Derivation of the Unruh Effect

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ABSTRACT

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We derive the Unruh effect using the method of Bogoliubov transformations. We begin by constructing the Bogoliubov transformations which relate different sets of QFT operators to each other. We then consider the specific case of comparing an inertial reference frame to an accelerating reference frame. We calculate the Bogoliubov coefficients for this situation and use these to show that the state which is a vacuum from the perspective of the inertial observer contains particles from the perspective of the accelerating observer. The particle density has the form of a thermal bath with a temperature proportional to the acceleration. We discuss the interpretation of this result and possible other applications of the techniques used in this derivation.

Keywords: Unruh, Quantum Field Theory, Bogoliubov Transformation, Vacuum

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Contents

Chapter 1

Introduction

In classical physics, the vacuum is a trivial thing. It contains nothing, and nothing happens there. In Quantum Field Theory (QFT), however, the vacuum has quite a bit of structure. It includes field fluctuations, and entanglement between different locations. We think of the vacuum as the absence of particles, but even the definition of a particle in quantum field theory depends on how you choose to describe your system. We consider one example of this: An accelerating observer uses different coordinates to describe space and time than an inertial observer, and as a consequence, the definition of a particle differs significantly between the two, so much so that the state which appears to be a vacuum to the inertial observer is flooded with particles from the perspective of the accelerating observer. This is called the Unruh effect, and the particles observed in the accelerated frame are called Unruh radiation. In this paper, we derive the Unruh effect from the basic principles of quantum field theory.

The calculation is divided into four chapters. In Chapter 2 we cover basic QFT and general Bogoliubov transformations. We also discuss how to work with quantum fields in general, without relying on any specific details of the problem. We motivate and introduce the Bogoliubov transformation, which allows us to move between different descriptions of the same system. We find that the important information about our system is captured by the classical solutions to the equations of motion, and the information required for Bogoliubov transformations can be described using a certain symplectic product on the classical solutions.

In chapter 3 we cover coordinates and classical solutions for inertial and accelerating observers. In particular, we perform all the classical field theory calculations that are necessary precursors to the quantum calculation. We construct the coordinate system of the inertial and accelerating observers and solve the classical field theory in each of these coordinates systems, thus obtaining the classical mode functions we need. We also write down the form of the symplectic product in each coordinate system, which will allow us to do explicit calculations later.

Chapter 4 deals with the actual calculation of the Bogoliubov coefficients. Here we combine the general principles of QFT derived in chapter 2 and the classical solutions derived in chapter 3 to find the specific Bogoliubov coefficients for our problem. Our final results and presented in chapter 5. We write the particle number density, as seen by the accelerating observer, in terms of Bogoliubov coefficients, and then use the results of Chapter 4 to extract a value. We discuss the interpretation of this particle number density as thermal radiation.

The main points I intend to cover in this thesis are those that would me most useful to someone who has been exposed to basic QFT and is hoping to solve problems about comparing different vacua in QFT. For example, after reading this, the audience should be able to see how to apply these principles to solve Unruh in the massive case or the case with more than 1 spatial dimension.

Chapter 2

Basic QFT and general Bogoliubov transformations

2.1 Classical field theory

In classical field theory, the objects of interest are fields, which are functions of spacetime which satisfy certain equations of motion. In the case where the equations of motion are linear, then the space of solutions has the form of a vector space with a symplectic product.

This symplectic product can be written in terms of the values of the functions at any single point in time as

$$(f, g) = -i \int f(x,t) \overleftrightarrow{\partial}_t g^*(x,t) dx \ . \tag{2.1}$$

Where $f \overleftrightarrow{\partial} g$ is defined to be

$$f \overleftrightarrow{\partial} g = f \partial g - (\partial f) g. \tag{2.2}$$

A useful feature of the symplectic product is

$$(f, g) = -i \int f(x,t) \overleftrightarrow{\partial}_t g^*(x,t) dx = i \int g^*(x,t) \overleftrightarrow{\partial}_t f(x,t) dx = -(g^*, f^*) \ . \tag{2.3}$$

To do calculations in this space of solutions, it is useful to choose a basis, ideally something that we can treat like an orthonormal basis. If we have available a definition of time translation, then we can distinguish solutions by their frequency ω with respect to this time variable, and we can choose a basis consisting of a collection of positive frequency solutions, $\alpha_\omega(x,t)$, along with their negative frequency complex conjugates, $\alpha^*_\omega(x,t)$. In particular, if we choose these functions so that they satisfy

$$(\alpha_\omega, \alpha_{\omega'}) = \delta(\omega, \omega') \ , \quad (\alpha^*_\omega, \alpha^*_{\omega'}) = -\delta(\omega, \omega') \ , \quad (\alpha_\omega, \alpha^*_{\omega'}) = 0 \ , \tag{2.4}$$

then we can think of this set of solutions as an orthonormal basis for the solution space. It is not obvious that basis with this property exists. In this section, we simply assume that such a basis has been found, and in later sections we discuss the problem of how this is done.

We refer to elements of such a basis as mode functions. Each mode function is a function of spacetime, and we index these mode functions with a parameter which includes ω, but may also include other information, since there may be multiple mode functions with each frequency. For example, as we will see later, in a 1+1 dimensional spacetime, there are right-moving and left-moving wave solutions for each frequency. We will discuss the problem of parameterizing the solution space in more detail in chapter 3. For now, we just assume that ω lives in some space that can be integrated over.

Since the mode functions form a basis for the solution space, we may write any solution ϕ of the classical EOMs as a linear combination of mode functions:

$$\phi = \int d\omega \left[\alpha_\omega(x,t)a(\omega) + \alpha^*_\omega(x,t)a^*(\omega)\right] \ , \tag{2.5}$$

Where the $a(\omega)$ gives the coefficients in a linear combination. We can extract these coefficients by projecting ϕ against the mode functions and using the symplectic product and the orthogonality relation:

$$a(\omega) = (\phi, \alpha_\omega) \ , \quad a^*(\omega) = -(\phi, \alpha^*_\omega) \ . \tag{2.6}$$

For more detail and rigor regarding this procedure, see pages 10-52 of [1].

2.2 Quantization

At this point, we are ready to turn our classical field theory into a quantum field theory. In quantum mechanics, physics is described using a vector space containing state vectors, and operators which act on that vector space. We denote vectors by $|v\rangle$ where v is simply a label for some vector. The conjugate vector to $|v\rangle$ is expressed as $\langle v|$. Operators are given hats, such as $\widehat{O}$.

When we quantize a classical theory, we associate each observable with a hermitian operator. For example, the position x of an object becomes an operator $\widehat{X}$. Similarly, to quantize a classical field theory, we must turn the field into an operator, i.e. the classical field $\phi(x,t)$ becomes an operator $\hat{\phi}(x,t)$. Notice that this operator depends on position and time, so it is really an operator-valued function of spacetime. For the field theories we address in this paper, the time derivative of ϕ is an independent observable, and thus we must have an additional operator $\hat{\pi} := \partial_t\hat{\phi}$, somewhat analogous to the momentum operator in quantum mechanics. These satisfy a commutation relation which is closely connected to the symplectic product:

$$[\phi(x,t), \pi(x',t)] = i\delta(x,x') \tag{2.7}$$

We now prove that

$$[(\hat{\phi}, f), (\hat{\phi}, g)] = -(g^*, f) \tag{2.8}$$

To begin we note that

$$\begin{aligned}
&\left[\left(\hat{\phi}, f\right), \left(\hat{\phi}, g\right)\right] \\
&= \left[-i\int \hat{\phi}(x,t)\overleftrightarrow{\partial}_t f^*(x,t)dx\,, -i\int \hat{\phi}(x',t)\overleftrightarrow{\partial}_t g^*(x',t)dx'\right] \\
&= \left[-i\int \left(\hat{\phi}(x)\dot{f}^*(x) - \hat{\pi}(x)f^*(x)\right)dx\,, -i\int \left(\hat{\phi}(x')\dot{g}^*(x') - \hat{\pi}(x')g^*(x')\right)dx'\right]\,,
\end{aligned} \tag{2.9}$$

where in the last step we have used the fact that $\hat{\pi} = \partial_t\hat{\phi}$ and suppressed the time dependence for brevity. We then use linearity of the commutator to pull everything but the operators out of the commutators:

$$\left[\left(\hat{\phi}, f\right), \left(\hat{\phi}, g\right)\right] = (-i)^2 \int dx \int dx' \Big(\dot{f}^*(x)\dot{g}^*(x') \left[\hat{\phi}(x), \hat{\phi}(x')\right] - \dot{f}^*(x) g^*(x') \left[\hat{\phi}(x), \hat{\pi}(x')\right] \\ - f^*(x)\dot{g}^*(x') \left[\hat{\pi}(x), \hat{\phi}(x')\right] + f^*(x) g^*(x') \left[\hat{\pi}(x), \hat{\pi}(x')\right]\Big) \tag{2.10}$$

We evaluate the commutators using Eq. (2.7)

$$\begin{aligned}[(\hat{\phi}, f), (\hat{\phi}, g)] &= (-i)^2 \int dx \int dx' \left(-\dot{f}^*(x) g^*(x') \left(i\delta(x,x')\right) - f^*(x)\dot{g}^*(x') \left(-i\delta(x,x')\right)\right) \\ &= -i \int dx \left(-\dot{f}^*(x) g^*(x) + f^*(x)\dot{g}^*(x)\right) = -i \int dx \left(f^*(x) \overleftrightarrow{\partial_t} g^*(x)\right) \\ &= (f^*, g) = -(g^*, f) \ . \end{aligned} \tag{2.11}$$

2.3 Modes and Particles

We wish to use our classical mode functions to simplify our description of the system. To this end, we generalize the mode expansion formula and apply it to the operator $\hat{\phi}$:

$$\hat{\phi} = \int d\omega \left(\alpha_\omega(x,t)\hat{a}(\omega) + \alpha^*_\omega(x,t)\hat{a}^\dagger(\omega)\right) \ . \tag{2.12}$$

One way to think of this is that you have defined $\hat{a}$ and $\hat{a}^\dagger$ by projecting $\hat{\phi}$ against the mode functions using the symplectic product:

$$\hat{a}(\omega) = (\hat{\phi}, \alpha_\omega) \ , \quad \hat{a}^\dagger(\omega) = -(\hat{\phi}, \alpha^*_\omega) \ . \tag{2.13}$$

Note that the operator $\hat{\pi}$ becomes part of $\hat{a}$ and $\hat{a}^\dagger$ because there are time derivatives in the symplectic product.

We can easily find the commutation relations between these newly defined operators: It follows immediately from Eq. (2.13), (2.11), and (2.4) that

$$[\hat{a}(\omega), \hat{a}(\omega')] = 0 \ , \quad \left[\hat{a}(\omega), \hat{a}^\dagger(\omega')\right] = \delta(\omega, \omega') \ . \tag{2.14}$$

These commutation relations imply that we can interpret $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ as annihilation and creation operators, respectively, and that the operator

$$\hat{N}(\omega) := \hat{a}^\dagger(\omega)\hat{a}(\omega) \ , \tag{2.15}$$

can be interpreted as a particle number operator. See standard texts on quantum field theory, such as [2], for introductions to these operators.

Although I have referenced time translations and frequency, these things are not required to arrive at Eq. (2.14) and the conclusion that $\hat{N}$ counts particle number. The only necessary assumption we made was that the space of solutions could be given a basis satisfying Eq. (2.4). The formula Eq. (2.1) appears to depend on a separation between time and space coordinates, however this formula is just one representation of the symplectic product, which is a coordinate-independent object.

Although we can interpret our operators in terms of particle creation and annihilation even in the absence of time translation symmetry, the "particles" described this way do not necessarily have definite energy. To get particles of definite energy, we need our classical solutions to have well-defined frequency with respect to some definition of time translation. Specifically, we need α_ω to satisfy

$$\alpha_\omega(x,t) = \alpha_\omega(x,t_0)e^{-i\omega(t-t_0)}. \tag{2.16}$$

This condition ensures that α is oscillating in time with a fixed frequency, or equivalently, fixed energy. Then $\hat{a}^\dagger$ and $\hat{a}$ respectively add and remove exactly this energy from the system, and we can think of this quantum of energy as a particle.

In addition to providing us with the notion of particles in general, the creation and annihilation operators allow us to define the vacuum state, $|0\rangle$. Naturally we expect the vacuum to contain no particles, so $|0\rangle$ should satisfy

$$\hat{N}(\omega)\,|0\rangle = 0 \ . \tag{2.17}$$

Multiplying on the left by $\langle 0|$, we find that

$$0 = \langle 0|\,\hat{a}^\dagger(\omega)\hat{a}(\omega)\,|0\rangle = (\hat{a}(\omega)\,|0\rangle)^\dagger\,\hat{a}(\omega)\,|0\rangle\,, \tag{2.18}$$

which means that $\hat{a}(\omega)\,|0\rangle$ is a vector of magnitude zero, so

$$\hat{a}(\omega)\,|0\rangle = 0 \tag{2.19}$$

for all ω. This argument can be made more rigorous, but for our purposes it suffices to take Eq. (2.19) as the definition of the vacuum state

We are ready to ask the question that this thesis is about: "Does that vacuum state depend on the choice of coordinates and mode functions?" Suppose that we have two distinct coordinate systems for spacetime: (x, t) and (χ, τ), and a corresponding set of mode functions for each coordinate system. That is, suppose that in addition to the $\alpha_\omega(x, t)$ we already found, there is also another collection of functions $\beta_\Omega(\chi, \tau)$. These are indexed by a new frequency parameter Ω and likewise satisfy an equation that has the same form as Eq. (2.4):

$$(\beta_\Omega, \beta_{\Omega'}) = \delta(\Omega, \Omega')\ , \quad (\beta^*_\Omega, \beta^*_{\Omega'}) = -\delta(\Omega, \Omega')\ , \quad (\beta_\Omega, \beta^*_{\Omega'}) = 0. \tag{2.20}$$

However, they oscillate with respect to a different time coordinate, τ rather than t, so

$$\beta_\Omega(\chi, \tau) = \beta_\Omega(\chi, \tau_0)e^{-i\Omega(\tau-\tau_0)}, \tag{2.21}$$

analogous to Eq. (2.16). As a reminder, the parameters ω and Ω include frequency but may also run over other degrees of freedom.

The conserved symplectic product can be written in terms of these new coordinates as

$$(f, g) = -i\int f(\chi, \tau)\overleftrightarrow{\partial}_\tau g^*(\chi, \tau)d\chi\ . \tag{2.22}$$

Note that this is the same symplectic product on the space of solutions to the equations of motion, we are just representing it differently. Indeed, each piece of machinery we developed using the mode functions $\alpha_\omega, \alpha^*_\omega$ can be duplicated. We have creation and annihilation operators:

$$\hat{b}(\Omega) = \left(\hat{\phi}, \beta_\Omega\right)\ , \quad \hat{b}^\dagger(\Omega) = -\left(\hat{\phi}, \beta^*_\Omega\right)\ , \tag{2.23}$$

the mode expansion:

$$\hat{\phi}(\chi, \tau) = \int d\Omega\left(\beta_\Omega(\chi, \tau)\hat{b}(\Omega) + \beta^*_\Omega(\chi, \tau)\hat{b}^\dagger(\Omega)\right)\ , \tag{2.24}$$

and the commutation relations:

$$\left[\hat{b}(\Omega), \hat{b}(\Omega')\right] = 0 \ , \quad \left[\hat{b}(\Omega), \hat{b}^\dagger(\Omega')\right] = \delta(\Omega, \Omega') \ , \tag{2.25}$$

analogous to Eq. (2.13), (2.12), and (2.14), respectively

Finally, each of these sets of operators produces its own number operator and its own definition of the vacuum:

$$\hat{N}_a(\omega) = \hat{a}^\dagger(\omega)\hat{a}(\omega) \ , \quad \hat{N}_b(\Omega) = \hat{b}^\dagger(\Omega)\hat{b}(\Omega) \ , \tag{2.26}$$

where $|0_a\rangle$ satisfies

$$\hat{a}(\omega) |0_a\rangle = 0 \ , \tag{2.27}$$

for all ω and $|0_b\rangle$ satisfies

$$\hat{b}(\Omega) |0_b\rangle = 0 \ , \tag{2.28}$$

for all Ω. These two vacuua are in general different! Most of this thesis is devoted to calculating the relationship between them for two particular coordinate systems.

2.4 Bogoliubov Transforms

At a very high level, the reason the vacua may be different is because although $\hat{a}, \hat{a}^\dagger$ and $\hat{b}, \hat{b}^\dagger$ span the same space of all possible operators which are linear in $\hat{\phi}$ and $\hat{\pi}$, the annihilation operators $\hat{a}$ span a different subspace than the $\hat{b}$'s. The annihilation operators in one basis may contain a component of the creation operators in the other basis, which means each the vacuum in each system fails to be in the null space of the annihilation operators of the other. We now make this idea of mixing between operators precise.

We wish to relate the creation and annihilation operators between the two perspectives. The way we do this is with something called a Bogoliubov transform, which has the form:

$$\hat{b}(\Omega) = \int d\omega \left[A_{\Omega\omega}\hat{a}(\omega) - B_{\Omega\omega}\hat{a}^\dagger(\omega)\right] \ . \tag{2.29}$$

it is not obvious from the outset that such a representation is even possible, but remember that by Eq. (2.23) we can write $\hat{b}$ as

$$\hat{b}(\Omega) = (\hat{\phi}, \beta_\Omega) \ , \tag{2.30}$$

and then by Eq. (2.12) and we can write ϕ in terms of a and $a^\dagger$, as follows:

$$\begin{aligned} \hat{b}(\Omega) &= (\hat{\phi}, \beta_\Omega) \\ &= \left(\int d\omega (\alpha_\omega \hat{a}(\omega) + \alpha_\omega^* \hat{a}^\dagger(\omega)) \, , \beta_\Omega \right) \\ &= \int d\omega \left((\alpha_\omega, \beta_\Omega) \, \hat{a}(\omega) + (\alpha_\omega^*, \beta_\Omega) \, \hat{a}^\dagger(\omega) \right) \ , \end{aligned} \tag{2.31}$$

using the linearity of the symplectic product. This allows us to immediately write down the Bogoliubov coefficients:

$$A_{\Omega\omega} = (\alpha_\omega, \beta_\Omega) \ , \quad B_{\Omega\omega} = -(\alpha_\omega^*, \beta_\Omega) \tag{2.32}$$

Note that we can evaluate these using any of the various representations we have for the conserved symplectic product.

There is another useful relation these coefficients satisfy, which is a way to write the dirac delta in frequency space in terms of the coefficients. We find this by evaluating the commutation relations that $\hat{b}(\Omega)$ and its conjugate must satisfy. We have:

$$\begin{aligned} \delta(\Omega, \Omega') &= \left[\hat{b}(\Omega), \hat{b}^\dagger(\Omega') \right] \\ &= \int d\omega \int d\omega' \left[\left(A_{\Omega\omega} \hat{a}(\omega) - B_{\Omega\omega} \hat{a}^\dagger(\omega) \right) , \left(A_{\Omega'\omega'}^* \hat{a}^\dagger(\omega') - B_{\Omega'\omega'}^* \hat{a}(\omega') \right) \right] \\ &= \int d\omega \int d\omega' \left(A_{\Omega\omega} A_{\Omega'\omega'}^* \left[\hat{a}(\omega), \hat{a}^\dagger(\omega') \right] + B_{\Omega\omega} B_{\Omega'\omega'}^* \left[\hat{a}^\dagger(\omega), \hat{a}(\omega') \right] \right) \\ &= \int d\omega \int d\omega' (A_{\Omega\omega} A_{\Omega'\omega'}^* \delta(\omega, \omega') - B_{\Omega\omega} B_{\Omega'\omega'}^* \delta(\omega, \omega')) \\ &= \int d\omega (A_{\Omega\omega} A_{\Omega'\omega}^* - B_{\Omega\omega} B_{\Omega'\omega}^*) \ . \end{aligned} \tag{2.33}$$

Finally, we can use the Bogoliubov coefficients to calculate the expectation value of $\hat{N}_b(\Omega)$ in the vacuum $|0_a\rangle$. Using Eq. (2.26) and (2.29), we have

$$\begin{aligned}\langle 0_a|\, \hat{N}_b(\Omega)\, |0_a\rangle &= \langle 0_a|\, \hat{b}^\dagger(\Omega)\hat{b}(\Omega)\, |0_a\rangle \\ &= \langle 0_a| \left(\int d\omega \left[A_{\Omega\omega}\hat{a}(\omega) - B_{\Omega\omega}\hat{a}^\dagger(\omega)\right]\right)^\dagger \left(\int d\omega' \left[A_{\Omega\omega'}\hat{a}(\omega') - B_{\Omega\omega'}\hat{a}^\dagger(\omega')\right]\right) |0_a\rangle \ .\end{aligned}$$

We use linearity to pull the integrals out of the expectation value:

$$\langle 0_a|\, \hat{N}_b(\Omega)\, |0_a\rangle = \int d\omega \int d\omega' \langle 0_a| \left(A^*_{\Omega\omega}\hat{a}^\dagger(\omega) - B^*_{\Omega\omega}\hat{a}(\omega)\right) \left(A_{\Omega\omega'}\hat{a}(\omega') - B_{\Omega\omega'}\hat{a}^\dagger(\omega')\right) |0_a\rangle \ . \tag{2.34}$$

the $\hat{a}$ acting on $|0_a\rangle$ and the $\langle 0_a|$ acting on $\hat{a}^\dagger$ go to zero by Eq. (2.27), leaving us with

$$\begin{aligned}\langle 0_a|\, \hat{N}_b(\Omega)\, |0_a\rangle = \int d\omega \int d\omega' B^*_{\Omega\omega} B_{\Omega\omega'} \langle 0_a|\, \hat{a}(\omega)\hat{a}^\dagger(\omega')\, |0_a\rangle &= \int d\omega \int d\omega' B^*_{\Omega\omega} B_{\Omega\omega'} \delta(\omega, \omega') \\ &= \int d\omega |B_{\Omega\omega}|^2 \ .\end{aligned} \tag{2.35}$$

We can already see that if any of the $B_{\Omega\omega}$ coefficients are nonzero, then this expected particle number must be nonzero. So the vacuum state with respect to one coordinate systems need not contain zero particles when viewed from the other coordinate system. However, this is as far as we can go while keeping the problem general. In order to make any more progress towards evaluating this, we must proceed by choosing specific coordinates and mode functions, which is the subject of the next chapter.

Chapter 3

Coordinates and Classical Solutions for Inertial and Accelerating Observers

In this section, we construct the Minkowski and Rindler coordinate systems to describe the motion of inertial and accelerating observers. We then solve the classical field theory in each coordinate system to find collections of mode functions for quantization.

3.1 Minkowski coordinates

We begin with Minkowski coordinates in 1+1 dimensions: there is a time coordinate, t, and a position coordinate, x, each of which can take on any real value. The most important feature of Minkowski space is spacetime metric, which allows us to find magnitudes of spacetime vectors. The Minkowski metric is given by

$$ds^2 = dt^2 - dx^2 \ . \tag{3.1}$$

It will frequently be useful to describe Minkowski space in terms of light-like directions, so we define an alternative coordinate system for Minkowski space as follows:

$$u = t - x \ , \quad v = t + x \ . \tag{3.2}$$

These are called light cone variables. These are not the alternative coordinates we use for quantization. We merely use to simplify certain calculations. We can invert Eq. (3.2) to find

$$t = \frac{v+u}{2} \ , \quad x = \frac{v-u}{2} \ . \tag{3.3}$$

The metric takes on a particularly simple form in terms of light cone variables:

$$ds^2 = dt^2 - dx^2 = (dt - dx)(dt + dx) = du dv \ . \tag{3.4}$$

When we write down equations of motion, we will actually use the inverse metric rather than the metric. The metric is used to take inner products between spacetime vectors, so if p^μ and q^μ are vectors, we may write down a Lorentz-independent scalar $g_{\mu\nu}p^\mu q^\nu$. If we represent p and q in terms of their t and x components, then we may represent $g_{\mu\nu}$ as a symmetric matrix:

$$g_{\mu\nu}p^\mu q^\nu = \begin{pmatrix} t_p & x_p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} t_q \\ x_q \end{pmatrix} \ . \tag{3.5}$$

This equation looks different in different bases. For example, in terms of light cone coordinates, we have

$$g_{\mu\nu}p^\mu q^\nu = \begin{pmatrix} u_p & v_p \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} u_q \\ v_q \end{pmatrix} \ . \tag{3.6}$$

This is all well and good for multiplying vectors together. But what about covectors, such as the derivative $\partial_\mu \phi$ of a field? For that, we need the inverse metric, $g^{\mu\nu}$, which satisfies $g_{\mu\nu}g^{\nu\sigma} = \delta_\mu^\sigma$. So for example, in light cone Minkowski coordinates, $g^{\mu\nu}$ as a matrix is

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \ . \tag{3.7}$$

In Minkowski coordinates, the path of an inertial observer through spacetime is a straight line, and we can always choose a reference frame in which the observer is stationary at the origin of space. That is, the path of the inertial observer, as a function of their proper time s_i, can be written as

$$(x_i(s_i), t_i(s_i)) = (0, s_i) \tag{3.8}$$

We can also write this in light cone coordinates:

$$(u_i(s_i), v_i(s_i)) = (s_i, s_i) \ . \tag{3.9}$$

3.2 An Accelerating Observer and Rindler Coordinates

An observer experiencing constant acceleration traces out a hyperbola in Minkowski coordinates, with asymptotes pointing along light-like directions. If we position the accelerating observer so that the origin of spacetime is the intersection of those asymptotes, then we may describe the accelerating observer's position as a function of their proper time as follows:

$$(x_a(s_a), t_a(s_a)) = \left(\tfrac{1}{a}\cosh(as_a), \tfrac{1}{a}\sinh(as_a)\right) \ . \tag{3.10}$$

Or, in light cone variables:

$$(u_a(s_a), v_a(s_a)) = \left(-\tfrac{1}{a}e^{-as_a}, \tfrac{1}{a}e^{as_a}\right) \ . \tag{3.11}$$

This doesn't look nearly as nice as the path of the inertial observer. Presumably, an accelerating observer would prefer coordinates in which their path resembles Eq. (3.8). There is a coordinate system, called Rindler coordinates, that plays nicely with the accelerating observer's path. If we define $u = -\frac{1}{a}e^{-a\bar{u}}$ and $v = \frac{1}{a}e^{a\bar{v}}$, then we may rewrite Eq. (3.11) in terms of $\bar{u}$ and $\bar{v}$ as

$$(\bar{u}_a(s_a), \bar{v}_a(s_a)) = (s_a, s_a) \, , \tag{3.12}$$

which closely resembles Eq. (3.9).

These are the coordinates we want, but there are some bugs to work out. For example, if $\bar{u}$ and $\bar{v}$ are both allowed to range over the reals, u will only take on negative values and v will only take on positive values, which means that the only points in Minkowski space which are accessible in this way are those satisfying $x > |t|$, which is a wedge on the right side of the coordinate plane. In order to use things like Eq. (2.1), we need to be able to access at the very least a line running from $x = -\infty$ to $x = \infty$. Can we expand the $\bar{u}, \bar{v}$ coordinate system to also describe the left-hand wedge, the one where $x < -|t|$? Yes, this can be done by simply flipping some signs in the definitions: $u = \frac{1}{a}e^{a\bar{u}}$ and $v = -\frac{1}{a}e^{-a\bar{v}}$. But now these coordinates don't reach the right wedge.

What we need to do is create a coordinate system that packages some discrete left-right information in along with the continuous variables. To accomplish this, we include an extra bit of information with $\bar{u}$. First, we define $\bar{\boldsymbol{u}} := (\bar{u}, D_u)$ where D_u can be either $+1$ or -1, and $\bar{u}$ can take on any real value. Then, we define the coordinate transformation by

$$u(\bar{\boldsymbol{u}}) = u(\bar{u}, D_u) = \begin{cases} -\dfrac{1}{a}e^{-a\bar{u}} & D_u = -1 \\ \dfrac{1}{a}e^{a\bar{u}} & D_u = 1 \end{cases}, \tag{3.13}$$

where $D_u = 1$ corresponds to points in the future wedge and the left wedge, while $D_u = -1$ corresponds to points in the past wedge and the right wedge. Likewise, we define $\bar{\boldsymbol{v}} := (\bar{v}, D_v)$ and

$$v(\bar{\boldsymbol{v}}) = v(\bar{v}, D_v) = \begin{cases} -\dfrac{1}{a}e^{-a\bar{v}} & D_v = -1 \\ \dfrac{1}{a}e^{a\bar{v}} & D_v = 1 \end{cases}, \tag{3.14}$$

where $D_v = 1$ corresponds to points in the future wedge and the right wedge, while $D_v = -1$ corresponds to points in the past wedge and the left wedge.

Just as we have x and t in Minkowski space, we often find it useful to use a timelike and spacelike pair of coordinates to describe Rindler space. Specifically, we define these coordinates on the left and right wedges, ignoring the future and past wedges since they are not used in our calculations. This means that our spacelike coordinate is doubled, similar to $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}}$, but our timelike coordinate can be shared between the two wedges. Specifically, we define

$$\tau(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) = \begin{cases} \dfrac{\bar{u} + \bar{v}}{2} & D_u D_v = -1 \\ \text{undefined} & D_u D_v = 1 \end{cases}, \tag{3.15}$$

and

$$\boldsymbol{\chi} = (\chi, D_\chi) = \begin{cases} \left(\frac{\bar{u}-\bar{v}}{2}, -1\right) & D_u = 1, D_v = -1 \\ \left(\frac{\bar{u}-\bar{v}}{2}, 1\right) & D_u = -1, D_v = 1 \\ \text{undefined} & D_u D_v = 1 \end{cases} . \tag{3.16}$$

From here on out, we usually use D_χ to indicate which wedge we are referring to rather than D_u and D_v. $D_\chi = 1$ denotes the right wedge, and $D_\chi = -1$ denotes the left wedge. If we wish represent the symplectic product in these coordinates, our spatial integral is over $\boldsymbol{\chi}$, and thus we need to add together two integrals, one for each value of D_χ.

We also need the metric in Rindler coordinates. Recall from Eq. (3.4) that the metric in Minkowski coordinates is given by

$$ds^2 = dudv \; . \tag{3.17}$$

For $D_\chi = 1$, we have $D_u = -1, D_v = 1$, and plugging in the formulas from Eq. (3.13), (3.14), we have

$$\begin{aligned} ds^2 = dudv &= d\left(-\frac{1}{a}e^{-a\bar{u}}\right) d\left(\frac{1}{a}e^{a\bar{v}}\right) = e^{a(\bar{u}-\bar{v})} d\bar{u} d\bar{v} \\ &= e^{2a\chi} d\bar{u} d\bar{v} \; . \end{aligned} \tag{3.18}$$

Similarly, for $D_\chi = -1$ we find

$$ds^2 = e^{-2a\chi} d\bar{u} d\bar{v} \; . \tag{3.19}$$

We can combine these equations together:

$$ds^2 = e^{2a\chi D_\chi} d\bar{u} d\bar{v} \; , \tag{3.20}$$

and we can write it in terms of $d\chi$ and $d\tau$ if we wish:

$$ds^2 = e^{2a\chi D_\chi}(d\tau^2 - d\chi^2) \; . \tag{3.21}$$

Notice that the Rindler metrics have the same form as the Minkowski metric, up to multiplication by a scalar-valued function of spacetime. Transformations with this property are called Conformal.

3.3 Classical Field Equations and Mode functions

For the purposes of this paper, we work with one of the simplest possible systems, a massless scalar field, described by the Klein-Gordon equation:

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi = \partial_t^2\phi - \partial_x^2\phi = 0, \tag{3.22}$$

where $g^{\mu\nu}$ is the inverse of the metric $g_{\mu\nu}$. This can be written in Minkowski light-cone coordinates as:

$$4\partial_u\partial_v\phi = 0, \tag{3.23}$$

and equivalently in Rindler coordinates

$$4e^{-2a\chi D_\chi}\partial_{\bar{u}}\partial_{\bar{v}}\phi = 0. \tag{3.24}$$

But thanks to the conformal nature of the transformation, this is just a scalar factor multiplied by the Minkowski version. We can cancel that out, leaving us with

$$4\partial_{\bar{u}}\partial_{\bar{v}}\phi = 0. \tag{3.25}$$

The solutions to the equations of motion are plane waves in both cases. In Minkowski coordinates, these can be written as:

$$e^{\pm i\omega u} \ , \quad e^{\pm i\omega v} \ , \tag{3.26}$$

where ω may take on any positive real value. The functions of u represent right-moving waves, and the functions of v represent left-moving waves. Much as we combined $\bar{u}, \bar{v}$ with discrete information to produce $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}$, we combine the real-valued ω with the discrete information about left-moving or right-moving solutions into a single variable

$$\boldsymbol{\omega} = (\omega, d_\omega) \ . \tag{3.27}$$

Here d_ω can take on the values $+1$ and -1, with $d_\omega = +1$ representing right-moving waves and $d_\omega = -1$ representing left-moving waves. Then our solution set for Minkowski coordinates, which we call $\alpha_{\boldsymbol{\omega}}(u,v)$, along with their complex conjugates, is

$$\alpha_{(\omega,+1)}(u,v) = C(\omega)e^{-i\omega u} \qquad \text{right-moving positive frequency waves} \tag{3.28}$$

$$\alpha^*_{(\omega,+1)}(u,v) = C(\omega)e^{i\omega u} \qquad \text{right-moving negative frequency waves} \tag{3.29}$$

$$\alpha_{(\omega,-1)}(u,v) = C(\omega)e^{-i\omega v} \qquad \text{left-moving positive frequency waves} \tag{3.30}$$

$$\alpha^*_{(\omega,-1)}(u,v) = C(\omega)e^{i\omega v} \qquad \text{left-moving negative frequency waves} \tag{3.31}$$

where the C's represent normalization constants which are as yet unknown. To find them, we evaluate Eq. (2.1) and enforce Eq. (2.4). Consider the case of two values $\boldsymbol{\omega}$ and $\boldsymbol{\omega'}$ which have $d_\omega = d_{\omega'} = 1$. We have

$$\begin{aligned}(\alpha_{\boldsymbol{\omega}}, \alpha_{\boldsymbol{\omega'}}) &= -i\int \alpha_{\boldsymbol{\omega}}(x,t)\overleftrightarrow{\partial_t}\alpha^*_{\boldsymbol{\omega'}}(x,t)dx \\ &= -i\int C(\omega)C^*(\omega')e^{-i(\omega-\omega')u}\left[i\omega(\partial_t u) - (-i\omega'(\partial_t u))\right]dx \ ,\end{aligned} \tag{3.32}$$

where $u = t - x$, so $\partial_t u = 1$. We then find

$$(\alpha_{\boldsymbol{\omega}}, \alpha_{\boldsymbol{\omega'}}) = (\omega+\omega')C(\omega)C^*(\omega')e^{-i(\omega-\omega')t}\int e^{i(\omega-\omega')x}dx \ . \tag{3.33}$$

The integral at the end is equivalent to $2\pi\delta(\omega-\omega')$, and by Eq. (2.4), the whole expression should equal $\delta(\omega-\omega')$, so we must have

$$(\omega+\omega')C(\omega)C^*(\omega')e^{-i(\omega-\omega')t}(2\pi)\delta(\omega-\omega') = \delta(\omega-\omega') \tag{3.34}$$

and since the delta functions enforce $\omega = \omega'$, this we must have

$$2\omega|C(\omega)|^2(2\pi) = 1 \ , \tag{3.35}$$

so we may take $C(\omega) = \frac{1}{\sqrt{4\pi\omega}}$. Similar aruments apply to the other cases. The mode functions with the correct normalizations are

$$\alpha_{(\omega,+1)}(u,v) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u} \qquad \text{right-moving positive frequency waves} \tag{3.36}$$

$$\alpha^*_{(\omega,+1)}(u,v) = \frac{1}{\sqrt{4\pi\omega}} e^{i\omega u} \qquad \text{right-moving negative frequency waves} \tag{3.37}$$

$$\alpha_{(\omega,-1)}(u,v) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v} \qquad \text{left-moving positive frequency waves} \tag{3.38}$$

$$\alpha^*_{(\omega,-1)}(u,v) = \frac{1}{\sqrt{4\pi\omega}} e^{i\omega v} \qquad \text{left-moving negative frequency waves} \tag{3.39}$$

We can write these in terms of x and t, and this is in fact more convenient, since d can be used directly to encode the sign information that distinguishes left-moving from right-moving:

$$\begin{aligned}\alpha_{\boldsymbol{\omega}}(t,x) &= \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega(t - x d_\omega)}\ , \\ \alpha^*_{\boldsymbol{\omega}}(t,x) &= \frac{1}{\sqrt{4\pi\omega}} e^{i\omega(t - x d_\omega)}\ ,\end{aligned} \tag{3.40}$$

We proceed in a similar fashion in Rindler coordinates, but with an added complication: There are separate solutions for $D_\chi = 1$ and $D_\chi = -1$. Thus we need

$$\boldsymbol{\Omega} = (\Omega, d_\Omega, D_\Omega)\ , \tag{3.41}$$

with d_Ω representing the direction of motion, and D_Ω representing the wedge on which the solution has its support. Our solution set is $\beta(\boldsymbol{\Omega})$ given by

$$\begin{aligned}\beta_{\boldsymbol{\Omega}}(\tau,\boldsymbol{\chi}) &= \begin{cases} \frac{1}{\sqrt{4\pi\Omega}} e^{-i\Omega(\tau - \chi d_\Omega)} & D_\chi = D_\Omega \\ 0 & D_\chi \neq D_\Omega \end{cases} \\ \beta^*_{\boldsymbol{\Omega}}(\tau,\boldsymbol{\chi}) &= \begin{cases} \frac{1}{\sqrt{4\pi\Omega}} e^{i\Omega(\tau - \chi d_\Omega)} & D_\chi = D_\Omega \\ 0 & D_\chi \neq D_\Omega \end{cases} .\end{aligned} \tag{3.42}$$

We have found our mode functions! Everything we did in Chapter 2 can now be applied to these α and β functions: We can now talk about creation and annihilation operators $\hat{a}^\dagger(\boldsymbol{\omega}), \hat{a}(\boldsymbol{\omega}), \hat{b}^\dagger(\boldsymbol{\Omega}), \hat{b}(\boldsymbol{\Omega})$, and we have access to everything we proved about their commutation relations and the Bogoliubov coefficients relating them.

Chapter 4

Calculating the Bogoliubov coefficients

We now have the Minkowski and Rindler mode functions functions respectively defined in Eq. (3.40) and (3.42). In this chapter we use these to evaluate the Bogoliubov coefficients as described in Eq. (2.32), which is rewritten below to match our current notation:

$$A_{\boldsymbol{\Omega}\boldsymbol{\omega}} = (\alpha_{\boldsymbol{\omega}}, \beta_{\boldsymbol{\Omega}}) \ , \quad B_{\boldsymbol{\Omega}\boldsymbol{\omega}} = -(\alpha^*_{\boldsymbol{\omega}}, \beta_{\boldsymbol{\Omega}}) \tag{4.1}$$

We begin by expanding out the symplectic product in Rindler coordinates, and splitting the integral over $\boldsymbol{\chi}$ into left-wedge and right-wedge parts:

$$\begin{aligned} A_{\boldsymbol{\Omega}\boldsymbol{\omega}} = (\alpha_{\boldsymbol{\omega}}, \beta_{\boldsymbol{\Omega}}) &= -i\int \alpha_{\boldsymbol{\omega}}(\boldsymbol{\chi},\tau)\overleftrightarrow{\partial}_\tau \beta^*_{\boldsymbol{\Omega}}(\boldsymbol{\chi},\tau)d\boldsymbol{\chi} \\ &= -i\int_{\substack{-\infty \\ D_\chi=-1}}^{\infty} \alpha_{\boldsymbol{\omega}}(\boldsymbol{\chi},\tau)\overleftrightarrow{\partial}_\tau \beta^*_{\boldsymbol{\Omega}}(\boldsymbol{\chi},\tau)d\chi - i\int_{\substack{-\infty \\ D_\chi=1}}^{\infty} \alpha_{\boldsymbol{\omega}}(\boldsymbol{\chi},\tau)\overleftrightarrow{\partial}_\tau \beta^*_{\boldsymbol{\Omega}}(\boldsymbol{\chi},\tau)d\chi \end{aligned} \tag{4.2}$$

Consider the case where $d_\omega = 1, d_\Omega = 1$, and $D_\Omega = 1$. Then $\beta_{\boldsymbol{\Omega}}$ is only nonzero for $D_\chi = 1$, so we may discard the left integral and expand out α, β, and u explicitly in terms of $\bar{u}$,

$$\begin{aligned} A_{\boldsymbol{\Omega}\boldsymbol{\omega}} &= -i\int_{\substack{-\infty \\ D_\chi=1}}^{\infty} \alpha_{\boldsymbol{\omega}}(\boldsymbol{\chi},\tau)\overleftrightarrow{\partial}_\tau \beta^*_{\boldsymbol{\Omega}}(\boldsymbol{\chi},\tau)d\chi = -i\int_{\substack{-\infty \\ D_\chi=1}}^{\infty} \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega u(\boldsymbol{\chi},\tau)}\overleftrightarrow{\partial}_\tau \frac{1}{\sqrt{4\pi\Omega}}e^{i\Omega\bar{u}}d\chi \\ &= \frac{-i}{4\pi\sqrt{\omega\Omega}}\int_{-\infty}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)}\overleftrightarrow{\partial}_\tau e^{i\Omega\bar{u}}d\chi \qquad (4.3) \\ &= \frac{-i}{4\pi\sqrt{\omega\Omega}}\int_{-\infty}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)}(-\overleftrightarrow{\partial}_\chi)e^{i\Omega\bar{u}}d\chi \ . \qquad (4.4) \end{aligned}$$

In the last step, we have replaced the τ derivative with a $-\chi$ derivative, since the only τ dependence occurs through $\bar{u}$, and $\partial_\tau \bar{u} = -\partial_\chi \bar{u}$.

Recall that $f\overleftrightarrow{\partial} g$ is defined to equal $f(\partial g) - (\partial f)g$. If we had only the first of these terms, $f\partial g$, under an integral, we would be tempted to integrate by parts. This would give us a boundary term, and a new integral containing precisely the second term $-g\partial f$. In other words, the two terms here are equivalent under integration by parts, with the caveat that there is a boundary term. Such boundary terms are typically permitted to be zero in these field theory calculations We might imagine that α and β are not pure frequencies, but rather wave packets that stretch across a large finite region and evaluate to zero on the infinitely-distant boundaries. In any case, proceeding under the assumption that we can ignore these boundary terms, we may replace $f\partial g$ with $-g\partial f$ or vice versa. In other words, we can replace $f\overleftrightarrow{\partial} g$ with simply $2f\partial g$, giving us

$$\begin{aligned} A_{\boldsymbol{\Omega\omega}} &= \frac{i}{4\pi\sqrt{\omega\Omega}} \int_{-\infty}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} 2\partial_\chi e^{i\Omega\bar{u}} d\chi = \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} \left(\frac{\partial\bar{u}}{\partial\chi}\right) e^{i\Omega\bar{u}} d\chi \\ &= -\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} e^{i\Omega\bar{u}} d\chi \ . \end{aligned} \tag{4.5}$$

Now changing variables from χ to $\bar{u}$ gives us

$$A_{\boldsymbol{\Omega\omega}} = -\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} e^{i\Omega\bar{u}} d\bar{u} \ . \tag{4.6}$$

In this calculation, we had $d_\omega = d_\Omega$. If we take, say, $d_\omega = 1$ and $d_\Omega = -1$, then the $\bar{u}$ inside of β becomes a $\bar{v}$. The calculation proceeds essentially the same way up through Eq. (4.3), which becomes

$$A_{\boldsymbol{\Omega\omega}} = \frac{-i}{4\pi\sqrt{\omega\Omega}} \int_{\substack{-\infty \\ D_\chi=1}}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} \overleftrightarrow{\partial}_\tau e^{i\Omega\bar{v}} d\chi \ . \tag{4.7}$$

Since this depends on both $\bar{u}$ and $\bar{v}$, we cannot simply replace the τ derivative with a χ derivative throughout the integral. However, we can do it term by term:

$$\begin{aligned}
A_{\mathbf{\Omega\omega}} &= \frac{-i}{4\pi\sqrt{\omega\Omega}} \int_{\substack{-\infty \\ D_\chi=1}}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} \partial_\tau e^{i\Omega\bar{v}} - \left(\partial_\tau e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)}\right) e^{i\Omega\bar{v}} d\chi \\
&= \frac{-i}{4\pi\sqrt{\omega\Omega}} \int_{\substack{-\infty \\ D_\chi=1}}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} \partial_\chi e^{i\Omega\bar{v}} - \left(-\partial_\chi e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)}\right) e^{i\Omega\bar{v}} d\chi \\
&= \frac{-i}{4\pi\sqrt{\omega\Omega}} \int_{\substack{-\infty \\ D_\chi=1}}^{\infty} \partial_\chi \left(e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} e^{i\Omega\bar{v}}\right) d\chi \qquad (4.8)
\end{aligned}$$

This integral is nothing but a boundary term! As before, we assume this vanishes. Thus there is no overlap between these left-moving and right-moving waves. This turns out to be true across all the mode functions: Unless $d_\omega = d_\Omega$, the symplectic product evaluates to zero.

D_Ω	d_ω	d_Ω	$A_{\mathbf{\Omega\omega}}$	$B_{\mathbf{\Omega\omega}}$
1	1	1	$-\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} e^{i\Omega\bar{u}} d\bar{u}$	$\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{i\omega\left(-\frac{1}{a}e^{-a\bar{u}}\right)} e^{i\Omega\bar{u}} d\bar{u}$
1	1	-1	0	0
1	-1	1	0	0
1	-1	-1	$-\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{-i\omega\left(\frac{1}{a}e^{a\bar{v}}\right)} e^{i\Omega\bar{v}} d\bar{v}$	$\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{i\omega\left(\frac{1}{a}e^{a\bar{v}}\right)} e^{i\Omega\bar{v}} d\bar{v}$
-1	1	1	$-\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{-i\omega\left(\frac{1}{a}e^{a\bar{u}}\right)} e^{i\Omega\bar{u}} d\bar{u}$	$\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{i\omega\left(\frac{1}{a}e^{a\bar{u}}\right)} e^{i\Omega\bar{u}} d\bar{u}$
-1	1	-1	0	0
-1	-1	1	0	0
-1	-1	-1	$-\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{-i\omega\left(-\frac{1}{a}e^{-a\bar{v}}\right)} e^{i\Omega\bar{v}} d\bar{v}$	$\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{i\omega\left(-\frac{1}{a}e^{-a\bar{v}}\right)} e^{i\Omega\bar{v}} d\bar{v}$

Table 4.1 Formulas for Bogoliubov coefficients

The calculation of $B_{\mathbf{\Omega\omega}}$ involves complex conjugating one of these functions, but that does not affect the argument about dependence on $\bar{u}$ and $\bar{v}$. The same process works, and produces things like Eq. (4.6) with minus signs in different places. Continuing like this, we can find formulas for all the coefficients, which are given in Table 4.1.

The reader may disagree, but I think it is easier to do one integral than eight. So it is worthwhile to merge these formulas together. In these formulas, $\bar{u}$ and $\bar{v}$ are being integrated over, and can be freely renamed. By changing variables with a sign flip in some cases, we can make the nested exponentials all have the same form, as shown in table 4.2 with the shared coefficient of $\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}$ removed, as shown in Table 4.2.

A or B	D_Ω	d_ω, d_Ω	change of variables	result
A	1	1	$\bar{u} \to -s$	$-\int_{-\infty}^{\infty} e^{i\omega\left(\frac{1}{a}e^{as}\right)} e^{-i\Omega s} ds$
A	1	-1	$\bar{v} \to s$	$-\int_{-\infty}^{\infty} e^{-i\omega\left(\frac{1}{a}e^{as}\right)} e^{i\Omega s} ds$
A	-1	1	$\bar{u} \to s$	$-\int_{-\infty}^{\infty} e^{-i\omega\left(\frac{1}{a}e^{as}\right)} e^{i\Omega s} ds$
A	-1	-1	$\bar{v} \to -s$	$-\int_{-\infty}^{\infty} e^{i\omega\left(\frac{1}{a}e^{as}\right)} e^{-i\Omega s} ds$
B	1	1	$\bar{u} \to -s$	$\int_{-\infty}^{\infty} e^{-i\omega\left(\frac{1}{a}e^{as}\right)} e^{-i\Omega s} ds$
B	1	-1	$\bar{v} \to s$	$\int_{-\infty}^{\infty} e^{i\omega\left(\frac{1}{a}e^{as}\right)} e^{i\Omega s} ds$
B	-1	1	$\bar{u} \to s$	$\int_{-\infty}^{\infty} e^{i\omega\left(\frac{1}{a}e^{as}\right)} e^{i\Omega s} ds$
B	-1	-1	$\bar{v} \to -s$	$\int_{-\infty}^{\infty} e^{-i\omega\left(\frac{1}{a}e^{as}\right)} e^{-i\Omega s} ds$

Table 4.2 Variable changes to simplify formulas for Bogoliubov coefficients

Many of these are related to each other by complex conjugation, and the rest can be summarized in terms of a certain sign flip. Specifically, if we let σ be ± 1, and define $X(\sigma)$ to be

$$X(\sigma) = \sigma \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} e^{\sigma i\omega\left(\frac{1}{a}e^{as}\right)} e^{i\Omega s} ds \tag{4.9}$$

then we can compress everything down to what is shown in Table 4.3.

We proceed by evaluating $X(\sigma)$, leaving σ undetermined until the end. Using $z = e^{as}$, and $dz = azds$, we have

$$\begin{aligned} 2\pi\sqrt{\frac{\omega}{\Omega}} X(\sigma) &= \sigma \int_{-\infty}^{\infty} ds e^{i\sigma\omega\frac{1}{a}e^{as} + i\Omega s} = \sigma \int_{-\infty}^{\infty} ds e^{i\sigma\omega\frac{1}{a}e^{as}} \left(e^{as}\right)^{\frac{i\Omega}{a}} = \sigma \int_0^{\infty} \frac{dz}{az} e^{i\sigma\omega\frac{z}{a}} z^{\frac{i\Omega}{a}} \\ &= \frac{\sigma}{a}\int_0^{\infty} dz\ z^{\frac{i\Omega}{a}-1} e^{i\frac{\sigma\omega}{a}z} = \frac{\sigma}{a}\int_0^{\infty} dz\ z^{\zeta-1} e^{-\xi z}\ , \end{aligned} \tag{4.10}$$

D_Ω	d_ω, d_Ω	$A_{\mathbf{\Omega\omega}}$	$B_{\mathbf{\Omega\omega}}$
1	1	$X^*(-1)$	$X^*(1)$
1	-1	$X(-1)$	$X(1)$
-1	1	$X(-1)$	$X(1)$
-1	-1	$X^*(-1)$	$X^*(1)$

Table 4.3 Extremely abreviated formulas for Bogoliubov coefficients

where in the last step we have defined the abbreviations $\zeta = \frac{i\Omega}{a}$ and $\xi = -i\frac{\sigma\omega}{a}$. The last integral in Eq. (4.10) is a form of the definition of the Gamma function:

$$\int_0^\infty dz\ z^{\zeta-1}e^{-\xi z} = \xi^{-\zeta}\Gamma\left(\zeta\right) \tag{4.11}$$

which allows us to evaluate the integral, and we find

$$\begin{aligned} 2\pi\sqrt{\frac{\omega}{\Omega}}X(\sigma) = \sigma\left(-i\frac{\sigma\omega}{a}\right)^{-\frac{i\Omega}{a}}\Gamma\left(\frac{i\Omega}{a}\right) &= \frac{\sigma}{a}\left(e^{-\sigma i\frac{\pi}{2}}e^{\ln\left(\frac{\omega}{a}\right)}\right)^{-\frac{i\Omega}{a}}\Gamma\left(\frac{i\Omega}{a}\right) \\ &= \frac{\sigma}{a}e^{-\sigma\frac{\pi\Omega}{2a}}e^{-\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)}\Gamma\left(\frac{i\Omega}{a}\right)\ . \end{aligned} \tag{4.12}$$

Combining Eq. (4.12) with Table 4.3, our nonzero Bogoliubov coefficients are given by table 4.4

D_Ω	d_ω	d_Ω	$A_{\mathbf{\Omega\omega}}$	$B_{\mathbf{\Omega\omega}}$
1	1	1	$-\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}e^{\frac{\pi\Omega}{2a}}e^{\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)}\Gamma\left(\frac{-i\Omega}{a}\right)$	$\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}e^{-\frac{\pi\Omega}{2a}}e^{\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)}\Gamma\left(\frac{-i\Omega}{a}\right)$
1	-1	-1	$-\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}e^{\frac{\pi\Omega}{2a}}e^{-\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)}\Gamma\left(\frac{i\Omega}{a}\right)$	$\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}e^{-\frac{\pi\Omega}{2a}}e^{-\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)}\Gamma\left(\frac{i\Omega}{a}\right)$
-1	1	1	$-\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}e^{\frac{\pi\Omega}{2a}}e^{-\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)}\Gamma\left(\frac{i\Omega}{a}\right)$	$\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}e^{-\frac{\pi\Omega}{2a}}e^{-\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)}\Gamma\left(\frac{i\Omega}{a}\right)$
-1	-1	-1	$-\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}e^{\frac{\pi\Omega}{2a}}e^{\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)}\Gamma\left(\frac{-i\Omega}{a}\right)$	$\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}e^{-\frac{\pi\Omega}{2a}}e^{\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)}\Gamma\left(\frac{-i\Omega}{a}\right)$

Table 4.4 Explicit forms of nonzero Bogoliubov coefficients

The fact which is most useful to us is that for any $\boldsymbol{\omega}$ and $\mathbf{\Omega}$,

$$|A_{\mathbf{\Omega\omega}}|^2 = e^{+\frac{2\pi\Omega}{a}}|B_{\mathbf{\Omega\omega}}|^2. \tag{4.13}$$

Chapter 5

Results and Interpretation

In this chapter we combine the results of the previous chapters to explicitly calculate the particle number density of the Minkowski vacuum according to the Rindler particle number operators. We then discuss the meaning of these results and some ways that these ideas can be extended to other problems.

5.1 Results

We have now done all the hard work to find the Bogoliubov coefficients, and the next part is relatively simple. Applying Eq. (2.35), we have

$$\langle 0_M| \hat{N}_{\mathbf{\Omega}} |0_M\rangle = \int d\boldsymbol{\omega} |B_{\mathbf{\Omega}\boldsymbol{\omega}}|^2 \ . \tag{5.1}$$

We now have a formula for B, and we could plug that in and churn through the integral. But there is a shortcut. We already saw in Eq. (4.13) that $A_{\mathbf{\Omega}\boldsymbol{\omega}}$ and $B_{\mathbf{\Omega}\boldsymbol{\omega}}$ are proportional to each other, and there is also a normalization condition the coefficients must satisfy that constrains their combined size.

To find the normalization condition, we take the identity relation from Eq. (2.33):

$$\delta(\mathbf{\Omega}, \mathbf{\Omega}') = \int d\boldsymbol{\omega} (A_{\mathbf{\Omega}\boldsymbol{\omega}} A^*_{\mathbf{\Omega}'\boldsymbol{\omega}} - B_{\mathbf{\Omega}\boldsymbol{\omega}} B^*_{\mathbf{\Omega}'\boldsymbol{\omega}}) \ , \tag{5.2}$$

and we plug in a single value of $\mathbf{\Omega}$ for both $\mathbf{\Omega}$ and $\mathbf{\Omega}'$. That is, we assume $d_\Omega = d'_\Omega$, $D_\Omega = D'_\Omega$, and $\Omega = \Omega'$. We find

$$\delta(0) = \int d\boldsymbol{\omega} |A_{\mathbf{\Omega}\boldsymbol{\omega}}|^2 - |B_{\mathbf{\Omega}\boldsymbol{\omega}}|^2 \; .$$

The left hand side here is the Dirac delta function evaluated at zero, which is normally regarded as infinite. However, as discussed in [3], we may think of $\delta(0)$ as representing V, the volume of space in which we are quantizing, and the volume which we must divide by to turn our particle number into a number density.

Now apply the proportionality of $A_{\mathbf{\Omega}\boldsymbol{\omega}}$ and $B_{\mathbf{\Omega}\boldsymbol{\omega}}$ from Eq. (4.13):

$$\begin{aligned} V = \delta(0) &= \int d\boldsymbol{\omega} |B_{\mathbf{\Omega}\boldsymbol{\omega}}|^2 e^{\frac{2\pi\Omega}{a}} - |B_{\mathbf{\Omega}\boldsymbol{\omega}}|^2 \\ &= \left(e^{\frac{2\pi\Omega}{a}} - 1\right) \int d\boldsymbol{\omega} |B_{\mathbf{\Omega}\boldsymbol{\omega}}|^2 \; , \\ \int d\boldsymbol{\omega} |B_{\mathbf{\Omega}\boldsymbol{\omega}}|^2 &= V \left(e^{\frac{2\pi\Omega}{a}} - 1\right)^{-1} \; . \end{aligned} \tag{5.3}$$

This means that

$$\langle 0_M | \hat{N}_{\mathbf{\Omega}} | 0_M \rangle = \int d\boldsymbol{\omega} |B_{\mathbf{\Omega}\boldsymbol{\omega}}|^2 = V \left(e^{\frac{2\pi\Omega}{a}} - 1\right)^{-1} \tag{5.4}$$

Which gives us an average particle density of

$$n_{\mathbf{\Omega}} = \frac{\langle 0_M | \hat{N}_{\mathbf{\Omega}} | 0_M \rangle}{V} = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} = \frac{1}{e^{\frac{2\pi\Omega}{a/c}} - 1} \; . \tag{5.5}$$

Note that I have thus far been using natural units, $\hbar = c = 1$, but now is a good time to restore factors of these constants to make the units match up. In this formula, the only needed change is to replace a with $\frac{a}{c}$, which has been done in the last step above. Compare this to the formula for Boltzmann distribution for particles obeying Bose-Einstein statistics at a temperature T:

$$n(E) = \frac{1}{e^{E/(k_B T)} - 1} \; . \tag{5.6}$$

These have exactly the same form, with

$$\frac{E}{k_b T} = \frac{2\pi\Omega c}{a} \; . \tag{5.7}$$

Using the fact that the energy of a particle is $E = \hbar\Omega$, we can rearrange this into

$$T = \frac{\hbar a}{2\pi c k_b} \; . \tag{5.8}$$

In other words, we have found that the accelerating observer will find themselves immersed in a bath of particles whose density spectrum exactly matches that of a thermal bath of particles at temperature $T = \frac{\hbar a}{2\pi c k_b}$.

5.2 Interpretation

What are we to make of the temperature? This system appears to be a thermal state, but it is possible to extract more information if you look closer. Indeed, [1, 4] derives an exact representation of the Minkowski vacuum in terms of the Rindler particle number states, and it turns out to be a state with strong entanglement between the left and right wedges of spacetime. When you assume one of these wedges is inaccessible for measurement, which is the case for the accelerating observer, then the entanglement between the wedges turns into classical uncertainty about the state of the observable wedge.

Taking a step back, the reason why the particle density is nonzero is because $B_{\boldsymbol{\Omega\omega}}$ is nonzero. Remember that $B_{\boldsymbol{\Omega\omega}}$ represents mixing between positive and negative frequency states. To be more precise, it describes the extent to which negative-frequency mode functions in one coordinate system have a component made of positive frequency modes in the other coordinate system. But the Bogoliubov transformations can be applied to much more than just coordinate systems. They simply relate different bases of creation and annihilation operators. So anytime you have a system that can be seen from two different perspectives, if you can construct positive and negative frequency mode functions, then you can build a framework of particles in each system and use the Bogoliubov coefficients to learn how they are related. Some examples of situations where this is useful:

- Black holes and Hawking radiation [5]. In a spacetime containing a black hole, take one coordinate system representing a distant stationary observer. In a result that is very analogous to the calculation done here, we find that the stationary observer detects particles leaving the black hole.

- Vacuum decay. Some field theories allow the universe to exist in a metastable false vacuum state, which decays to the true vacuum during some finite time interval. Before the decay, mode functions and corresponding particles can be defined relative to the false vacuum, and after the decay a very different collection of mode functions and particles can be defined relative to the true vacuum. The method of Bogoliubov transformations can be used to relate these and calculate what particles should be produced by the vacuum decay process.

Consider how remarkable it is that the particle density is not zero. Our everyday intuition tells us that the existence of a particle should be a concrete fact, independent of how it is being measured. Yet in quantum field theory, this is evidently not the case. Observers may disagree on how many particles are present in spacetime. The concept of a vacuum state is dependent on your particular coordinate system. In more general geometries of spacetime, where it is impossible to choose preferred coordinates, the concept of particles may not be useful at all for describing the system. [1] is all about how to work with quantum field theory in these situations.

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