigg|\right\}\delta p_i\right) + \frac{\imath\hbar}{2!}\sum_{i,j}\left(\left\{\frac{\partial^2 F}{\partial x_i\partial x_j}\bigg|,G|\right\}\delta x_i\delta x_j\right.\\
&+\left\{F|,\frac{\partial^2 G}{\partial x_i\partial x_j}\bigg|\right\}\delta x_i\delta x_j + \left\{F|,\frac{\partial^2 G}{\partial x_i\partial p_j}\bigg|\right\}\delta x_i\delta p_j
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\frac{\partial^2 F}{\partial x_i \partial p_j}\bigg|, G|\right\}\delta x_i\delta p_j+\left\{F|, \frac{\partial^2 G}{\partial p_i \partial x_j}\bigg|\right\}\delta p_i\delta x_j\\
&+\left\{\frac{\partial^2 F}{\partial p_i \partial x_j}\bigg|, G|\right\}\delta p_i\delta x_j+\left\{F|, \frac{\partial^2 G}{\partial p_i \partial p_j}\bigg|\right\}\delta p_i\delta p_j\\
&+\left.\left\{\frac{\partial^2 F}{\partial p_i \partial p_j}\bigg|, G|\right\}\delta p_i\delta p_j\right)\\
&+\imath\hbar\sum_{j,k}\left(\left\{\frac{\partial F}{\partial x_j}\bigg|, \frac{\partial G}{\partial x_k}\bigg|\right\}\delta x_j\delta x_k\right.\\
&+\left\{\frac{\partial F}{\partial x_j}\bigg|, \frac{\partial G}{\partial p_k}\bigg|\right\}\left(\frac{\delta x_j\delta p_k+\delta p_k\delta x_j}{2}\right)\\
&+\left\{\frac{\partial F}{\partial p_j}\bigg|, \frac{\partial G}{\partial x_k}\bigg|\right\}\left(\frac{\delta p_j\delta x_k+\delta x_k\delta p_j}{2}\right)\\
&+\left.\left\{\frac{\partial F}{\partial p_j}\bigg|, \frac{\partial G}{\partial p_k}\bigg|\right\}\delta p_j\delta p_k\right)+\cdots. \qquad (39)
\end{aligned}
$$

A.3 Expectation Value of the Commutation Relation between Two 3-Dimensional General Functions

$$
\begin{aligned}
\langle[F(x_i,p_i),G(x_i,p_i)]\rangle \;=\; &\imath\hbar\sum_i\{F|,G|\}+\frac{\imath\hbar}{2!}\sum_{i,j}\left(\left\{\frac{\partial^2 F}{\partial x_i \partial x_j}\bigg|, G|\right\}\langle\delta x_i\delta x_j\rangle\right.\\
&+\left\{F|, \frac{\partial^2 G}{\partial x_i \partial x_j}\bigg|\right\}\langle\delta x_i\delta x_j\rangle+\left\{F|, \frac{\partial^2 G}{\partial x_i \partial p_j}\bigg|\right\}\langle\delta x_i\delta p_j\rangle\\
&+\left\{\frac{\partial^2 F}{\partial x_i \partial p_j}\bigg|, G|\right\}\langle\delta x_i\delta p_j\rangle+\left\{F|, \frac{\partial^2 G}{\partial p_i \partial x_j}\bigg|\right\}\langle\delta p_i\delta x_j\rangle\\
&+\left\{\frac{\partial^2 F}{\partial p_i \partial x_j}\bigg|, G|\right\}\langle\delta p_i\delta x_j\rangle+\left\{F|, \frac{\partial^2 G}{\partial p_i \partial p_j}\bigg|\right\}\langle\delta p_i\delta p_j\rangle\\
&+\left.\left\{\frac{\partial^2 F}{\partial p_i \partial p_j}\bigg|, G|\right\}\langle\delta p_i\delta p_j\rangle\right)\\
&+\imath\hbar\sum_{j,k}\left(\left\{\frac{\partial F}{\partial x_j}\bigg|, \frac{\partial G}{\partial x_k}\bigg|\right\}\langle\delta x_j\delta x_k\rangle\right.\\
&+\left\{\frac{\partial F}{\partial x_j}\bigg|, \frac{\partial G}{\partial p_k}\bigg|\right\}\left(\frac{\langle\delta x_j\delta p_k\rangle+\langle\delta p_k\delta x_j\rangle}{2}\right)
\end{aligned}
$$

$$
+\left\{ \frac{\partial F}{\partial p_j}\bigg|, \frac{\partial G}{\partial x_k}\bigg|\right\} \left(\frac{\langle \delta p_j \delta x_k\rangle + \langle \delta x_k \delta p_j\rangle}{2}\right.
$$
$$
\left. + \left\{ \frac{\partial F}{\partial p_j}\bigg|, \frac{\partial G}{\partial p_k}\bigg|\right\} \langle \delta p_j \delta p_k\rangle \right) + \cdots . \tag{40}
$$

A.4 Running Matlab sources

In order to utilize the code provided below, it is necessary to create the two m-files in the same directory. Then code them as instructed. In order to run them, you double-click the spin.m file. It will query you as to an appropriate initial momentum value, enter a value (a reasonable value might be 100-1500), and it will provide the results for that value. Built into the code is a disallowance for the abrogation of the physical laws of nature (i.e. when the outputs of the code violate the Heisenberg Uncertainty Principle the code quits).In the figures that pop up after successfully running the code, the axes represent the longitudinal axis of the longitudinal Stern-Gerlach magnet (i.e. current ring).

A.5 Source 1: rhs17.m

```
function F=rhs18(t,y)
global m;
global pphi;
global rho;
global Bo;
global R;
global q;

w=q*Bo/2/m*(R/(R^2+y(1)^2)^(1/2)))^3;
w1=-(3/2)*q*Bo/m*R^3*y(1)/((R^2+y(1)^2)^(5/2));
w2=-(3/2)*q*Bo*R^3/m*(R^2-4*y(1)^2)/((R^2+y(1)^2)^(7/2);
w3=(15/2)*q*Bo*R^3/m*y(1)*(3*R^2-4*y(1)^2)/((R^2+y(1)^2)^(9/2));
Hz1=-pphi*w1+m*w*w1*rho^2;
Hz3=-pphi*w3+3*m*w1*w2*rho^2+m*w*w3*rho^2;
Hz2p_phi=-w2;
Hzrho2=2*m*w*w1;
Hz2rho=2*m*w1^2*rho+2*m*w*w2*rho;
A=-pphi*w2+m*w1^2*rho^2+m*w*w2*rho^2;
```

```
B=-w1;
C=2*m*w*w1*rho;
D=3*pphi^2/m/(rho^4)+m*w^2;
E=-2*pphi/m/(rho^3);
F=zeros(length(y),1);
F(1)=y(2)/m;
F(2)=-Hz1-(1/2)*Hz3*y(3)-Hz2pphi*y(6)-(1/2)*Hzrho2*y(13)-Hz2rho*y(7);
F(3)=2*y(4)/m;
F(4)=y(5)/m-A*y(3)-b*y(6)-C*y(7);
F(5)=-2*A*y(4)-2*B*y(8)-2*C*y(14);
F(6)=y(8)/m;
F(7)=y(10)/m+y(14)/m;
F(8)=-A*y(6)-B*y(11)-C*y(12);
F(9)=y(14)/m-A*y(7)-B*y(12)-C*y(13);
F(10)=y(14)/m-C*y(3)-D*y(7)-E*y(6);
F(11)=0;
F(12)=y(16)/m;
F(13)=y(15)/m;
F(14)=-A*y(10)-B*y(16)-c*y(15)-C*y(4)-D*y(9)-E*y(8);
F(15)=y(17)/m-C*y(7)-D*y(13)-E*y(11);
F(16)=-C*y(6)-D*y(12)-E*y(11);
F(17)=-2*C*y(10)-2*D*y(15);
```

A.6 Source 2: spin.m

```
clear;close all;
global m;
global pphi;
global rho;
global Bo;
global R;
global q;

initial conditions
m=1;pphi=0;rho=5;Bo=10;R=100;q=1;mu=1;

y0=zeros(17,1);
y0(1)=-500;
```

```
y0(2)=input(' initial momentum - ');
y0(3)=1;
y0(4)=1;
y0(5)=1;
y0(6)=1;
y0(7)=1;
y0(8)=1;
y0(9)=1;
y0(10)=1;
y0(11)=0;
y0(12)=1;
y0(13)=1;
y0(14)=1;
y0(15)=1;
y0(16)=1;
y0(17)=1;

options=odeset('RelTol',1e-8);
tstart=0;tfinal=10;
N=1024;
taue=(tfinal-tstart)/(N-1);
te=tstart:taue:tfinal;
[t,y]=ode45(@rhs_17,te,y0,options);

spin separation animation



h1=0:.002:.15;
h2=0:.005:.15;
for n=1:N-1

   spin down (backward)
   xdown(1)=y0(1);
   vdown(1)=y0(2)/m;
   adown(1)=(mu/m)*(-3*Bo*R^3)*xdown(1)/((R^2+(xdown(1))^2)^(5/2));
   vdown(n+1)=vdown(n)+adown(n)*taue;
   xdown(n+1)=xdown(n)+vdown(n)*taue+(1/2)*adown(n)*(taue)^2;
```

```
    adown(n+1)=(mu/m)*(-3*Bo*R^3)*xdown(n+1)/((R^2+(xdown(n+1))^2)^(5/2));

    no spin
    xzero(1)=y0(1);
    vzero(1)=y0(2)/m;
    azero(1)=0;
    vzero(n+1)=vzero(n)+azero(n)*taue;
    xzero(n+1)=xzero(n)+vzero(n)*taue+(1/2)*azero(n)*(taue)^2;
    azero(n+1)=0;

    spin up (forward)
    xup(1)=y0(1);
    vup(1)=y0(2)/m;
    aup(1)=(mu/m)*(3*Bo*R^3)*xup(1)/((R^2+(xup(1))^2)^(5/2));
    vup(n+1)=vup(n)+aup(n)*taue;
    xup(n+1)=xup(n)+vup(n)*taue+(1/2)*aup(n)*(taue)^2;
    aup(n+1)=(mu/m)*(3*Bo*R^3)*xup(n+1)/((R^2+(xup(n+1))^2)^(5/2));
end

square wave animation & blurring

n=0;
set(gca,'NextPlot','replacechildren')
for j=1:32:N
    mu=y(j,1);
    sigma=y(j,3);
    delta=y(j,5);
    if sigma<0
        fprintf(' position dispersion violation \%g \n',y(j,3));
        break
    elseif delta<0
        fprintf(' momentum dispersion violation \%g \n',y(j,5));
        break
    elseif delta*sigma<.25
        fprintf(' heisenberg uncertainty violation \%g \n',y(j,5)*y(j,3));
        break
    else
        if sigma>250
```

```
            sigma=250;
        end
        x=(mu-sigma):(mu+sigma);
        s=1/2/sigma;
        a=0:.0005:s;
        plot(x,s,'k-',mu-sigma,a,'k-',mu+sigma,a,'k-',xdown(j),h2,'b-',xup(j),h1,'
        axis([y(j,1)-150 y(j,1)+150 0 .15])
        n=n+1;
        Q(:,n)=getframe;
    end
end
movie(Q,0)
```