Physical Insights into Black Hole Uniqueness

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ABSTRACT

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We present an alternative method for writing the Einstein-Maxwell equations as a set of five divergence-free, two-dimensional vector fields. These fields have close ties to the conserved quantities of a black hole. This formulation has the potential to provide new understanding of black hole uniqueness, based upon a black hole's conserved quantities.

Keywords: classical black holes, mathematical physics, general relativity, gravitation

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Chapter 1

Introduction

In 1915, Albert Einstein first published what is now known as the general theory of relativity [1, 2]. The first nontrivial solution to the Einstein field equations was made the very next year by Karl Schwarzschild [3]. This solution described a strange sort of mathematical object that had absurd physical implications. It was a region of space with gravity so impossibly strong that not even light could escape. Despite its many strange properties, it appeared to be fully consistent with the theory. This object, having since been observed and detected many times, is now known as a black hole.

Black holes come in several different classifications, though all have mass. If they have no electric charge or angular momentum, they are of a Schwarzschild type. If they have electric charge but no angular momentum, they are of a Reissner-Nordström type and require the inclusion of the Maxwell equations. This solution was derived from 1916 to 1918 by a few separate physicists [4]. If they have angular momentum but no electric charge, they are of the Kerr type, derived by Roy Kerr in 1963 [5]. Finally, the most general case is a black hole that has mass, charge, and angular momentum. These are known as Kerr-Newman black holes; the solution was found when Ezra Newman and some students extended the Kerr solution to include electric charge [6, 7].

	Electrically Neutral	Charged
No Angular Momentum	Schwarzschild	Reissner-Nordström
Angular Momentum	Kerr	Kerr-Newman

Table 1.1 Comparison between parameters present in different black hole types. All black holes have a nonzero mass. This indicates the presence of other parameters for the various types of black hole.

So long as we remain in three spatial dimensions and one temporal dimension, everything about a black hole can be completely characterized by just these three parameters. They constitute a uniqueness set. This is remarkable. Given how dynamic, how furious, and how frankly messy the process of gravitational collapse is, the whole system settling down into something that is completely described by only three quantities seems unlikely. The uniqueness of Kerr-Newman black holes was proven by Mazur in 1982 as the culminating paper in a large effort from the community [8, 9]. However, the collected proof does not explicitly relate to the physical properties of the black hole. Here we present an alternative formulation of the Einstein-Maxwell equations that has the potential to prove uniqueness in a much more concise manner and provide further insight into the physical nature of black holes.

Before proceeding, prudence dictates a note on the utility of this problem. In other words, why do we care? After all, we already have a proof that the solution is uniquely defined based on the given parameters, and that itself does not seem especially useful. The answer to this is twofold.

First, as mentioned, the current proof of black hole metric uniqueness is extraordinarily long and relies exclusively on the mathematical structure of general relativity without elucidating many physical principles. This new approach is much more closely tied to the physical conserved quantities of a black hole, allowing us to construct fields that reproduce values equivalent to these quantities.

The second and perhaps more fundamental reason gets at the heart of why we care about black hole uniqueness at all. Much of the corpus of modern cosmology is based upon the observations of dramatic astronomical events. Arguably the most extreme known objects in our universe, the places where our theories are pushed to the limits of their validity, are black holes. This means black holes are an excellent tool for detecting events and generating results. A fantastic example of an understanding which leveraged black holes was the detection of gravitational waves in 2015, providing further evidence in support of general relativity [10]. In effect, characterizing a black hole is analogous to calibrating our instrument. Extracting useful information from the data collected requires that we know the dynamics and structure of spacetime around these objects. While a new understanding of black holes is not going to change how we process gravitational wave data, it may allow us to gain further understanding from the data we do have. The fact that there are only three parameters determining uniqueness makes a black hole an unusually valuable instrument—arbitrarily high complexity can be reduced to three values, telling us everything we would need to know about our instrument of analysis.

Though much of the modern work in relativity is concentrated on numerical schemes for solving more complex systems [11, 12], there is still a sector of research in analytic relativity attempting to develop further methods of understanding the fundamental relationships between mass and spacetime dynamics [13, 14]. In particular, there has been some research into the reduction of the usual four-dimensional problem into two dimensions by assuming that the spacetime is axisymmetric and stationary; that is, it has an axis of rotational symmetry, and it does not change in time [15]. We are making a similar dimensional reduction here, but following this reduction to construct combinations of base mathematical objects that have very close ties to the conserved quantities of spacetime. We believe the approach presented here, wherein new fields are constructed, shown to be divergence-free, and able to largely reproduce the classical conserved quantities of a black hole, is wholly unique.

As stated, we build up a method of recasting the Einstein-Maxwell equations into a set of five two-dimensional divergence-free vector fields. One of these is solved exactly for all black hole types. The other four are closely related to conservation of energy, angular momentum, electric charge, and magnetic charge. This construction is made under the assumption that the system dynamics are axisymmetric and stationary, reducing our usual degrees of freedom from four to two. We show how these fields may be used to recover the classical conserved quantities of angular momentum, mass, and charge.

It should be noted that we have not as yet been able to completely reconstruct the metric from first principles, despite having made substantial progress. Nonetheless, this is a solid first step and a strong characterization of the way forward. While some of the details in these calculations are mathematically demanding to the uninitiated, we hope this discussion provides some of the necessary background for further attempts at tackling this problem.

Chapter 2

Mathematical Methods

In this chapter, we introduce some of the mathematical concepts used in this analysis which are not typically covered in an undergraduate physics curriculum. We assume the reader has some knowledge of index notation and at least a passing notion of tensors. For reference on these topics, see [16, 17]. We don't attempt to derive or prove any of these results, but simply introduce and motivate them.

2.1 Metric

General relativity ascribes the phenomenon of gravity to the curvature of spacetime, so we need some background on how to compute things when space bends. Before getting in to the calculus on curved surfaces, we need to understand what it means for something to be curved. In math, we describe many things as **manifolds**. A manifold is a space that is "smooth." If you get close enough to any spot on the manifold, it looks identical to flat space. This might sound like some surface placed in a higher-dimensional flat space that is easier to conceptualize, but unfortunately it is a bit more abstract. We define a manifold without the notion of an embedding space, simply by how different the space we have is from flat space.

Reassuringly, we don't need the full power of abstract manifolds for general relativity. We can immediately restrict to manifolds with a special way of describing them, manifolds that have a metric.

The **metric** is an additional kind of structure that we can place on a manifold. In normal geometry, it is what defines distance and angles on a particular surface. Geometry is how we get gravity in the framework of general relativity. Our normal notions of flat geometry disappear in any space that is curved, which space we call non-Euclidean. To illustrate this, consider the example of a sphere. We usually think of any two great circles as being parallel. Does that mean they should have an angle between them of 180°? But they meet at a point (in fact, two points). Fortunately, the metric can tell us how to resolve this. The metric is what we consider to be the "solution" of a problem in general relativity.

There are a few ways we can construct a metric. The first and likely more familiar is to look at how the change in one coordinate impacts the physical distances associated with that coordinate. For instance, on a flat sheet, a fixed angle with a given radius will generate a specific arc length. However, if you change the radius while keeping the same angle, the arc length will also change in concert with the radius. A larger radius gives a larger physical distance for the same angle. The physical distance associated with the angular coordinate θ therefore needs to include some dependence on the radius r. When constructing a metric in this way, its components are the squares of the physical distances per unit coordinate. However, this method often relies upon some notion of "true" length, which arises from an embedding space.

The more common method in general relativity is to construct a metric as a solution to the Einstein field equations. If the distribution of matter and energy is known, then the field equations provide a set of coupled partial differential equations that, when solved, allow one to recover the metric. This is why we refer to the metric of a given spacetime as a solution;

it both directly reflects how it is obtained and demonstrates that all of the information of the spacetime curvature (gravity) is contained within the metric.

In relativity, we wrap the spatial components of our coordinates together with the time coordinate to create spacetime. This four-dimensional object is called the **metric tensor** (also just called the metric) and is, perhaps predictably, a rank-2 tensor. We write it as

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} , \tag{2.1}$$

where the zeroth component is time. Note that the metric is a symmetric object, so $g_{\alpha\beta} = g_{\beta\alpha}$.

If we were to have perfectly Euclidean spacetime (that is, our normal sense of space where there is no curvature), then the metric in Cartesian coordinates would be the metric we use in special relativity,

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.2}$$

This particular metric has what we call positive signature, because there are more positive values than negative (it has a specific signature $s = +2$, the number of positive eigenvalues minus the number of negative eigenvalues). The metric can be defined just as consistently if all of the components are negative of their current values so $g_{\mu\nu} = -g_{\alpha\beta}$, in which case the signature is negative. If the symbol g is ever given without indices, that is the determinant of $g_{\alpha\beta}$, so for the above metric we have

$$g = \det g_{\alpha\beta} = -1 . \tag{2.3}$$

Any time the metric appears with indices up, that is the inverse metric, defined such that

$$g^{\alpha\gamma} g_{\gamma\beta} = \delta^{\alpha}{}_{\beta} \ . \tag{2.4}$$

In the specific case above we see $g^{\alpha\beta} = g_{\alpha\beta}$. Furthermore, the definition (2.4) demonstrates that $g^{\alpha}{}_{\beta} = g^{\alpha\gamma} g_{\gamma\beta} = \delta^{\alpha}{}_{\beta}$, which will be a useful identity in some of the later derivations.

We also use the metric to define what is called the line element,

$$\mathrm{d}s^2 = g_{\alpha\beta}\, \mathrm{d}x^{\alpha}\, \mathrm{d}x^{\beta} \ . \tag{2.5}$$

Recall that in index notation, a repeated index is summed over, so this expression will end up being multiple terms added together. For example, if we wanted to express the metric of a sphere in spherical coordinates (r, θ, ϕ), it would be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \ . \tag{2.6}$$

In this case, the line element is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2 + r^2 \sin^2\theta\, \mathrm{d}\phi^2 \ . \tag{2.7}$$

Finally, the metric allows us to differentiate between a few kinds of vectors: **spacelike** and **timelike**. These depend upon the sign of the interval $(\Delta s)^2$, which is another way of describing the line element of Minkowski spacetime,

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \ . \tag{2.8}$$

If the interval of a vector is positive, it is called spacelike, and nothing on one end of a spacelike vector can affect anything on the other end because that would require superluminal travel. If the interval is negative, it is called timelike, and things happening at one end of a timelike vector can affect things happening at the other end.

2.2 Covariant Derivative

General relativity describes spacetime as being curved. As stated above, this does not mean gravity is described by some space that is curved when embedded in a larger Euclidean space—rather, it means that the very space in which we express motion and understand distance is curved. As motion is described in the language of calculus, it is necessary to introduce some notions of calculus on curved surfaces prior to doing any calculations. Most prominent for this construction is the concept of the covariant derivative.

When we first learn calculus, we talk about changing coordinates in one dimension. As we move along in one direction, say x, we can measure how the function f changes. By looking at the instantaneous rate of this functional change (f'), we defined the derivative. Then, when doing multivariable calculus, we start by looking at the change in one variable and define partial derivatives in the same way as the single-variable derivative, but this time holding all other variables constant. This assumes that we are able to decouple the change in one variable from the change in any others. However, what about the instances where we cannot? For instance, if you were confined to be on the surface of the sphere and, starting at the top, move a tiny bit in any direction, you must also move down. Any direction you move necessarily involves a change in at least one other direction.

Those familiar with multivariable calculus may be internally shouting at this point, just waiting for the term "Jacobian" to appear. While that is ultimately where this road goes, we are looking for something a bit more general. If we go continue with the analogy of a sphere, we can imagine placing a vector at our location, pointed in the direction we want to go. If we then move in that direction a tiny bit and draw a new arrow still pointed along the same path, we see that the direction of the arrow has changed, as shown in Fig. 2.1:

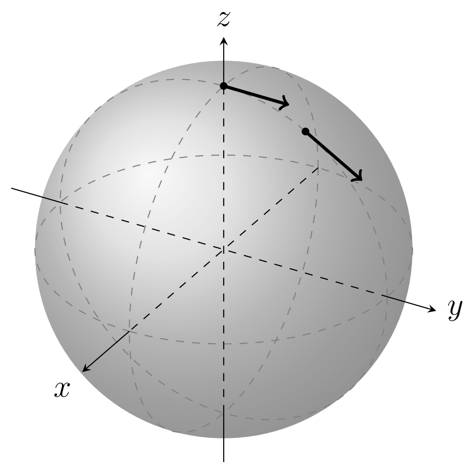

Figure 2.1 Visualization of how the direction of motion changes as the path is traversed. Note that the tangent arrow further along the path is pointed more down relative to the starting position at the top of the sphere, despite following the direction of the first arrow perfectly along the manifold.

The derivative of a vector field confined to this manifold must therefore involve not only a change in the coordinate itself, but also a way of moving the coordinate system so that our derivative stays the same across regions of the manifold where we expect our path to have the same kind of direction change. Since ordinary derivatives are about comparing one point of a function to all of the neighboring points, if the space itself curves, we need a coefficient to correct for this curvature and accurately compare neighboring points on this curved manifold. One way we solve this is with something called the **Christoffel symbols** (specifically of the second kind, also known as the **metric connection**), ${\Gamma^\alpha}_{\beta\gamma}$. We show how to compute these below.

With all this in mind, we can define the **covariant derivative**, so called because it varies with the curvature of the manifold on which the vector space lies. In index notation, it is

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + V^\gamma {\Gamma^\alpha}_{\beta\gamma} \ , \tag{2.9}$$

where

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma} \right) \ . \tag{2.10}$$

The reader may verify that the connection is symmetric in its lower indices, i.e. $\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$. A nifty party trick demonstrated in many general relativity textbooks [16, 18, 19] is for the special case of a repeated index:

$$\Gamma^{\alpha}{}_{\alpha\beta} = \partial_\beta \ln \sqrt{|g|} \ . \tag{2.11}$$

This will become useful in a few of the following derivations.

We can use this result immediately to define the divergence of a vector field in curved spacetime. Since index notation sums over repeated indices and the divergence of a vector field is the sum of the derivatives of the vector components, it seems natural to say that $\nabla_\alpha V^\alpha$ is the divergence. This is in fact how we define it, and it ends up giving us a neat little formula:

$$\nabla_\alpha V^\alpha = \partial_\alpha V^\alpha + V^\beta \Gamma^{\alpha}{}_{\alpha\beta} = \partial_\alpha V^\alpha + \frac{1}{\sqrt{|g|}} V^\beta \partial_\beta \sqrt{|g|} \tag{2.12}$$

$$= \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|} V^\alpha \right) \tag{2.13}$$

We have mentioned several times that the fields we construct have no divergence. When something is emphasized repeatedly, it is wise to assume that it is a rather important characteristic. As can be understood from the concept of divergence, a field which globally has no divergence is one in which a given quantity is conserved: just as much appears (positive divergence) as goes away (negative divergence). Going further, if a field has no divergence locally, that means there is no production or loss at all. There's not just a net conservation, but nothing appears and nothing goes away. There are no sources or sinks. This connects very nicely with conservation laws, which form the cornerstone of modern physics.

2.3 Killing Vectors

We make a brief note on Killing vector fields, or more simply **Killing vectors**. They are vectors X_i which satisfy Killing's equation,

$$\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0 \; . \tag{2.14}$$

This makes the covariant derivative of a Killing vector an **antisymmetric** object, since swapping the indices switches the sign,

$$\nabla_\alpha X_\beta = -\nabla_\beta X_\alpha \; . \tag{2.15}$$

Physically, Killing vectors are vector fields over a manifold with the property that if each point is moved an infinitesimal amount in the direction of the associated vector along the manifold, the shape of the manifold doesn't change and the metric is preserved. This makes these directions what we call in mathematics an **isometry**. In terms of physics, after we move along the direction of a Killing vector, we cannot distinguish (either physically or geometrically) whether we are at the first point or the second. Everything stays the same. These isometries have a very special property relating to conserved quantities, which arises from Noether's Theorem.

Along a geodesic (the equivalent of a "straight line" in Euclidean space, the shortest distance between any two points), each Killing vector corresponds to a conserved quantity. The quantity happens to be the inner product between the Killing vector and the vector tangent to the geodesic. This is very relevant when talking about Komar integrals and conserved quantities, as well as in our initial construction of our alternative formulation.

One other property of Killing vectors that bears mentioning is their divergence-free nature. This hopefully makes intuitive sense, since they correspond to conserved quantities. However, demonstrating it mathematically is rather instructive and illustrates a general point: the contraction of a symmetric quantity with an antisymmetric quantity will always give zero. Starting from Killing's equation (2.14) and applying the metric, we see

$$\begin{aligned} g^{\alpha\beta}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) &= 0 \\ \nabla_\alpha X^\alpha + \nabla_\beta X^\beta = 2\nabla_\alpha X^\alpha &= 0 \\ \nabla_\alpha X^\alpha &= 0 \ . \end{aligned} \tag{2.16}$$

The last thing that we need to know about Killing vectors is that, in specific coordinate systems, their ordinary derivative is zero,

$$\partial_\alpha X^\beta = 0 \ . \tag{2.17}$$

This is because when we choose coordinates adapted to these vectors, they become constant with respect to all variables, essentially being categorized as a basis vector in this coordinate system. To give an example, consider two-dimensional Cartesian coordinates, spanned by the vectors $\hat{x}$ and $\hat{y}$. These are fixed vectors, always pointing the same way so long as we choose a coordinate system that is constant with respect to them. Thus their derivative is zero, because they simply do not change as we move along a coordinate. This is the same with arbitrary Killing vectors. In coordinate systems adapted to the direction generated by these vectors, their ordinary derivative is zero.

2.4 Komar Integrals

Now that we have an idea of how differentiation works in curved space, it would be nice to define its operator pair, integration in curved space. We won't delve into the full general form of integration here, but instead restrict ourselves to the specific case that we need for the conserved quantity calculation. In the study of black holes, there is a special kind of integral called a **Komar integral** that integrates over a spherical shell S_t at infinite distance from the black hole, capturing all of the "something" (mass, charge, angular momentum, etc.) within the spacetime. In electrostatics we have Gauss's law, which states that the net flux through a closed surface is proportional to the amount of charge contained within that surface. One could imagine doing something similar with sources of mass and angular momentum, if only we had a way to talk about the flux of these sources.

Fortunately, there is a way. We won't derive the expressions here, but instead invite the reader to go to [20] for more information. The full integrals are given by

$$\mathcal{M} = -\frac{1}{8\pi} \lim_{S_t \to \infty} \oint_{S_t} \mathrm{d}S_{\alpha\beta}\, \nabla^\alpha t^\beta \tag{2.18}$$

$$\mathcal{J} = \frac{1}{16\pi} \lim_{S_t \to \infty} \oint_{S_t} \mathrm{d}S_{\alpha\beta}\, \nabla^\alpha \phi^\beta \tag{2.19}$$

$$\mathcal{Q} = \frac{1}{8\pi} \oint_{S_t} \mathrm{d}S_{\alpha\beta}\, F^{\alpha\beta} \ , \tag{2.20}$$

where the surface element is

$$\mathrm{d}S_{\alpha\beta} = (n_\beta r_\alpha - n_\alpha r_\beta)\sqrt{\sigma}\,\mathrm{d}^2\sigma \ . \tag{2.21}$$

In much the same way as Gauss's law is used in electrostatics, we are getting the mass and angular momentum from the relevant Killing vector and the charge from the field strength. We integrate over a sphere enclosing our whole source, and the flux through that sphere tells us how much of the associated quantity is inside our sphere (the charge integral (2.20) actually is just Gauss's Law in several trench coats). The vectors n_α and r_α are the timelike and spacelike unit vectors normal to the surface S_t. We define these explicitly when we need them in Chapter 4. The value σ is what is called the **induced metric**, but when under the square root it's really just the Jacobian for the integration. It is the determinant of the metric of the sphere S_t, and we likewise define it when needed.

Taking the limit as $r \to \infty$ seems like it could be difficult to tame if any of the resulting integrals contain a direct functional dependence on r. Additionally, it seems odd that to capture all of the mass or angular momentum of a black hole we must go out all the way to infinity. As one may hope, both of these conundrums (conundra?) are solved by the same recognition. The parameters $\mathcal{M}$ and $\mathcal{J}$ are not just the mass and angular momentum of the actual black hole, as otherwise these integrals could be evaluated on the event horizon and we would be done. They are instead characteristic of the entire structure of the spacetime, involving contributions from both the central object and the external spacetime extending out to infinity. This allows us to break up the integral into two parts, hopefully taming the pernicious infinities.

Again appealing to the derivations in [20], the actual mass and angular momentum of the black hole itself are computed using the above integrals and evaluating them at the event horizon. The remaining portions of these quantities contained within the exterior spacetime are dealt with by adding another term, like so:

$$\mathcal{M} = -\frac{1}{8\pi} \oint_{S_t} \mathrm{d}S_{\alpha\beta}\, \nabla^\alpha t^\beta \Big|_{r_h} + 2 \int_{\Sigma(r_h)} \mathrm{d}^3y \sqrt{h} \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) n^\alpha t^\beta \ , \tag{2.22}$$

$$\mathcal{J} = \frac{1}{16\pi} \oint_{S_t} \mathrm{d}S_{\alpha\beta}\, \nabla^\alpha \phi^\beta \Big|_{r_h} - \int_{\Sigma(r_h)} \mathrm{d}^3y \sqrt{h} \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) n^\alpha \phi^\beta \ . \tag{2.23}$$

Here $\Sigma(r_h)$ is a spacelike hypersurface extending from r_h out to infinity, n^α is the spacelike normal to the hypersurface, and $T_{\alpha\beta}$ is the stress-energy tensor to be considered below. Additionally, h is the determinant of the induced metric on the hypersurface, so $\mathrm{d}^3y \sqrt{h}$ is just the Jacobian of the hypersurface as before. We again define it as needed. The $\mathcal{Q}$ integral really is the charge of the black hole itself and is thus defined at any spatial radius as all of the charge is contained within the horizon. For simplicity we define it at the horizon, making the integral entirely governable without exploding as radius increases.

Chapter 3

Construction of Fields

Now that we have the necessary mathematical machinery to begin work, let us do just that. As stated in the introduction, we are working up to constructing fields that hopefully contain all of the information of the Einstein-Maxwell equations, but also have no divergence. However, to actually get to that point, we need to do a substantial amount of prep work. These next few sections do a few things: first, we define the space and build up the Ricci tensor necessary for the Einstein equations, then we develop the stress-energy tensor from the vector potential, and finally we see how all of these definitions work together with the Einstein-Maxwell equations to motivate the definition of our desired fields. These fields are fully contained in the external region of the black hole, and we do not consider anything within the event horizon.

3.1 Ricci Tensor

The first thing we tackle is understanding how the spacetime around a black hole actually warps. We leverage the symmetry of a stationary, axisymmetric, isolated black hole to express this curvature as simply as possible. In differential geometry, the object that describes how much a manifold deviates from flat is called the **Ricci tensor**, and deriving it is the main point of this section.

3.1.1 Defining the Metric

As mentioned earlier, we are assuming that the spacetime is both stationary (we can move forward or backward in time and the system doesn't change) and axisymmetric (there is an axis of symmetry around which we can rotate without changing the system). These properties must involve conserved quantities, and therefore correspond to Killing vectors. We call them t^α and ϕ^α for the timelike and rotational symmetries respectively. These have norms

$$\phi^\alpha \phi_\alpha = s^2 \qquad \text{and} \qquad t^\alpha t_\alpha = -c^2 \; . \tag{3.1}$$

The signs differ because we are using a metric of positive signature, meaning the time component has a negative sign and the three spatial components are positive.

To make use of these symmetries, we choose coordinates that follow the flow of t^α and ϕ^α. Since these directions correspond to symmetries of the system, the physical quantities cannot depend on these coordinates. We call such a coordinate scheme **adapted coordinates**, meaning they are adapted to the symmetries of the system. This leaves us with two coordinates remaining, two degrees of freedom. The particular symmetries of this spacetime lend themselves to description via spherical coordinates with time attached, so we notate these remaining coordinates as r and θ.

For the sake of convenience, we also define some scaled vectors to keep derivations and results cleaner. These are constructed to be normalized when contracted with the original Killing vector, not the scaled vector. We also use a Gram-Schmidt process to ensure the two are orthogonal. This leaves us with

$$Y^\alpha = \frac{1}{s^2}\phi^\alpha \qquad \text{and} \qquad T^\alpha = \frac{1}{Q^2}\left(t^\alpha - \phi_t Y^\alpha\right) \ , \tag{3.2}$$

where $Q^2 = c^2 + \dfrac{\phi_t^2}{s^2}$ and $\phi_t = \phi_\alpha t^\alpha$. Details about the construction of these vectors and verification of their orthogonality are provided in Appendix A. For convenience, we also introduce the notational shorthand

$$\omega = \frac{\phi_t}{s^2} \ . \tag{3.3}$$

We can then express the general metric of the spacetime, $g_{\alpha\beta}$, as

$$g_{\alpha\beta} = \sigma_{\alpha\beta} + s^2 Y_\alpha Y_\beta - Q^2 T_\alpha T_\beta \ , \tag{3.4}$$

where we have just pulled out the t and ϕ dependence, leaving $\sigma_{\alpha\beta}$ to be the metric on the manifold which is orthogonal to both the t and ϕ coordinates, meaning it is just a function of r and θ.

As we go further and further from the black hole, it should bend spacetime less and less. In the limiting case where we go out infinitely far from the singularity, we should get the normal metric and line element for flat space,

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \, . \tag{3.5}$$

This kind of spacetime is called asymptotically flat. Immediately, this shows us the limiting forms for a few quantities:

$$Q^2\Big|_\infty = 1 \ , \qquad s^2\Big|_\infty = r^2\sin^2\theta \ . \tag{3.6}$$

Via a rather technical bit of mathematics called the Uniformization Theorem [21], we know that every two-dimensional manifold must be conformally flat, meaning there exists a special kind of function that maps our metric to the metric of flat spacetime. We don't know what this conformal factor is, so we denote it with an exponential of an unknown function f. That is,

$$\sigma_{\alpha\beta} = e^{2f} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \ . \tag{3.7}$$

Alternatively, we can write this as the line element to get

$$\mathrm{d}\sigma^2 = e^{2f}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) \ . \tag{3.8}$$

3.1.2 Computing the Ricci Tensor

Using the metric of the full four-dimensional spacetime, we can derive the form of the Christoffel symbols using Eq. (2.10). See Appendix B for the derivation. We get

$$\begin{aligned} \Gamma^{\alpha}{}_{\beta\gamma} = {}& {}^{(2)}\Gamma^{\alpha}{}_{\beta\gamma} + \frac{1}{2}\sigma^{\alpha\delta}\left[s^2(Y_\beta T_\gamma + Y_\gamma T_\beta)\Delta_\delta \omega - Y_\beta Y_\gamma \Delta_\delta s^2 + T_\beta T_\gamma \Delta_\delta Q^2\right] \\ & + \frac{1}{2}Y^\alpha\left[\partial_\beta(s^2 Y_\gamma) + \partial_\gamma(s^2 Y_\beta)\right] \\ & - \frac{1}{2}T^\alpha\left[\partial_\beta(Q^2 T_\gamma) + \partial_\gamma(Q^2 T_\beta) + s^2\left(Y_\beta \Delta_\gamma \omega + Y_\gamma \Delta_\beta \omega\right)\right] \ . \end{aligned} \tag{3.9}$$

Here, ${}^{(2)}\Gamma^{\alpha}_{\beta\gamma}$ is the Christoffel symbol (also called the connection) on the two-manifold with $\sigma_{\alpha\beta}$, and $\sigma^{\alpha\beta}$ is the inverse metric of this same two-manifold. We will call this two-manifold the **submanifold**, and the other two directions the isometry directions. We also use the notation of Δ_α for the covariant derivative restricted to the submanifold of r and θ, meaning $\Delta_\alpha V^\alpha = \Delta_r V^r + \Delta_\theta V^\theta$. When applied to a scalar function, it is just the ordinary derivative but restricted to taking derivatives with respect to r or θ. Since all of the constructed scalar quantities are independent of t and ϕ, we may freely swap between ∂_α and Δ_α whenever acting on scalar functions.

With this in hand, we define the Ricci tensor via

$$R_{\alpha\beta} = \partial_\gamma \Gamma^{\gamma}{}_{\alpha\beta} - \partial_\beta \Gamma^{\gamma}{}_{\alpha\gamma} + \Gamma^{\gamma}{}_{\alpha\beta}\Gamma^{\delta}{}_{\gamma\delta} - \Gamma^{\gamma}{}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} \ . \tag{3.10}$$

Computing this quantity is no small task, but is a relatively-straightforward exercise in endurance. An outline of this derivation is included in Appendix C. When the dust eventually settles and our eyes refocus, we obtain

$$
\begin{aligned}
R_{\alpha\beta} =& \sigma^{\gamma}{}_{\alpha}\sigma^{\delta}{}_{\beta}\left[{}^{(2)}R_{\gamma\delta} - \frac{1}{s}\Delta_{\gamma}\Delta_{\delta}s - \frac{1}{Q}\Delta_{\gamma}\Delta_{\delta}Q + \frac{s^2}{2Q^2}(\Delta_{\gamma}\omega)\Delta_{\delta}\omega\right] \\
&- Y_{\alpha}Y_{\beta}\left[\frac{s}{Q}\Delta_{\gamma}(Q\Delta^{\gamma}s) + \frac{s^4}{2Q^2}(\Delta_{\gamma}\omega)\Delta^{\gamma}\omega\right] + (Y_{\alpha}T_{\beta} + Y_{\beta}T_{\alpha})\left[\frac{Q}{2s}\Delta_{\gamma}\left(\frac{s^3}{Q}\Delta^{\gamma}\omega\right)\right] \\
&+ T_{\alpha}T_{\beta}\left[\frac{Q}{s}\Delta_{\gamma}(s\Delta^{\gamma}Q) - \frac{s^2}{2}(\Delta_{\gamma}\omega)\Delta^{\gamma}\omega\right] .
\end{aligned}
\tag{3.11}
$$

This is unlikely to look very intuitive or obvious. Fortunately, the keys of general relativity are not contained within the Ricci tensor itself, but instead with the relationship between the Ricci tensor and the stress-energy tensor.

3.2 Stress-Energy Tensor

The next big quantity we need is the aforementioned stress-energy tensor. The great John Wheeler said, “Space tells matter how to move, matter tells space how to curve.” [18] The stress-energy tensor is the matter (and energy) in space, and the Ricci tensor is the curvature of space in response. Since we are in the external region of the black hole and we have assumed a spacetime where the only thing in it is the black hole, there are no sources of any mass or charge within our defined space, but if the black hole is charged there is still an electric field. This kind of space is called **electrovac**, meaning we are in a perfect vacuum except for the presence of an electric field. The stress-energy tensor of electrovac space comes from the field strength tensor, which in turn is derived from the vector potential that is likely familiar from electromagnetism.

3.2.1 Vector Potential

We call the vector potential A^α. We actually want this to be in the index-down covector form $A_\alpha = g_{\alpha\beta}A^\beta$, as that is what will be used in the derivation of the field strength tensor. Interestingly, the portion of A_α that is confined to the submanifold vanishes [22, 23]. What remains is

$$A_\alpha = s^2 Y_\alpha Y_\beta A^\beta - Q^2 T_\alpha T_\beta A^\beta = s^2 Y_\alpha A_Y - Q^2 T_\alpha A_T \ . \tag{3.12}$$

Note that $A_Y = A_\alpha Y^\alpha$ is the component of the vector potential in the Y^α direction, which is a scalar and could have r or θ dependence. The same is true for $A_T = A_\alpha T^\alpha$. From this form we can get to the field strength tensor, which describes the electric and magnetic fields of the system and should therefore have the same symmetries (stationary and axisymmetric) as the system.

3.2.2 Field Strength Tensor

The definition of the field strength tensor, assuming there is zero net magnetic charge, is

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \ . \tag{3.13}$$

To tackle this computation, we first identify the derivatives of the scaled vectors (3.2). We obtain

$$\begin{aligned}\partial_\alpha T_\beta &= \delta_\beta{}^\gamma \partial_\alpha T_\gamma = g_\beta{}^\gamma \partial_\alpha T_\gamma \\ &= (\sigma_\beta{}^\gamma + s^2 Y_\beta Y^\gamma - Q^2 T_\beta T^\gamma)\partial_\alpha T_\gamma \ .\end{aligned} \tag{3.14}$$

As shown in [24], the portion of the derivative of the scaled vectors confined to the submanifold vanishes, leaving us with

$$\begin{aligned}\partial_\alpha T_\beta &= (Y_\beta \phi^\gamma)\partial_\alpha T_\gamma - T_\beta (t^\gamma - \omega\phi^\gamma)\partial_\alpha T_\gamma \\ &= Y_\beta \partial_\alpha \phi^\gamma T_\gamma - T_\beta \partial_\alpha t^\gamma T_\gamma + T_\beta \omega \partial_\alpha \phi^\gamma T_\gamma \ .\end{aligned} \tag{3.15}$$

Since $\phi^\alpha T_\alpha = 0$ and $t^\alpha T_\alpha$ is a constant of -1, everything goes to zero:

$$\partial_\alpha T_\beta = 0 \ . \tag{3.16}$$

We can now do the same thing for $\partial_\alpha Y_\beta$ and find

$$\begin{aligned}
\partial_\alpha Y_\beta &= ({\sigma_\beta}^\gamma + s^2 Y_\beta Y^\gamma - Q^2 T_\beta T^\gamma)\partial_\alpha Y_\gamma \\
&= Y_\beta \partial_\alpha \phi^\gamma Y_\gamma - T_\beta (t^\gamma - \omega\phi^\gamma)\partial_\alpha Y_\gamma \\
&= -T_\beta \partial_\alpha \frac{1}{s^2} t^\gamma \phi_\gamma + T_\beta \omega \partial_\alpha \phi^\gamma Y_\gamma \\
&= -T_\beta \partial_\alpha \omega \ .
\end{aligned} \tag{3.17}$$

Now leveraging these results to compute the field strength tensor, we obtain

$$\begin{aligned}
\partial_\alpha A_\beta &= \partial_\alpha (s^2 Y_\beta A_Y - Q^2 T_\beta A_T) \\
&= Y_\beta \partial_\alpha s^2 A_Y - T_\beta \partial_\alpha Q^2 A_T + s^2 A_Y \partial_\alpha Y_\beta - Q^2 A_T \partial_\alpha T_\beta \\
&= Y_\beta \partial_\alpha s^2 A_Y - T_\beta \partial_\alpha Q^2 A_T - s^2 A_Y T_\beta \partial_\alpha \omega \ .
\end{aligned} \tag{3.18}$$

We can subtract off the same thing but with swapped indices, which brings us to our full expression of

$$F_{\alpha\beta} = (Y_\beta \partial_\alpha - Y_\alpha \partial_\beta) s^2 A_Y - (T_\beta \partial_\alpha - T_\alpha \partial_\beta) Q^2 A_T - s^2 A_Y (T_\beta \partial_\alpha - T_\alpha \partial_\beta)\omega \ . \tag{3.19}$$

As shown in Appendix D, we can factor this into

$$F_{\alpha\beta} = \frac{1}{sQ}\left(\phi_\beta B_\alpha - \phi_\alpha B_\beta - \frac{s}{Q}(t_\beta E_\alpha - t_\alpha E_\beta)\right) \ , \tag{3.20}$$

where we have defined

$$E_\alpha = \Delta_\alpha Q^2 A_T + s^2 A_Y \Delta_\alpha \omega \qquad \text{and} \qquad B_\alpha = \frac{Q}{s}\Delta_\alpha s^2 A_Y + \frac{s}{Q}\omega E_\alpha \ . \tag{3.21}$$

Here we again use the notation of Δ_α for the covariant derivative restricted to the submanifold of r and θ. Note that both E_α and B_α are functions of only r and θ, because the only vectorial part of these functions is a derivative with respect to one of these two variables. We will come back to this factorization soon.

3.2.3 Computation of Stress-Energy Tensor

In a source-free region, the stress-energy tensor is a special case called the Hilbert stress-energy tensor, and it is given by

$$4\pi T_{\alpha\beta} = g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \ . \tag{3.22}$$

Thus, from this definition and the form of our field strength tensor, we obtain (see Appendix E)

$$\begin{aligned} 4\pi T_{\alpha\beta} = {} & \sigma^{\gamma}{}_{\alpha} \sigma^{\delta}{}_{\beta} \left[\frac{1}{s^2} \left((\Delta_\gamma s^2 A_Y) \Delta_\delta s^2 A_Y - \frac{1}{2} \sigma_{\gamma\delta} (\Delta^\epsilon s^2 A_Y) \Delta_\epsilon s^2 A_Y \right) - \frac{1}{Q^2} \left(E_\gamma E_\delta - \frac{1}{2} \sigma_{\gamma\delta} E_\epsilon E^\epsilon \right) \right] \\ & + \frac{1}{2} Y_\alpha Y_\beta \left[(\Delta^\gamma s^2 A_Y) \Delta_\gamma s^2 A_Y + \frac{s^2}{Q^2} E_\gamma E^\gamma \right] - (Y_\alpha T_\beta + Y_\beta T_\alpha)[E^\gamma \Delta_\gamma s^2 A_Y] \\ & + \frac{1}{2} T_\alpha T_\beta \left[\frac{Q^2}{s^2} (\Delta^\gamma s^2 A_Y) \Delta_\gamma s^2 A_Y + E^\gamma E_\gamma \right] \ . \end{aligned} \tag{3.23}$$

Computing the trace of the stress-energy tensor is, as usual, the contraction of the quantity with the metric, so

$$\begin{aligned} 4\pi T & = 4\pi g^{\alpha\beta} T_{\alpha\beta} \\ & = g^{\alpha\beta} \sigma^{\gamma}{}_{\alpha} \sigma^{\delta}{}_{\beta} \left[\frac{1}{s^2} \left((\Delta_\gamma s^2 A_Y) \Delta_\delta s^2 A_Y - \frac{1}{2} \sigma_{\gamma\delta} (\Delta^\epsilon s^2 A_Y) \Delta_\epsilon s^2 A_Y \right) - \frac{1}{Q^2} \left(E_\gamma E_\delta - \frac{1}{2} \sigma_{\gamma\delta} E_\epsilon E^\epsilon \right) \right] \\ & \quad + \frac{1}{2} g^{\alpha\beta} Y_\alpha Y_\beta \left[(\Delta^\gamma s^2 A_Y)(\Delta_\gamma s^2 A_Y) + \frac{s^2}{Q^2} E_\gamma E^\gamma \right] - g^{\alpha\beta} (Y_\alpha T_\beta + Y_\beta T_\alpha)[E^\gamma \Delta_\gamma s^2 A_Y] \\ & \quad + \frac{1}{2} g^{\alpha\beta} T_\alpha T_\beta \left[\frac{Q^2}{s^2} (\Delta^\gamma s^2 A_Y) \Delta_\gamma s^2 A_Y + E_\gamma E^\gamma \right] \ . \end{aligned} \tag{3.24}$$

We take these terms one at a time, starting with the cross term. Expanding this out, we find

$$g^{\alpha\beta} (Y_\alpha T_\beta + Y_\beta T_\alpha) = Y^\beta T_\beta + Y^\alpha T_\alpha = 0 \ , \tag{3.25}$$

because Y^α and T^α are orthogonal as shown in Appendix A.

Moving to the $Y_\alpha Y_\beta$ and $T_\alpha T_\beta$ terms, we see

$$g^{\alpha\beta} Y_\alpha Y_\beta = Y^\beta Y_\beta = \frac{1}{s^2} \ , \tag{3.26}$$

and

$$g^{\alpha\beta}T_\alpha T_\beta = T^\beta T_\beta = \frac{1}{Q^2}(t^\beta - \omega\phi^\beta)T_\beta = -\frac{1}{Q^2}\ . \tag{3.27}$$

Plugging these back in to their respective terms, we obtain

$$\begin{aligned}&\frac{1}{2}g^{\alpha\beta}Y_\alpha Y_\beta\left[(\Delta^\gamma s^2 A_Y)\Delta_\gamma s^2 A_Y + \frac{s^2}{Q^2}E_\gamma E^\gamma\right] + \frac{1}{2}g^{\alpha\beta}T_\alpha T_\beta\left[\frac{Q^2}{s^2}(\Delta^\gamma s^2 A_Y)\Delta_\gamma s^2 A_Y + E_\gamma E^\gamma\right]\\ &= \left(\frac{1}{2s^2}(\Delta^\gamma s^2 A_Y)\Delta_\gamma s^2 A_Y + \frac{1}{2Q^2}E_\gamma E^\gamma\right) - \left(\frac{1}{2s^2}(\Delta^\gamma s^2 A_Y)\Delta_\gamma s^2 A_Y + \frac{1}{2Q^2}E_\gamma E^\gamma\right) = 0\ .\end{aligned} \tag{3.28}$$

Finally, we examine the first term. We note

$$g^{\alpha\beta}\sigma^\gamma{}_\alpha\sigma^\delta{}_\beta = \sigma^{\gamma\beta}\sigma^\delta{}_\beta = \sigma^{\delta\gamma}\ . \tag{3.29}$$

Inserting this leaves us with

$$\begin{aligned}4\pi T &= \sigma^{\delta\gamma}\left(\frac{1}{s^2}(\Delta_\gamma s^2 A_Y)(\Delta_\delta s^2 A_Y) - \frac{1}{2s^2}\sigma_{\gamma\delta}(\Delta^\epsilon s^2 A_Y)(\Delta_\epsilon s^2 A_Y) - \frac{1}{Q^2}E_\gamma E_\delta - \frac{1}{2Q^2}\sigma_{\gamma\delta}E_\epsilon E^\epsilon\right)\\ &= \frac{1}{s^2}\left((\Delta^\gamma s^2 A_Y)(\Delta_\gamma s^2 A_Y) - \frac{1}{2}{}^{(2)}\delta^\gamma{}_\gamma(\Delta^\epsilon s^2 A_Y)(\Delta_\epsilon s^2 A_Y)\right) - \frac{1}{Q^2}\left(E_\gamma E^\gamma - \frac{1}{2}{}^{(2)}\delta^\gamma{}_\gamma E_\epsilon E^\epsilon\right)\\ &= \frac{1}{s^2}(\Delta^\gamma s^2 A_Y)(\Delta_\gamma s^2 A_Y)\left(1 - \frac{1}{2}{}^{(2)}\delta^\gamma{}_\gamma\right) - \frac{1}{Q^2}E_\gamma E^\gamma\left(1 - \frac{1}{2}{}^{(2)}\delta^\gamma{}_\gamma\right)\ .\end{aligned} \tag{3.30}$$

Here we have specified that we are in the submanifold by the superscript (2), meaning ${}^{(2)}\delta^\gamma{}_\gamma = 2$, so both parentheses are zero. 4 and π are constants, giving us

$$T = 0. \tag{3.31}$$

Now that we have both the stress-energy tensor and the Ricci tensor, we can move on to the Einstein equations and start seeing how the fields we construct arise.

3.3 Divergence-free Fields

With all of the preparatory work in place, we can really begin to harvest the fruits of expressing everything in this highly-symmetric form. The Einstein equations break apart into four different equations thanks to the orthogonality of the coordinate system, and from there we can start combining things to develop fields that have no divergence. These fields comprise our system from which we can extract physically-relevant quantities.

3.3.1 Einstein-Maxwell Equations

The Einstein equations are

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi G T_{\alpha\beta}\ . \tag{3.32}$$

While this may look like one equation, it is actually 10 coupled equations thanks to the magic of index notation. By contracting both sides with the inverse metric $g^{\alpha\beta}$, we compute the trace. Letting D be the number of dimensions in which we are working (4 in this case), we get

$$\begin{aligned}
g^{\alpha\beta}R_{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta}R &= 8\pi G g^{\alpha\beta}T_{\alpha\beta} \\
R^{\beta}{}_{\beta} - \frac{1}{2}\delta^{\beta}{}_{\beta}R &= 8\pi G T^{\beta}{}_{\beta} \\
R - \frac{1}{2}DR &= 8\pi G T \\
R\left(\frac{2-D}{2}\right) &= 0 \\
R &= 0\ .
\end{aligned} \tag{3.33}$$

Plugging this result in, the Einstein equations become a little bit simpler, giving us

$$R_{\alpha\beta} = 8\pi G T_{\alpha\beta}\ . \tag{3.34}$$

From this, combining the Ricci tensor (3.11) and the stress-energy tensor (3.23), we find

$$\begin{aligned}
&\sigma^{\gamma}{}_{\alpha}\sigma^{\delta}{}_{\beta}\left[{}^{(2)}R_{\gamma\delta}-\frac{1}{s}\Delta_\gamma\Delta_\delta s-\frac{1}{Q}\Delta_\gamma\Delta_\delta Q+\frac{s^2}{2Q^2}(\Delta_\gamma\omega)\Delta_\delta\omega\right]\\
&-Y_\alpha Y_\beta\left[\frac{s}{Q}\Delta_\gamma(Q\Delta^\gamma s)+\frac{s^4}{2Q^2}(\Delta^\gamma\omega)\Delta_\gamma\omega\right]+(Y_\alpha T_\beta+Y_\beta T_\alpha)\left[\frac{Q}{2s}\Delta_\gamma\left(\frac{s^3}{Q}\Delta^\gamma\omega\right)\right]\\
&+T_\alpha T_\beta\left[\frac{Q}{s}\Delta_\gamma(s\Delta^\gamma Q)-\frac{s^2}{2}(\Delta^\gamma\omega)\Delta_\gamma\omega\right]\\
=\;&2G\sigma^{\gamma}{}_{\alpha}\sigma^{\delta}{}_{\beta}\left[\frac{1}{s^2}\left((\Delta_\gamma s^2A_Y)\Delta_\delta s^2A_Y-\frac{1}{2}\sigma_{\gamma\delta}(\Delta^\epsilon s^2A_Y)\Delta_\epsilon s^2A_Y\right)-\frac{1}{Q^2}\left(E_\gamma E_\delta-\frac{1}{2}\sigma_{\gamma\delta}E_\epsilon E^\epsilon\right)\right]\\
&+GY_\alpha Y_\beta\left[\frac{1}{2}(\Delta^\gamma s^2A_Y)\Delta_\gamma s^2A_Y+\frac{s^2}{2Q^2}E_\gamma E^\gamma\right]-2G(Y_\alpha T_\beta+Y_\beta T_\alpha)[E^\gamma\Delta_\gamma s^2A_Y]\\
&+GT_\alpha T_\beta\left[\frac{Q^2}{s^2}(\Delta^\gamma s^2A_Y)\Delta_\gamma s^2A_Y+E_\gamma E^\gamma\right]\;.
\end{aligned}\tag{3.35}$$

At first blush, this looks aggressively terrible. However, we can simplify this by recalling that the way we construct the vectors T_α and Y_α, using the Gram-Schmidt process, forces these vectors to be orthogonal. We can therefore set the quantities multiplied by the same vectors out front equal to each other, and get a set of four equations:

$$\begin{aligned}
&{}^{(2)}R_{\gamma\delta}-\frac{1}{s}\Delta_\gamma\Delta_\delta s-\frac{1}{Q}\Delta_\gamma\Delta_\delta Q+\frac{s^2}{2Q^2}(\Delta_\gamma\omega)(\Delta_\delta\omega)\\
=\;&\frac{2G}{s^2}\left((\Delta_\gamma s^2A_Y)\Delta_\delta s^2A_Y-\frac{1}{2}\sigma_{\gamma\delta}(\Delta^\epsilon\; s^2A_Y)\Delta_\epsilon s^2A_Y\right)-\frac{2G}{Q^2}\left(E_\gamma E_\delta-\frac{1}{2}\sigma_{\gamma\delta}E_\epsilon E^\epsilon\right)\;,
\end{aligned}\tag{3.36}$$

$$-\frac{s}{Q}\Delta_\gamma(Q\Delta^\gamma s)-\frac{s^4}{2Q^2}(\Delta^\gamma\omega)\Delta_\gamma\omega=G(\Delta^\gamma s^2A_Y)\Delta_\gamma s^2A_Y+\frac{Gs^2}{Q^2}E_\gamma E^\gamma\;,\tag{3.37}$$

$$\frac{Q}{2s}\Delta_\gamma\left(\frac{s^3}{Q}\Delta^\gamma\omega\right)=-2GE^\gamma\Delta_\gamma s^2A_Y\;,\tag{3.38}$$

$$\frac{Q}{s}\Delta_\gamma(s\Delta^\gamma Q)-\frac{s^2}{2}(\Delta^\gamma\omega)\Delta_\gamma\omega=\frac{GQ^2}{s^2}(\Delta^\gamma s^2A_Y)\Delta_\gamma s^2A_Y+GE_\gamma E^\gamma\;.\tag{3.39}$$

We reorganize these equations to solve for some quantities that will be of interest below, such as turning Eq. (3.37) into

$$\Delta_\gamma(Q\Delta^\gamma s)=-\frac{s^3}{2Q}(\Delta^\gamma\omega)\Delta_\gamma\omega-G\left(\frac{Q}{s}(\Delta^\gamma s^2A_Y)\Delta_\gamma s^2A_Y+\frac{s}{Q}E_\gamma E^\gamma\right)\;.\tag{3.40}$$

Similarly, we turn Eq. (3.39) into

$$\Delta_\gamma(s\Delta^\gamma Q) = \frac{s^3}{2Q}(\Delta^\gamma\omega)\Delta_\gamma\omega + G\left(\frac{Q}{s}(\Delta^\gamma s^2 A_Y)\Delta_\gamma s^2 A_Y + \frac{s}{Q}E_\gamma E^\gamma\right) , \tag{3.41}$$

and Eq. (3.38) becomes

$$\Delta_\gamma\left(\frac{s^3}{Q}\Delta^\gamma\omega\right) = -4G\frac{s}{Q}E^\gamma\Delta_\gamma s^2 A_Y . \tag{3.42}$$

Immediately, we see that the full structure of (3.40) and (3.41) are strikingly similar. In fact, adding these together gives us

$$\Delta_\gamma(Q\Delta^\gamma s) + \Delta_\gamma(s\Delta^\gamma Q) = 0 , \tag{3.43}$$

which may be written more compactly as

$$\Delta_\gamma\Delta^\gamma(sQ) = 0 . \tag{3.44}$$

We have constructed one of our five fields.

To get the other four, we turn to the Maxwell equations. As a reminder, we are considering the electrovac case, so there are no sources of mass or charge external to the event horizon. There is, however, an electric field in this region if the black hole is charged. We start with half of the source-free Maxwell equations,

$$\nabla_\alpha F^{\alpha\beta} = 0 . \tag{3.45}$$

We note that for an arbitrary direction α, we can expand the covariant derivative via its definition to get

$$\nabla_\alpha F^{\alpha\beta} = \partial_\alpha F^{\alpha\beta} + F^{\alpha\mu}\Gamma^\beta{}_{\alpha\mu} + F^{\mu\beta}\Gamma^\alpha{}_{\alpha\mu} . \tag{3.46}$$

However, the field strength tensor is antisymmetric while the Christoffel symbol is symmetric in its lower indices, so the second term on the right-hand side is zero. Next, using the party trick of Eq. (2.11) on the final term, we can express the full contracted covariant derivative of an antisymmetric rank-2 tensor as

$$\begin{aligned}
\nabla_\alpha F^{\alpha\beta} &= \partial_\alpha F^{\alpha\beta} + F^{\alpha\beta}\partial_\alpha \ln\sqrt{|g|} \\
&= \partial_\alpha F^{\alpha\beta} + F^{\alpha\beta}\partial_\alpha \ln\!\left(sQ\sqrt{\sigma}\right) \\
&= \partial_\alpha F^{\alpha\beta} + F^{\alpha\beta}\left(\frac{1}{\sqrt{\sigma}}\partial_\alpha\sqrt{\sigma} + \frac{1}{sQ}\partial_\alpha sQ\right) \\
&= \partial_\alpha F^{\alpha\beta} + F^{\alpha\beta}\left({}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} + \frac{1}{sQ}\partial_\alpha sQ\right) \ .
\end{aligned} \tag{3.47}$$

To aid in the computation of this quantity, we first calculate the partial derivatives of our scaled vectors (3.2), but with the index up instead of down as was done in (3.16) and (3.17). We find

$$\begin{aligned}
\partial_\alpha Y^\beta &= \partial_\alpha \phi^\beta \frac{1}{s^2} = \phi^\beta \partial_\alpha \frac{1}{s^2} \\
&= -\frac{1}{s^4}\phi^\beta \partial_\alpha s^2 \\
&= -\frac{2}{s} Y^\beta \partial_\alpha s \ ,
\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}
\partial_\alpha T^\beta &= \partial_\alpha \frac{1}{Q^2}\left(t^\beta - \omega\phi^\beta\right) \\
&= \frac{1}{Q^2}\partial_\alpha\left(t^\beta - \omega\phi^\beta\right) + \left(t^\beta - \omega\phi^\beta\right)\partial_\alpha \frac{1}{Q^2} \\
&= -\frac{1}{Q^2}\phi^\beta\partial_\alpha\omega - \frac{1}{Q^2}T^\beta\partial_\alpha Q^2 \\
&= -\frac{s^2}{Q^2}Y^\beta\partial_\alpha\omega - \frac{2}{Q}T^\beta\partial_\alpha Q \ .
\end{aligned} \tag{3.49}$$

With this, we may compute the covariant derivative of the field strength tensor. We begin with the connection term. Combining Eq. (3.20) with Eq. (3.21) and recalling that all of our relevant quantities are functions of only r and θ, our most useful form of the field strength is

$$F^{\alpha\beta} = \left(Y^\beta\Delta^\alpha - Y^\alpha\Delta^\beta\right)s^2 A_Y - \left(T^\beta E^\alpha - T^\alpha E^\beta\right) \ , \tag{3.50}$$

from which we proceed.

The submanifold connection term is best expressed in its most trivial form,

$$\begin{aligned}{}^{(2)}\Gamma^{\gamma}{}_{\gamma\alpha}F^{\alpha\beta} &= {}^{(2)}\Gamma^{\gamma}{}_{\gamma\alpha}Y^{\beta}\Delta^{\alpha}s^2A_Y - {}^{(2)}\Gamma^{\gamma}{}_{\gamma\alpha}Y^{\alpha}\Delta^{\beta}s^2A_Y - {}^{(2)}\Gamma^{\gamma}{}_{\gamma\alpha}T^{\beta}E^{\alpha} + {}^{(2)}\Gamma^{\gamma}{}_{\gamma\alpha}T^{\alpha}E^{\beta} \\ &= {}^{(2)}\Gamma^{\gamma}{}_{\gamma\alpha}Y^{\beta}\Delta^{\alpha}s^2A_Y - {}^{(2)}\Gamma^{\gamma}{}_{\gamma\alpha}T^{\beta}E^{\alpha}\ . \qquad (3.51)\end{aligned}$$

We have dropped the terms where a scaled vector is contracted with the connection because the connection is solely confined to the submanifold, which is orthogonal to both of our vectors Y^α and T^α by construction, making these terms zero. Moving on to the isometry directions of the connection, we find

$$\begin{aligned}\frac{1}{sQ}(\partial_\alpha sQ)F^{\alpha\beta} &= \frac{1}{sQ}(\partial_\alpha sQ)\left[\left(Y^{\beta}\Delta^{\alpha} - Y^{\alpha}\Delta^{\beta}\right)s^2A_Y - \left(T^{\beta}E^{\alpha} - T^{\alpha}E^{\beta}\right)\right] \\ &= Y^{\beta}\frac{1}{sQ}(\partial_\alpha sQ)\Delta^{\alpha}s^2A_Y - T^{\beta}E^{\alpha}\frac{1}{sQ}\partial_\alpha sQ \\ &= Y^{\beta}\left(\frac{1}{Q}\partial_\alpha Q + \frac{1}{s}\partial_\alpha s\right)\Delta^{\alpha}s^2A_Y - T^{\beta}E^{\alpha}\left(\frac{1}{Q}\partial_\alpha Q + \frac{1}{s}\partial_\alpha s\right)\ . \qquad (3.52)\end{aligned}$$

Here we have dropped the terms that are the Y^α or T^α contracted with the derivative of sQ, because sQ is only a function of r and θ, and those components of the vectors are zero. This is a process we may apply generally going forward. The contraction of one of our scaled vectors with a partial derivative of a scalar function must be zero. The same is true for the partial derivative of any vector function with components only dependent upon r and θ, such as the E^α and B^α functions.

Pushing onward, we now calculate the partial derivative of the field strength, which is where (3.48) and (3.49) come in handy. We find

$$\begin{aligned}\partial_\alpha F^{\alpha\beta} &= \partial_\alpha\left[\left(Y^{\beta}\Delta^{\alpha} - Y^{\alpha}\Delta^{\beta}\right)s^2A_Y - \left(T^{\beta}E^{\alpha} - T^{\alpha}E^{\beta}\right)\right] \\ &= \partial_\alpha\left(Y^{\beta}\Delta^{\alpha}s^2A_Y\right) - \partial_\alpha\left(Y^{\alpha}\Delta^{\beta}s^2A_Y\right) - \partial_\alpha T^{\beta}E^{\alpha} + \partial_\alpha T^{\alpha}E^{\beta} \\ &= -\frac{2}{s}Y^{\beta}(\partial_\alpha s)\Delta^{\alpha}s^2A_Y + Y^{\beta}\partial_\alpha\Delta^{\alpha}s^2A_Y + \frac{2}{s}Y^{\alpha}(\partial_\alpha s)\Delta^{\beta}s^2A_Y - Y^{\alpha}\partial_\alpha\Delta^{\beta}s^2A_Y - T^{\beta}\partial_\alpha E^{\alpha} \\ &\qquad + \frac{2}{Q}T^{\beta}E^{\alpha}\partial_\alpha Q + \frac{s^2}{Q^2}E^{\alpha}Y^{\beta}\partial_\alpha\omega + T^{\alpha}\partial_\alpha E^{\beta} - \frac{2}{Q}T^{\alpha}E^{\beta}\partial_\alpha Q - \frac{s^2}{Q^2}E^{\beta}Y^{\alpha}\partial_\alpha\omega\end{aligned}$$

$$
\begin{aligned}
&= Y^\alpha \left(\frac{2}{s}(\partial_\alpha s)\Delta^\beta s^2 A_Y - \partial_\alpha \Delta^\beta s^2 A_Y - \frac{s^2}{Q^2} E^\beta \Delta_\alpha \omega \right) \\
&\quad + Y^\beta \left(-\frac{2}{s}(\partial_\alpha s)\Delta^\alpha s^2 A_Y + \partial_\alpha \Delta^\alpha s^2 A_Y + \frac{s^2}{Q^2} E^\alpha \Delta_\alpha \omega \right) \\
&\quad + T^\alpha \left(\partial_\alpha E^\beta - \frac{2}{Q} E^\beta \partial_\alpha Q \right) - T^\beta \left(\partial_\alpha E^\alpha - \frac{2}{Q} E^\alpha \partial_\alpha Q \right) \ .
\end{aligned} \tag{3.53}
$$

Now making use of the property of the contraction of the vectors with partial derivatives, we see that the entire Y^α term and T^α term go to zero. This leaves us with

$$
\partial_\alpha F^{\alpha\beta} = Y^\beta \left[\partial_\alpha \Delta^\alpha s^2 A_Y - \frac{2}{s}(\partial_\alpha s)\Delta^\alpha s^2 A_Y + \frac{s^2}{Q^2} E^\alpha \Delta_\alpha \omega \right] - T^\beta \left[\partial_\alpha E^\alpha - \frac{2}{Q} E^\alpha \partial_\alpha Q \right] \ . \tag{3.54}
$$

By adding together Eqs. (3.54), (3.52), and (3.51), we may obtain the full form of the covariant derivative of the field strength,

$$
\begin{aligned}
\nabla_\alpha F^{\alpha\beta} = Y^\beta &\left[\partial_\alpha \Delta^\alpha s^2 A_Y - \frac{1}{s}(\partial_\alpha s)\Delta^\alpha s^2 A_Y + \frac{1}{Q}(\partial_\alpha Q)\Delta^\alpha s^2 A_Y + \frac{s^2}{Q^2} E^\alpha \Delta_\alpha \omega + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} \Delta^\alpha s^2 A_Y \right] \\
&- T^\beta \left[\partial_\alpha E^\alpha - E^\alpha \frac{1}{Q} \partial_\alpha Q + E^\alpha \frac{1}{s} \partial_\alpha s + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} E^\alpha \right] \ .
\end{aligned} \tag{3.55}
$$

A series of algebraic manipulations casts this in a better light:

$$
\begin{aligned}
\nabla_\alpha F^{\alpha\beta} &= Y^\beta \left[\partial_\alpha \Delta^\alpha s^2 A_Y - \frac{1}{s}(\partial_\alpha s)\Delta^\alpha s^2 A_Y + \frac{1}{Q}(\partial_\alpha Q)\Delta^\alpha s^2 A_Y + \frac{s^2}{Q^2} E^\alpha \Delta_\alpha \omega + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} \Delta^\alpha s^2 A_Y \right] \\
&\quad - T^\beta \left[\partial_\alpha E^\alpha - E^\alpha \frac{1}{Q} \partial_\alpha Q + E^\alpha \frac{1}{s} \partial_\alpha s + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} E^\alpha \right] \\
&= Y^\beta \left[\frac{s}{Q}(\Delta^\alpha s^2 A_Y) \left(-\frac{Q}{s^2} \partial_\alpha s + \frac{1}{s} \partial_\alpha Q \right) + \partial_\alpha \Delta^\alpha s^2 A_Y + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} \Delta^\alpha s^2 A_Y + \frac{s^2}{Q^2} E^\alpha \Delta_\alpha \omega \right] \\
&\quad - T^\beta \left[\partial_\alpha E^\alpha + \frac{Q}{s} E^\alpha \left(-\frac{s}{Q^2} \partial_\alpha Q + \frac{1}{Q} \partial_\alpha s \right) + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} E^\alpha \right] \\
&= \frac{s}{Q} Y^\beta \left[(\Delta^\alpha s^2 A_Y) \partial_\alpha \frac{Q}{s} + \frac{Q}{s} \partial_\alpha \Delta^\alpha s^2 A_Y + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} \frac{Q}{s} \Delta^\alpha s^2 A_Y + \frac{s}{Q} E^\alpha \Delta_\alpha \omega \right] \\
&\quad - \frac{Q}{s} T^\beta \left[\frac{s}{Q} \partial_\alpha E^\alpha + E^\alpha \partial_\alpha \frac{s}{Q} + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} \frac{s}{Q} E^\alpha \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{s}{Q} Y^\beta \left[\partial_\alpha \left(\frac{Q}{s} \Delta^\alpha s^2 A_Y \right) + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} \frac{Q}{s} \Delta^\alpha s^2 A_Y + \frac{s}{Q} E^\alpha \Delta_\alpha \omega \right] \\
&\qquad\qquad - \frac{Q}{s} T^\beta \left[\partial_\alpha \left(\frac{s}{Q} E^\alpha \right) + {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} \frac{s}{Q} E^\alpha \right] \\
&= \frac{s}{Q} Y^\beta \left[\Delta_\alpha \left(\frac{Q}{s} \Delta^\alpha s^2 A_Y \right) + \frac{s}{Q} E^\alpha \Delta_\alpha \omega \right] - \frac{Q}{s} T^\beta \Delta_\alpha \left(\frac{s}{Q} E^\alpha \right) \\
&= \frac{s}{Q} Y^\beta \left[\Delta_\alpha B^\alpha - \omega \Delta_\alpha \left(\frac{s}{Q} E^\alpha \right) \right] - \frac{Q}{s} T^\beta \Delta_\alpha \left(\frac{s}{Q} E^\alpha \right) = 0 \ , \qquad (3.56)
\end{aligned}
$$

where we used the definition of the B^α field as it appears in Eq. (3.21).

Now we have found something truly striking. Since the Y^α and T^α vectors are orthogonal, their associated terms must each be individually zero to satisfy the Maxwell equations, because they cannot add to cancel out. This shows that we must have

$$
\Delta_\alpha \left(\frac{s}{Q} E^\alpha \right) = 0 \qquad (3.57)
$$

in order to make the T^β term be zero. This is our second divergence-free field. Additionally, since the field of (3.57) has no divergence, to make the Y^β term zero we also need to have

$$
\Delta_\alpha B^\alpha = 0 \ . \qquad (3.58)
$$

This is our third of the five constructed fields. With these properties determined, we may turn back to the Einstein equations and develop our last two equations.

3.3.2 Collecting the Fields

Since we have managed to show the divergence of $\frac{s}{Q} E^\alpha$ is 0, we can rewrite Eq. (3.42) as

$$
\begin{aligned}
\Delta_\alpha \left(\frac{s^3}{Q} \Delta^\alpha \omega \right) + 4G \frac{s}{Q} E^\alpha \Delta_\alpha s^2 A_Y &= 0 \\
\Delta_\alpha \left(\frac{s^3}{Q} \Delta^\alpha \omega + 4G \frac{s}{Q} E^\alpha s^2 A_Y \right) &= 0 \ . \qquad (3.59)
\end{aligned}
$$

This motivates us defining a new quantity,

$$J^\alpha = \frac{s^3}{Q}\Delta^\alpha\omega + 4G\frac{s}{Q}E^\alpha s^2 A_Y \ . \tag{3.60}$$

This is our fourth field,

$$\Delta_\alpha J^\alpha = 0 \ . \tag{3.61}$$

Leveraging the form of Eq. (3.60) with the divergence-free nature of J^α and $\frac{s}{Q}E^\alpha$, we may rewrite Eq. (3.41) as

$$\begin{aligned}
0 &= \Delta_\alpha(s\Delta^\alpha Q) - \frac{1}{2}\left(J^\alpha - 4G\frac{s}{Q}E^\alpha s^2 A_Y\right)\Delta_\alpha\omega - G\left(\frac{Q}{s}(\Delta^\alpha s^2 A_Y)\Delta_\alpha s^2 A_Y + \frac{s}{Q}E_\alpha E^\alpha\right)\\
&= \Delta_\alpha(s\Delta^\alpha Q) - \Delta_\alpha\frac{1}{2}\omega J^\alpha + G\left(2\frac{s}{Q}E^\alpha s^2 A_Y\Delta_\alpha\omega - \frac{Q}{s}(\Delta^\alpha s^2 A_Y)\Delta_\alpha s^2 A_Y - \frac{s}{Q}E_\alpha E^\alpha\right)\\
&= \Delta_\alpha(s\Delta^\alpha Q) - \Delta_\alpha\frac{1}{2}\omega J^\alpha + G\left(2s^2 A_Y\Delta_\alpha\frac{s}{Q}E^\alpha\omega - \frac{Q}{s}(\Delta^\alpha s^2 A_Y)\Delta_\alpha s^2 A_Y\right.\\
&\qquad\left. -\Delta_\alpha\frac{s}{Q}E^\alpha Q^2 A_T - s^2 A_Y\Delta_\alpha\frac{s}{Q}E^\alpha\omega\right)\\
&= \Delta_\alpha(s\Delta^\alpha Q) - \Delta_\alpha\frac{1}{2}\omega J^\alpha - G\left(\Delta_\alpha\frac{s}{Q}E^\alpha Q^2 A_T - s^2 A_Y\Delta_\alpha\frac{s}{Q}E^\alpha\omega + \frac{Q}{s}(\Delta^\alpha s^2 A_Y)\Delta_\alpha s^2 A_Y\right) \ .
\end{aligned} \tag{3.62}$$

Since $\Delta_\alpha B^\alpha = 0$, we also have $-Gs^2 A_Y\Delta_\alpha B^\alpha = 0$, meaning we can add it to the above expression to get

$$\begin{aligned}
&\Delta_\alpha(s\Delta^\alpha Q) - \Delta_\alpha\frac{1}{2}\omega J^\alpha - G\left(\Delta_\alpha\frac{s}{Q}E^\alpha Q^2 A_T - s^2 A_Y\Delta_\alpha\frac{s}{Q}E^\alpha\omega + \frac{Q}{s}(\Delta^\alpha s^2 A_Y)\Delta_\alpha s^2 A_Y + s^2 A_Y\Delta_\alpha B^\alpha\right)\\
&= \Delta_\alpha(s\Delta^\alpha Q) - \Delta_\alpha\frac{1}{2}\omega J^\alpha - G\left(\Delta_\alpha\frac{s}{Q}E^\alpha Q^2 A_T - s^2 A_Y\Delta_\alpha\frac{s}{Q}E^\alpha\omega + \frac{Q}{s}(\Delta^\alpha s^2 A_Y)\Delta_\alpha s^2 A_Y\right.\\
&\qquad\left. + s^2 A_Y\Delta_\alpha\frac{Q}{s}\Delta^\alpha s^2 A_Y + sQA_Y\Delta_\alpha\Delta^\alpha s^2 A_Y + s^2 A_Y\Delta_\alpha\frac{s}{Q}E^\alpha\omega\right)\\
&= \Delta_\alpha(s\Delta^\alpha Q) - \Delta_\alpha\frac{1}{2}\omega J^\alpha - G\left(\Delta_\alpha\frac{s}{Q}E^\alpha Q^2 A_T + sQA_Y\Delta_\alpha\Delta^\alpha s^2 A_Y\right.\\
&\qquad\left. + \frac{Q}{s}(\Delta^\alpha s^2 A_Y)\Delta_\alpha s^2 A_Y + s^2 A_Y\Delta_\alpha\frac{Q}{s}\Delta^\alpha s^2 A_Y\right)
\end{aligned}$$

$$= \Delta_\alpha(s\Delta^\alpha Q) - \Delta_\alpha \frac{1}{2}\omega J^\alpha - G\left(\Delta_\alpha \frac{s}{Q} E^\alpha Q^2 A_T + sQA_Y \Delta_\alpha \Delta^\alpha s^2 A_Y + (\Delta_\alpha sQA_Y)\Delta^\alpha s^2 A_Y\right)$$
$$= \Delta_\alpha(s\Delta^\alpha Q) - \Delta_\alpha \frac{1}{2}\omega J^\alpha - G\left(\Delta_\alpha \frac{s}{Q} E^\alpha Q^2 A_T + \Delta_\alpha(sQA_Y \Delta^\alpha s^2 A_Y)\right)$$
$$= \Delta_\alpha\left(s\Delta^\alpha Q - \frac{1}{2}\omega J^\alpha - G\left[\frac{s}{Q} E^\alpha Q^2 A_T + sQA_Y \Delta^\alpha s^2 A_Y\right]\right) = 0 \ , \tag{3.63}$$

giving us our final defined quantity

$$M^\alpha = s\Delta^\alpha Q - \frac{1}{2}\omega J^\alpha - G\left[\frac{s}{Q} E^\alpha Q^2 A_T + sQA_Y \Delta^\alpha s^2 A_Y\right] \ . \tag{3.64}$$

Pulling everything together, we get our set of five divergence-free fields,

$$\begin{aligned} \Delta_\alpha \Delta^\alpha sQ &= 0 \\ \Delta_\alpha \left(\frac{s}{Q} E^\alpha\right) &= 0 \\ \Delta_\alpha B^\alpha &= 0 \\ \Delta_\alpha J^\alpha &= 0 \\ \Delta_\alpha M^\alpha &= 0 \ . \end{aligned} \tag{3.65}$$

This form is very indicative that these fields are related to conserved quantities of spacetime —a divergence-free vector field implies no net production or annihilation, but a flow from point to point. Each field (aside from the first) has a classically conserved quantity relating to it. The E^α field is in some way related to the electric charge of the black hole, the J^α to its angular momentum, the B^α to the magnetic charge (which is only nontrivial if we assume the black hole has a nonzero net magnetic charge, implying the existence of magnetic monopoles), and the M^α has something to do with the total energy and, therefore, the total mass.

3.4 Exact Solution

The first function of (3.65) seems to be the odd one out. Instead of the divergence of a vector field, it is the Laplacian of a scalar field. Additionally, we claim that it is not related to a classically conserved quantity. This statement deserves some qualification. As far as we are aware, this function does not directly relate to a conserved quantity. Not only that, but it is directly solvable for all black holes.

Recalling the asymptotic forms of both Q and s found in Eq. (3.6), we set up a functional form as some unknown function, $F(r, \theta)$, multiplied by these asymptotic expressions. For reasons that would take us too far afield to discuss here, Q must vanish on the event horizon of the black hole. This gives us the boundary conditions on F, which must be 0 at the horizon and approach 1 as r grows without bound. We thus get the ansatz

$$sQ = r \sin\theta F \ , \tag{3.66}$$

where $F\Big|_{r_h} = 0$, r_h being the event horizon, and $F\Big|_{\infty} = 1$.

We can now plug this into our equation. The first (inner) derivative gets applied to sQ, a scalar function, so the covariant derivative is just a normal partial derivative. Then using the special case of a derivative with contracted indices as explained in Eq. (2.13), we obtain

$$\Delta_\alpha \Delta^\alpha sQ = \frac{1}{\sqrt{\sigma}} \partial_\alpha [\sqrt{\sigma} \sigma^{\alpha\beta} \partial_\beta sQ] \ . \tag{3.67}$$

Since the submanifold metric is diagonal (as shown in Eq. (3.8)), the summation expands to just two terms, and we get the following:

$$\begin{aligned}
0 &= \frac{1}{\sqrt{\sigma}}\partial_r\left[\sqrt{\sigma}\sigma^{rr}\partial_r sQ\right] + \frac{1}{\sqrt{\sigma}}\partial_\theta\left[\sqrt{\sigma}\sigma^{\theta\theta}\partial_\theta sQ\right] \\
&= \frac{e^{-2f}}{r}\partial_r\left[re^{2f}e^{-2f}\partial_r(r\sin\theta\, F)\right] + \frac{e^{-2f}}{r}\partial_\theta\left[re^{2f}\frac{e^{-2f}}{r^2}\partial_\theta(r\sin\theta\, F)\right] \\
&= \frac{e^{-2f}}{r}\partial_r\left[r(r\sin\theta\,\partial_r F + \sin\theta\, F)\right] + \frac{e^{-2f}}{r}\partial_\theta\left[\frac{1}{r}(r\sin\theta\,\partial_\theta F + r\cos\theta\, F)\right] \\
&= \frac{e^{-2f}}{r}\sin\theta\,\partial_r\left(r^2F' + F\right) + \frac{e^{-2f}}{r}\partial_\theta\left(\sin\theta\,\dot{F} + \cos\theta\, F\right)\ ,
\end{aligned} \tag{3.68}$$

where we have used the notation F' for the derivative of F with respect to r and $\dot{F}$ for the derivative with respect to θ.

We know that e^{-2f} cannot be identically zero, because that would imply f is infinite everywhere, which would in turn make our metric infinite, which is not reasonable. We can therefore multiply both sides by re^{2f} and continue to expand, finding

$$\begin{aligned}
0 &= \sin\theta\,\partial_r\left(r^2F' + F\right) + \partial_\theta\left(\sin\theta\,\dot{F} + \cos\theta\, F\right) \\
&= \sin\theta\left(2rF' + r^2F'' + rF' + F\right) + \sin\theta\,\ddot{F} + \cos\theta\,\dot{F} - \sin\theta\, F + \cos\theta\,\dot{F} \\
&= \sin\theta\left(3rF' + r^2F'' + \ddot{F} + 2\cot\theta\,\dot{F}\right)\ .
\end{aligned} \tag{3.69}$$

Finally, since this relationship must hold for all $\theta \in [0, \pi]$, we see

$$3rF' + r^2F'' + \ddot{F} + 2\cot\theta\,\dot{F} = 0\ . \tag{3.70}$$

Employing the method of separation of variables, we let $F = R(r)T(\theta)$. Running through the process and calling our separation constant k, we get

$$\underbrace{3r\frac{R'}{R} + r^2\frac{R''}{R}}_{-k} + \underbrace{\frac{T''}{T} + 2\cot\theta\,\frac{T'}{T}}_{k} = 0\ . \tag{3.71}$$

Here we are using the prime to designate the derivative of the function with respect to its only argument. This gives us two equations,

$$3rR' + r^2R'' + kR = 0 \ , \tag{3.72}$$

$$T'' + 2\cot\theta\, T' - kT = 0 \ . \tag{3.73}$$

Let us go back to the boundary conditions for F. We see that, out to infinity, F can have no θ-dependence, because $F\big|_{\infty} = 1$, which is independent of θ. It can can be shown more rigorously that no solution for R consistent with its differential equation (3.72) will decay fast enough to kill all the angular dependence. We must then have $T(\theta)$ be some constant, meaning $T' = T'' = 0$. Putting these into Eq. (3.73), we get $kT = 0$. While the trivial solution $T = 0$ does technically solve this, it is the physically uninteresting case. We therefore let $k = 0$, and T be a nonzero constant.

Setting $k = 0$ in Eq. (3.72) makes it especially simple,

$$3rR' + r^2R'' = 0 \ . \tag{3.74}$$

This is solved using the usual methods of differential equations, and we find

$$R = d_1 - \frac{d_2}{r^2} \ . \tag{3.75}$$

Multiplying $R(r)$ by our constant $T(\theta)$ function, we get

$$F = c_1 - \frac{c_2}{r^2} \ , \tag{3.76}$$

where c_1 and c_2 are arbitrary constants. Applying our boundary conditions lets us determine these constants. We first let r tend to ∞, and find

$$F\Big|_{\infty} = c_1 = 1 \ .$$

Next setting $r = r_h$, we find

$$F\Big|_{r_h} = 1 - \frac{c_2}{r_h^2} = 0 \ ,$$

so $c_2 = r_h^2$. This means

$$F = 1 - \frac{r_h^2}{r^2} \ . \tag{3.77}$$

Putting this back into Eq. (3.66), we have solved the first field,

$$sQ = r \sin\theta \left(1 - \frac{r_h^2}{r^2}\right) \ . \tag{3.78}$$

It may be surprising to see that we made no assumption as to the form of the black hole to get this result. This is the solution for every stationary, axisymmetric black hole. While it may not obviously correspond to a physical observable, the delightfully simple form of Eq. (3.78) supports the notion that there is an underlying level of homogeneity to all types of black holes.

Chapter 4

Conserved Quantities

Since the form of Eq. (3.65) (our five divergence-free fields) is so suggestive of conservation, it would be wise to see if it reproduces the conserved quantities of our system. Using the definition of the Komar integrals, Eq. (2.18)–(2.20), we can calculate this. The surface over which we are integrating, S_t, is a spherical shell that encloses the black hole. In the surface element (2.21) (which, as a reminder, is $\mathrm{d}S_{\alpha\beta} = (n_\beta r_\alpha - n_\alpha r_\beta)\sqrt{\sigma_s}\,\mathrm{d}^2\sigma_s$), the vectors n_α and r_α are respectively the timelike and spacelike unit normals to the sphere of integration, with the sign chosen to match the convention in [20],

$$n_\alpha = -T_\alpha \frac{1}{\sqrt{-g^{tt}}} = -QT_\alpha \ , \qquad r_\alpha = \delta^r_\alpha \frac{1}{\sqrt{g^{rr}}} = \delta^r_\alpha e^f \ . \tag{4.1}$$

The sphere enclosing the black hole is described by the variables θ and ϕ, which gives us the metric

$$\mathrm{d}s^2 = e^{2f} r^2 \,\mathrm{d}\theta^2 + s^2 \,\mathrm{d}\phi^2 \, . \tag{4.2}$$

Thus $\sqrt{\sigma_s} = e^f rs$.

4.1 Angular Momentum Integral

We first tackle the angular momentum integral, Eq. (2.19). This quantity (along with all of the others we are considering here) has no dependence on ϕ, so that part just integrates to 2π. We see

$$\begin{aligned}\mathcal{J} &= \frac{1}{16\pi}\lim_{S_t\to\infty}\oint_{S_t} \mathrm{d}S_{\alpha\beta}\,\nabla^\alpha\phi^\beta \\ &= -\frac{1}{16\pi}\lim_{r\to\infty}\int \mathrm{d}\theta\,\mathrm{d}\phi\, e^{2f} rs(\delta^r_\alpha QT_\beta - \delta^r_\beta QT_\alpha)\nabla^\alpha\phi^\beta \\ &= -\frac{1}{8}\lim_{r\to\infty}\int \mathrm{d}\theta\, e^{2f} rs(\delta^r_\alpha QT_\beta - \delta^r_\beta QT_\alpha)\nabla^\alpha\phi^\beta \\ &= -\frac{1}{8}\lim_{r\to\infty}\int \mathrm{d}\theta\, e^{2f}\frac{rs}{Q}\left[\delta^r_\alpha(t_\beta-\omega\phi_\beta) - \delta^r_\beta(t_\alpha-\omega\phi_\alpha)\right]\nabla^\alpha\phi^\beta\ .\end{aligned} \tag{4.3}$$

Using Killing's equation (2.14), this can be reduced considerably. Since $\nabla_\alpha\phi_\beta = -\nabla_\beta\phi_\alpha$ and all of the indices are dummy indices, we may write

$$\begin{aligned}\Big(\delta^r_\alpha(t_\beta-\omega\phi_\beta) - \delta^r_\beta(t_\alpha-\omega\phi_\alpha)\Big)\nabla^\alpha\phi^\beta &= (t_\alpha-\omega\phi_\alpha)\delta^r_\beta(\nabla^\beta\phi^\alpha - \nabla^\alpha\phi^\beta) \\ &= 2(t_\alpha-\omega\phi_\alpha)\delta^r_\beta\nabla^\beta\phi^\alpha \\ &= 2(t_\alpha-\omega\phi_\alpha)\delta^r_\beta g^{\beta\gamma}\nabla_\gamma\phi^\alpha \\ &= 2(t_\alpha-\omega\phi_\alpha)g^{r\gamma}\nabla_\gamma\phi^\alpha \\ &= 2(t_\alpha-\omega\phi_\alpha)e^{-2f}\nabla_r\phi^\alpha\ .\end{aligned} \tag{4.4}$$

We plug this result into the integral Eq. (4.3) to get

$$\begin{aligned}\mathcal{J} &= -\frac{1}{4}\lim_{r\to\infty}\int \mathrm{d}\theta\,\frac{rs}{Q}(t_\alpha-\omega\phi_\alpha)\nabla_r\phi^a \\ &= -\frac{1}{4}\lim_{r\to\infty}\int \mathrm{d}\theta\,\frac{rs}{Q}(t_\alpha-\omega\phi_\alpha)(\partial_r\phi^\alpha + \Gamma^\alpha{}_{r\beta}\phi^\beta) \\ &= -\frac{1}{4}\lim_{r\to\infty}\int \mathrm{d}\theta\,\frac{rs}{Q}(t_\alpha-\omega\phi_\alpha)\Gamma^\alpha{}_{r\beta}\phi^\beta\ .\end{aligned} \tag{4.5}$$

This contraction with the Christoffel symbol is remarkably straightforward, since most of the quantities become zero immediately. A quick examination shows us

$$\begin{aligned}
\Gamma^{\alpha}{}_{r\beta}\phi^{\beta} = {}^{(2)}\Gamma^{\alpha}{}_{r\beta}\phi^{\beta} &+ \frac{1}{2}\sigma^{\alpha\gamma}[s^2(Y_rT_\beta + Y_\beta T_r)\Delta_\gamma\omega - Y_rY_\beta\Delta_\gamma s^2 + T_rT_\beta\Delta_\gamma Q^2]\phi^\beta \\
&+ \frac{1}{2}Y^\alpha[\partial_r(s^2Y_\beta) + \partial_\beta(s^2Y_r)]\phi^\beta \\
&- \frac{1}{2}T^\alpha[\partial_r(Q^2T_\beta) + \partial_\beta(Q^2T_r) + s^2(Y_r\Delta_\beta\omega + Y_\beta\Delta_r\omega)]\phi^\beta \\
= -\frac{1}{2}&T^\alpha\phi^\beta Y_\beta s^2\partial_r\omega \\
= -\frac{1}{2}&T^\alpha s^2\partial_r\omega \ . \qquad (4.6)
\end{aligned}$$

Thus we get

$$\begin{aligned}
\mathcal{J} &= \frac{1}{8}\lim_{r\to\infty}\int \mathrm{d}\theta\,\frac{rs^3}{Q}(t_\alpha - \omega\phi_\alpha)T^\alpha\partial_r\omega \\
&= -\frac{1}{8}\lim_{r\to\infty}\int \mathrm{d}\theta\,\frac{rs^3}{Q}\partial_r\omega \ . \qquad (4.7)
\end{aligned}$$

However, we are running into the same problem that we brought up when first discussing Komar integrals. This has functional dependence on r, but r is tending to infinity, so we need to use the alternative formulation in Eq. (2.23). As a reminder, this is

$$\mathcal{J} = \frac{1}{16\pi}\oint_{S_t}\mathrm{d}S_{\alpha\beta}\,\nabla^\alpha\phi^\beta\Big|_{r_h} - \int_{\Sigma(r_h)}\mathrm{d}^3y\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)n^\alpha\phi^\beta\sqrt{h} \ , \qquad (4.8)$$

where $\sqrt{h}$ is the induced metric on the spacelike hypersurface $\Sigma(r_h)$ extending from r_h to infinity. The spacelike portion of the metric is

$$\mathrm{d}\ell^2 = \sigma\,\mathrm{d}\sigma^2 + s^2\,\mathrm{d}\phi^2 = e^{2f}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) + s^2\,\mathrm{d}\phi^2 \ , \qquad (4.9)$$

meaning $\mathrm{d}^3y\,\sqrt{h} = \mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r\,s\sqrt{\sigma}$. We already calculated that $T = 0$, so we now need to evaluate $T_{\alpha\beta}n^\alpha\phi^\beta$. Recall $n^\alpha = -QT^\alpha$, and T^α is orthogonal to the submanifold and Y_α. Furthermore, ϕ^β is orthogonal to T_β. That means the only term that survives from $T_{\alpha\beta}$ in the contraction is

$$4\pi T_{\alpha\beta}n^\alpha\phi^\beta = \phi^\beta QT^\alpha Y_\beta T_\alpha E^\gamma\Delta_\gamma s^2 A_Y = -\frac{1}{Q}E^\gamma\Delta_\gamma s^2 A_Y \ . \qquad (4.10)$$

Plugging this result in, we get

$$\mathcal{J} = -\frac{1}{8}\int \mathrm{d}\theta \, \frac{rs^3}{Q}\partial_r\omega\bigg|_{r_h} + \frac{1}{4\pi}\int \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}r \, \sqrt{\sigma}\Delta_\gamma \left(\frac{s}{Q}E^\gamma s^2 A_Y\right) \ . \tag{4.11}$$

We can expand out the sum over γ and deal with each part individually.

First looking at the $\gamma = \theta$ term, we have

$$\frac{1}{4\pi}\int \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}r \, \sqrt{\sigma}\Delta_\theta \left(\frac{s}{Q}E^\theta s^2 A_Y\right) \ . \tag{4.12}$$

None of the quantities depend on ϕ, so that portion immediately integrates to 2π. Employing Eq. (2.13), we get

$$\frac{1}{2}\int \mathrm{d}\theta \, \mathrm{d}r \, \sqrt{\sigma}\Delta_\theta \left(\frac{s}{Q}E^\theta s^2 A_Y\right) = \frac{1}{2}\int \mathrm{d}\theta \, \mathrm{d}r \, \partial_\theta \left(\sqrt{\sigma}\frac{s}{Q}E^\theta s^2 A_Y\right) \ . \tag{4.13}$$

The θ integral can now be evaluated using the fundamental theorem of calculus, and we get

$$\frac{1}{2}\int_{r_h}^{\infty} \mathrm{d}r \left(\sqrt{\sigma}\frac{s}{Q}E^\theta s^2 A_Y\bigg|_0^\pi\right) \ . \tag{4.14}$$

In order for there to be no magnetic monopole, we require that $A_Y(\pi) = A_Y(0)$, meaning this portion of the integral is zero.

Now moving on to the $\gamma = r$ portion, we have

$$\begin{aligned}
\frac{1}{4\pi}\int \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}\phi \, \sqrt{\sigma}\Delta_r \left(\frac{s}{Q}E^r s^2 A_Y\right) &= \frac{1}{4\pi}\int \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \partial_r \left(\sqrt{\sigma}g^{rb}\frac{s}{Q}E_b s^2 A_Y\right) \\
&= \frac{1}{2}\int \mathrm{d}\theta \, \mathrm{d}r \, \partial_r \left(\sqrt{\sigma}e^{-2f}\frac{s}{Q}E_r s^2 A_Y\right) \\
&= \frac{1}{2}\int \mathrm{d}\theta \, \mathrm{d}r \, \partial_r \left(\frac{rs}{Q}E_r s^2 A_Y\right) \ .
\end{aligned} \tag{4.15}$$

Again applying the fundamental theorem of calculus, this becomes

$$-\frac{1}{2}\int \mathrm{d}\theta \, \frac{rs}{Q}E_r s^2 A_Y\bigg|_{r_h} \ , \tag{4.16}$$

where the term evaluated at $r = \infty$ goes to zero because we assumed asymptotic flatness so E_r and A_Y should go to zero at infinite radius. The full expression for $\mathcal{J}$ is then

$$\mathcal{J} = -\frac{1}{8}\int \mathrm{d}\theta \, \frac{rs^3}{Q}\partial_r\omega\bigg|_{r_h} - \frac{1}{2}\int \mathrm{d}\theta \, \frac{rs^3}{Q}E_r A_Y\bigg|_{r_h} \ . \tag{4.17}$$

Something quite nice happens when we looks at this first quantity; recalling that the definition of J_α is (in geometrized units where $G = c = 1$)

$$J_\alpha = \frac{s^3}{Q}\Delta_\alpha\omega + 4\frac{s}{Q}E_\alpha s^2 A_Y \ , \tag{4.18}$$

we may write

$$\frac{rs^3}{Q}\partial_r\omega = rJ_r - 4\frac{rs}{Q}E_r A_Y \ . \tag{4.19}$$

Inserting this into our integral gives us

$$\begin{aligned}\mathcal{J} &= -\frac{1}{8}\int \mathrm{d}\theta\, rJ_r\Big|_{r_h} + \frac{1}{2}\int \mathrm{d}\theta\, \frac{rs^3}{Q}E_r A_Y\Big|_{r_h} - \frac{1}{2}\int \mathrm{d}\theta\, \frac{rs^3}{Q}E_r A_Y\Big|_{r_h} \\ &= -\frac{1}{8}\int \mathrm{d}\theta\, rJ_r\Big|_{r_h} \ . \end{aligned} \tag{4.20}$$

This is quite an intriguing result. The conserved quantity of angular momentum is just the integral of our angular momentum field over θ at the event horizon with a constant prefactor. Somehow, a single radius of our angular momentum field J_r seems to encode a global property of the spacetime. We now turn our investigation to the other parameters and see if such a relationship holds for the other fields in (3.65).

4.2 Mass Integral

Of particular interest is the mass. If we can recover the mass parameter from these fields, it will enforce that what we are doing is real physics, not just pushing some symbols around until we get something pretty. To this end, we take a look at Eq. (2.18).

Doing the same thing that we did to get to Eq. (4.7) with the only modification being to use the t^α vector instead of ϕ^α, this comes out to

$$\mathcal{M} = -\frac{1}{2}\lim_{r\to\infty}\int \mathrm{d}\theta\, \frac{rs}{Q}(t_\alpha - \omega\phi_\alpha){\Gamma^\alpha}_{r\beta}t^\beta \ , \tag{4.21}$$

as the reader may verify. The same terms in the connection go to zero, except for the first term in the T^α term, $\partial_r(Q^2 T_\beta) = 2QT_\beta \partial_r Q$. This one went to zero previously because the T_β is orthogonal to ϕ^β, but it not orthogonal to t^β. This contraction then gives us

$$\Gamma^\alpha{}_{r\beta} t^\beta = -T^\alpha t^\beta T_\beta Q \partial_r Q - \frac{1}{2} T^\alpha t^\beta Y_\beta s^2 \partial_r \omega = T^\alpha \left(Q \partial_r Q - \frac{1}{2} \omega s^2 \partial_r \omega \right) . \tag{4.22}$$

Putting this into the integral (4.21), we obtain

$$\mathcal{M} = -\frac{1}{2} \lim_{r\to\infty} \int \mathrm{d}\theta \, r \left(s \partial_r Q - \frac{s^3}{2Q} \omega \partial_r \omega \right) . \tag{4.23}$$

We now employ the same method of hypersurfaces as used in the angular momentum integral, this time in the form of Eq. (2.22),

$$\mathcal{M} = -\frac{1}{8\pi} \oint_{S_t} \mathrm{d}S_{\alpha\beta} \, \nabla^\alpha t^\beta \Big|_{r_h} + 2 \int_{\Sigma(r_h)} \mathrm{d}^3 y \sqrt{h} \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) n^\alpha t^\beta . \tag{4.24}$$

Computing $T_{\alpha\beta} n^\alpha t^\beta$, we get

$$\begin{aligned} 4\pi T_{\alpha\beta} n^\alpha t^\beta &= -Q T^\alpha T_\alpha t^\beta T_\beta \left(\frac{Q^2}{2s^2} (\Delta^\gamma s^2 A_Y) \Delta_\gamma s^2 A_Y + \frac{1}{2} E_\gamma E^\gamma \right) & (4.25) \\ &= -\frac{Q}{2s^2} (\Delta^\gamma s^2 A_Y) \Delta_\gamma s^2 A_Y - \frac{1}{2Q} E_\gamma E^\gamma . & (4.26) \end{aligned}$$

Using the definitions of B^α and E^α from Eq. (3.21), we may rewrite this as

$$\begin{aligned} 4\pi T_{\alpha\beta} n^\alpha t^\beta &= -\frac{1}{2s} \left(B^\gamma - \frac{s}{Q} \omega E^\gamma \right) \Delta_\gamma s^2 A_Y - \frac{1}{2Q} E^\gamma \left[\Delta_\gamma \left(Q^2 A_T + s^2 \omega A_Y \right) - \omega \Delta_\gamma s^2 A_Y \right] \\ &= -\frac{1}{2s} B^\gamma \Delta_\gamma s^2 A_Y - \frac{1}{2Q} E^\gamma \Delta_\gamma \left(Q^2 A_T + s^2 \omega A_Y \right) + \frac{1}{Q} E^\gamma \omega \Delta_\gamma s^2 A_Y . \qquad (4.27) \end{aligned}$$

Plugging this in and performing the ϕ integral, this becomes

$$\begin{aligned} \mathcal{M} = -\frac{1}{2} \int \mathrm{d}\theta \, r \left(s \partial_r Q - \frac{1}{2} \omega \frac{s^3}{Q} \partial_r \omega \right) \Big|_{r_h} &- \frac{1}{2} \int \mathrm{d}\sigma \sqrt{\sigma} \left[\frac{s}{Q} E^\alpha \Delta_\alpha (Q^2 A_T + s^2 \omega A_Y) + B^\alpha \Delta_\alpha s^2 A_Y \right] \\ &+ \int \mathrm{d}\sigma \sqrt{\sigma} \omega \frac{s}{Q} E^\alpha \Delta_\alpha s^2 A_Y . \qquad (4.28) \end{aligned}$$

After a somewhat-hefty chain of integration by parts and expanding out the sums in the index contraction, this may be further reduced. Recalling that $A_Y(\theta = 0) = A_Y(\theta = \pi)$ and throwing away any terms evaluated as $r \to \infty$ due to the assumption of asymptotic flatness, this all condenses down to

$$\mathcal{M} = -\frac{1}{2}\int \mathrm{d}\theta\, r M_r\Big|_{r_h}\ , \tag{4.29}$$

as the reader can check. This is again a suggestive result. Integrating our mass field over θ, multiplied by a constant prefactor, gives us the mass parameter. Clearly our constructed fields have a deep relationship with the conserved quantities of spacetime.

4.3 Charge Integral

Having gotten the angular momentum and mass integrals out of our fields, the next step would be to see if we also get the conserved charge as an integral of our E^α field. For the sake of simplicity, we assume the black hole has zero magnetic charge, though it would be interesting to consider one with nonzero magnetic charge. This is also why we do not attempt to reconstruct the magnetic charge from the B^α field.

Once more unto the breach, we turn to the final foreshadowed integral, Eq. (2.20). This one is slightly different (read: simpler), because it does not rely on the limit as we go out to infinity, it just needs to have $r > r_h$. Plugging in $F^{\alpha\beta}$ from Eq. (3.13) and using the fact that it is an antisymmetric tensor, we find

$$\begin{aligned}
\mathcal{Q} &= \frac{1}{8\pi}\oint_{S_t} \mathrm{d}S_{\alpha\beta}\, F^{\alpha\beta} \\
&= -\frac{1}{8\pi}\int \mathrm{d}\theta\, \mathrm{d}\phi\, rse^{2f}(\delta^r_\alpha QT_\beta - \delta^r_\beta QT_\alpha)F^{\alpha\beta} \\
&= -\frac{1}{4}\int \mathrm{d}\theta\, \frac{rs}{Q}e^{2f}\left[(t_\beta - \omega\phi_\beta)F^{r\beta} - (t_\alpha - \omega\phi_\alpha)F^{\alpha r}\right] \\
&= -\frac{1}{4}\int \mathrm{d}\theta\, \frac{rs}{Q}e^{2f}\left[(t_\beta - \omega\phi_\beta)F^{r\beta} + (t_\alpha - \omega\phi_\alpha)F^{r\alpha}\right] \\
&= -\frac{1}{2}\int \mathrm{d}\theta\, \frac{rs}{Q}e^{2f}(t_\alpha - \omega\phi_\alpha)F^{r\alpha}\ .
\end{aligned} \tag{4.30}$$

This contraction over α warrants a touch of foresight, recognizing that we will want the derivatives to have the index down as usual. Doing this, exploiting the orthogonality of the ϕ_α and T^α vectors, and recognizing that $Y_r = T_r = 0$, we obtain

$$\begin{aligned}
(t_\alpha - \omega\phi_\alpha)F^{r\alpha} &= g^{rb}(t_\alpha - \omega\phi_\alpha)F_b{}^\alpha \\
&= g^{rr}(t_\alpha - \omega\phi_\alpha)F_r{}^\alpha \\
&= e^{-2\alpha}(t_\alpha - \omega\phi_\alpha)\Big[(Y^\alpha\partial_r - Y_r\partial^\alpha)s^2A_Y - (T^\alpha\partial_r - T_r\partial^\alpha)Q^2A_T \\
&\qquad\qquad -s^2A_Y(T^\alpha\partial_r - T_r\partial^\alpha)\omega\Big] \\
&= e^{-2\alpha}(t_\alpha - \omega\phi_\alpha)\Big(Y^\alpha\partial_r s^2A_Y - T^\alpha\partial_r Q^2A_T - s^2A_Y T^\alpha\partial_r\omega\Big) \\
&= e^{-2\alpha}\Big[\Big(t_\alpha Y^\alpha\partial_r s^2A_Y - t_\alpha T^\alpha\partial_r Q^2A_T - s^2A_Y t_\alpha T^\alpha\partial_r\omega\Big) \\
&\qquad -\omega\Big(\phi_\alpha Y^\alpha\partial_r s^2A_Y - \phi_\alpha T^\alpha\partial_r Q^2A_T - s^2A_Y\phi_\alpha T^\alpha\partial_r\omega\Big)\Big] \\
&= e^{-2f}\Big(\omega\partial_r s^2A_Y + \partial_r Q^2A_T + s^2A_Y\partial_r\omega - \omega\partial_r s^2A_Y\Big) \\
&= e^{-2f}\Big(\partial_r Q^2A_T + s^2A_Y\partial_r\omega\Big) \\
&= e^{-2f}E_r\ . \qquad (4.31)
\end{aligned}$$

Setting this in our integral Eq. (4.30), we find

$$\mathcal{Q} = -\frac{1}{2}\int \mathrm{d}\theta\, \frac{rs}{Q}E_r\Big|_{r_h}\ . \qquad (4.32)$$

As stated, we do not need to rely on the large-r limit, so there is no need for the hypersurface construction in charge as there was with the other two quantities. We have thus managed to express the conserved quantity of electric charge as an integral of our divergence-free electric field over θ, multiplied by a constant prefactor, just like we did for the other two parameters. Our fields certainly have a strong association with the parameters of uniqueness of black holes.

This problem, black hole uniqueness, has a property that affords us greater insight; the solution is known. We can compare what we derived to the known solution and read off our quantities and fields in terms of its parameters. We can also verify that the integrals for the conserved quantities reproduce those same parameters. Doing so may provide further understanding of our construction and illuminate the physical nature of these fields.

4.4 Known Solution Comparison

The Kerr-Newman solution is most commonly given in Boyer-Lindquist coordinates, and frequently appears as [19]

$$\begin{aligned} \mathrm{d}s^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)\mathrm{d}t^2 - \frac{2a\sin^2\theta(R^2+a^2-\Delta)}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi \\ &+\left[\frac{(R^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,\mathrm{d}\phi^2 + \Sigma\left(\frac{1}{\Delta}\,\mathrm{d}R^2 + \mathrm{d}\theta^2\right)\ , \end{aligned} \tag{4.33}$$

$$A_\alpha = -\frac{qR}{\Sigma}(t_\alpha - a\sin^2\theta\phi_\alpha)\ , \tag{4.34}$$

where

$$\Sigma = R^2 + a^2\cos^2\theta \quad \text{and} \quad \Delta = R^2 + a^2 + q^2 - 2mR\ . \tag{4.35}$$

The values of m, a, and q represent the total mass of the system, the angular momentum parameter of the system, and the total charge of the system respectively.

To compare with our metric form Eq. (3.4), we want to express everything in terms of its explicit coordinate, not using our scaled vectors (3.2). Doing so, we get

$$\begin{aligned} \mathrm{d}s^2 &= -c^2\,\mathrm{d}t^2 + 2\phi_t\,\mathrm{d}t\,\mathrm{d}\phi + s^2\,\mathrm{d}\phi^2 + e^{2f}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) \\ &= -(Q^2 - \omega^2 s^2)\,\mathrm{d}t^2 + 2s^2\omega\,\mathrm{d}t\,\mathrm{d}\phi + s^2\,\mathrm{d}\phi^2 + e^{2f}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)\ . \end{aligned} \tag{4.36}$$

We also need the vector potential in these coordinates,

$$\begin{aligned} A_\alpha &= t_\alpha t_\beta A^\beta + \phi_\alpha\phi_\beta A^\beta \\ &= t_\alpha(Q^2 T_\beta + s^2\omega Y_\beta)A^\beta + \phi_\alpha(s^2 Y_\beta)A^\beta \\ &= t_\alpha(Q^2 A_T + s^2\omega A_Y) + \phi_\alpha s^2 A_Y\ . \end{aligned} \tag{4.37}$$

However, there is a slight problem. The known solution uses a different kind of r coordinate than what we assumed, which is why it is denoted as R instead. We made the assumption in Eq. (3.7) that our spacetime was conformally mapped to flat space. This made our r isotropic, meaning the distance between a given change in r was the same no matter where in space you are. That is not the case for Boyer-Lindquist coordinates (in fact, it is not possible to construct a conformal map between the full three-dimensional space of Boyer-Lindquist coordinates and flat space [25]. Fortunately, since all of our quantities of interest are on the submanifold, we may avoid this problem. Just note that the map we derive will not work for the $g_{\phi\phi}$ term, due to an effect known as inertial frame-dragging). The physical length associated with a given change in r changes as you move away from the horizon. We therefore need a way to map from r to R.

Comparing the dθ terms in Eq. (4.36) to Eq. (4.33), we see that $\Sigma = e^{2f} r^2$. Now comparing the dr and dR terms, our transformation between the two must obey

$$\begin{aligned} \frac{\Sigma}{\Delta}\,\mathrm{d}R^2 &= e^{2f}\,\mathrm{d}r^2 \ , \qquad \text{or} \\ \frac{\mathrm{d}R}{\sqrt{\Delta}} &= \frac{\mathrm{d}r}{r} \ . \end{aligned} \tag{4.38}$$

We can solve this differential equation by direct integration. It's not immediately trivial, but a few layers of u-substitution shows that

$$\ln\!\left(R - m + \sqrt{\Delta}\right) = \ln(r) + D \ , \qquad \text{or} \qquad r = C\left(R - m + \sqrt{\Delta}\right) \ . \tag{4.39}$$

We want r to match up with R as they both go to infinity, which determines our choice of normalization. Expanding out $\sqrt{\Delta}$, we find

$$\sqrt{\Delta} = R\sqrt{1 - \frac{2m}{R} + \mathcal{O}\left(\frac{1}{R^2}\right)} \approx R\left(1 - \frac{m}{R}\right) = R - m \tag{4.40}$$

as $R \to \infty$. Thus, in this same limit,

$$r = C(R - m + R - m) = C(2R - 2m) \to 2CR \ , \tag{4.41}$$

so we choose $C = \frac{1}{2}$. Now solving for R in terms of r, we get

$$r = \frac{1}{2}\left(R - m + \sqrt{\Delta}\right) \tag{4.42}$$

$$R = r + m + \frac{m^2 - a^2 - q^2}{4r} \ . \tag{4.43}$$

This gives us our prescription for mapping between different quantities. Any time we see an R in an equation, we take it to be a function of r given by

$$R = r + m + \frac{R_h^2}{r} \ , \tag{4.44}$$

where we have defined

$$R_h = \frac{\sqrt{m^2 - a^2 - q^2}}{2} \ . \tag{4.45}$$

For instance, directly substituting the definition Eq. (4.44) into Eq. (4.42), we find

$$\sqrt{\Delta} = 2r + m - R = r\left(1 - \frac{R_h^2}{r^2}\right) \ . \tag{4.46}$$

Plugging this result into Eq. (4.38) shows us

$$\frac{\mathrm{d}R}{\mathrm{d}r} = 1 - \frac{R_h^2}{r^2} \ , \tag{4.47}$$

a result that will prove necessary when we go about performing our integrations. This provides a dictionary for mapping between our quantities and the known values, including the parameters for mass, angular momentum, and charge. Some of these quantities can be immediately read off of the comparison between the two metrics (4.33) and (4.36), such as f and s^2. Others require a few lines of simplification, like Q^2 and A_T. To assist with this, we introduce the function

$$\rho = (R^2 + a^2)^2 - \Delta a^2 \sin^2\theta \ . \tag{4.48}$$

In total, we find

$$f = \frac{1}{2}\ln\left(\frac{\Sigma}{R^2}\right) , \tag{4.49}$$

$$s^2 = \frac{\rho \sin^2\theta}{\Sigma} , \tag{4.50}$$

$$\omega = -\frac{a(2mR - q^2)}{\rho} , \tag{4.51}$$

$$Q^2 = \frac{\Delta\Sigma}{\rho} , \tag{4.52}$$

$$A_Y = \frac{aqR}{\rho} , \tag{4.53}$$

$$A_T = -\frac{qR(R^2 + a^2)}{\Delta\Sigma} . \tag{4.54}$$

They look nice in this form, but given that Σ, Δ, and ρ are all functions of R and all of our fields in the set have derivatives with respect to r which is itself a function of R, the integrations can start to look rather ugly very quickly. We start with the $\mathcal{Q}$ integral as it is the simplest to expand fully.

The charge integral becomes

$$\int \mathrm{d}\theta \, \frac{rs}{Q} E_r = r \int \mathrm{d}\theta \, \frac{s}{Q} \left(\partial_r Q^2 A_T + s^2 A_Y \partial_r \omega\right) . \tag{4.55}$$

By the product rule,

$$\partial_r \equiv \frac{\partial}{\partial r} = \frac{\partial R}{\partial r}\frac{\partial}{\partial R} = \left(1 - \frac{R_h^2}{r^2}\right)\frac{\partial}{\partial R} = \frac{\sqrt{\Delta}}{r}\partial_R . \tag{4.56}$$

To find $\partial_R \omega$, we first compute $\partial_R \rho$. It is

$$\begin{aligned}
\partial_R \rho &= \partial_R \left[\left(R^2 + a^2\right)^2 - R^2 a^2 \sin^2\theta + 2mRa^2 \sin^2\theta\right] \\
&= 2(R^2 + a^2)2R - 2Ra^2 \sin^2\theta + 2ma^2 \sin^2\theta \\
&= 2\left[2R(R^2 + a^2) - a^2 \sin^2\theta (R - m)\right] .
\end{aligned} \tag{4.57}$$

Then we may calculate

$$\partial_R \omega = -a\partial_R(2mR - q^2)\rho^{-1} = -2a\left(\frac{m}{\rho} - \frac{(2mR-q^2)}{\rho^2}\left[2R(R^2+a^2) - a^2\sin^2\theta(R-m)\right]\right) . \tag{4.58}$$

Plugging in our values to compute half of the charge integral, we find

$$\begin{aligned} r\int_0^\pi \mathrm{d}\theta\, \frac{s^3}{Q} A_Y \partial_r \omega = -2r\left(1 - \frac{R_h^2}{r^2}\right)\int \mathrm{d}\theta\, \frac{\rho\sin\theta}{\Sigma\sqrt{\Delta}} \frac{a^2 qR\sin^2\theta}{\Sigma}\left(\frac{m}{\rho} - \frac{(2mR-q^2)}{\rho^2}\right. \\ \left.\left[2R(R^2+a^2) - a^2\sin^2\theta(R-m)\right]\right) \\ = -2a^2qR\left[m\int_0^\pi \mathrm{d}\theta\, \frac{\sin^3\theta}{\Sigma^2} - 2R(2mR-q^2)(R^2+a^2)\int_0^\pi \frac{\sin^3\theta}{\rho\Sigma^2}\right. \\ \left. + a^2(2mR-q^2)(R-m)\int_0^\pi \mathrm{d}\theta\, \frac{\sin^5\theta}{\rho\Sigma^2}\right] . \end{aligned} \tag{4.59}$$

We now examine the first part of the charge integral, which necessitates finding $\partial_R Q^2 A_T$. Using our dictionary, we get

$$\partial_R Q^2 A_T = -q\frac{3R^2+a^2}{\rho} + q\frac{R(R^2+a^2)}{\rho^2}\partial_R\rho . \tag{4.60}$$

Substituting this expression in, we obtain

$$\begin{aligned} r\int \mathrm{d}\theta\, \frac{s}{Q}\partial_r Q^2 A_T = 4qR^2(R^2+a^2)^2\int_0^\pi \mathrm{d}\theta\, \frac{\sin\theta}{\rho\Sigma} - 2qa^2R(R^2+a^2)(R-m)\int_0^\pi \mathrm{d}\theta\, \frac{\sin^3\theta}{\rho\Sigma} \\ - q(3R^2+a^2)\int_0^\pi \mathrm{d}\theta\, \frac{\sin\theta}{\Sigma} . \end{aligned} \tag{4.61}$$

These integrals are not pretty and are something best done in the privacy of one's chamber. They employ virtually every elementary trick for integral evaluation, as well as a number of more advanced methods. We simply quote the answer here and leave the verification to the most dedicated reader. When all is said and done, we find

$$\int_0^\pi \mathrm{d}\theta\, \frac{rs}{Q}E_r = 2q , \tag{4.62}$$

meaning

$$\mathcal{Q} = -\frac{1}{2}\int_0^\pi \mathrm{d}\theta\, \frac{rs}{Q}E_r = -q . \tag{4.63}$$

This is exactly the charge parameter we had hoped to recover, albeit with a negative sign. It is feasible that more care should be taken with the signs in each step of the computation, and there is an error present in the calculation. It could be that the convention in [20] gives us the incorrect sign for the timelike normal to the sphere over which we integrate in this particular instance. In any case, we have recovered the usual conserved quantity up to a minus sign. Now comes the natural question, can we get any other conserved quantities?

We may answer that question in the affirmative, with some stipulations as seen below. The angular momentum integral expands out as

$$\int \mathrm{d}\theta\, r J_r = r \int \mathrm{d}\theta \left[\frac{s^3}{Q}\partial_r\omega + 4\frac{s}{Q}E_r s^2 A_Y\right] \; . \tag{4.64}$$

Plugging in the definitions from our dictionary and Eq. (4.58), the first term in this integration becomes

$$\begin{aligned} r \int \mathrm{d}\theta\, \frac{s^3}{Q}\partial_r\omega &= r \int \mathrm{d}\theta\, \frac{\sqrt{\Delta}}{r}\frac{s^3}{Q}\left\{-2a\left[\frac{m}{\rho} - \frac{(2mR-q^2)}{\rho^2}\Big(2R(R^2+a^2) - a^2\sin^2\theta(R-m)\Big)\right]\right\} \\ &= -2am\int_0^\pi \mathrm{d}\theta\,\frac{\rho\sin^2\theta}{\Sigma^2} + 4aR(2mR-q^2)(R^2+a^2)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^3\theta}{\Sigma^2} \\ &\qquad - 2a^3(2mR-q^2)(R-m)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^5\theta}{\Sigma^2} \; . \end{aligned} \tag{4.65}$$

These integrals are again unpleasant, but fear not, it gets much worse. Again inserting our definitions to find the second term of the angular momentum integration, it expands out to be

$$\begin{aligned} r\int \mathrm{d}\theta\, 4\frac{s}{Q}E_r s^2 A_Y &= 4aqR\frac{r}{\sqrt{\Delta}}\int \mathrm{d}\theta\,\frac{\rho\sin^2\theta}{\Sigma^2}\frac{\sqrt{\Delta}}{r}\Big(\partial_R Q^2 A_T + s^2 A_Y \partial_R\omega\Big) \\ &= -4aq^2R(3R^2+a^2)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^3\theta}{\Sigma^2} + 16aq^2R^3(R^2+a^2)^2\int_0^\pi \mathrm{d}\theta\,\frac{\sin^3\theta}{\rho\Sigma^2} \\ &\quad - 8a^3q^2R^2(R^2+a^2)(R-m)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^5\theta}{\rho\Sigma^2} - 8a^3q^2mR^2\int_0^\pi \mathrm{d}\theta\,\frac{\sin^5\theta}{\Sigma^3} \\ &\quad + 16a^3q^2R^3(2mR-q^2)(R^2+a^2)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^5\theta}{\rho\Sigma^3} \\ &\quad - 8a^5q^2R^2(2mR-q^2)(R-m)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^7\theta}{\rho\Sigma^3} \; . \end{aligned} \tag{4.66}$$

This is downright heinous. Some of these integrals involve rational expressions of trig functions to the eighth power without cancellation. Such integrals, while having an analytic solution, would take a well-practiced mathematician the better part of a day to compute. We just quote the overall result, not the values of individual integrals. In total, we may verify that the expression remarkably collapses into

$$\int_0^\pi \mathrm{d}\theta\, r J_r = 8am \ , \tag{4.67}$$

meaning our overall conserved quantity is

$$\mathcal{J} = -\frac{1}{8}\int_0^\pi \mathrm{d}\theta\, r J_r = -am \ . \tag{4.68}$$

Once again, this is precisely the typical angular momentum value of $J = am$, but negative. At this point, it seems likely that there is something built into the construction of the fields that gives us a negative sign. This warrants further investigation.

The natural next step is to examine the mass integral, see if we get the usual mass parameter, and if it too is negative. Pulling apart the mass integral, we get

$$\int \mathrm{d}\theta\, r M_r = r\int \left[s\partial_r Q - \frac{1}{2}\omega J_r - \frac{s}{Q}E_r Q^2 A_T - sQA_Y\partial_r s^2 A_Y\right] \ . \tag{4.69}$$

We take these terms one at a time.

The first term is fairly simple to expand, especially since we already found $\partial_R\omega$ in Eq. (4.57). It yields

$$\begin{aligned} r\int \mathrm{d}\theta\, s\partial_r Q &= \Delta\int \mathrm{d}\theta\, s\partial_R \left(\frac{\Sigma}{\rho}\right)^{1/2} \\ &= \Delta R\int_0^\pi \mathrm{d}\theta\,\frac{\sin\theta}{\Sigma} - 2R(R^2+a^2)\Delta\int_0^\pi \mathrm{d}\theta\,\frac{\sin\theta}{\rho} + a^2(R-m)\Delta\int_0^\pi \mathrm{d}\theta\,\frac{\sin^3\theta}{\rho} \ . \end{aligned} \tag{4.70}$$

The next term is a bit more challenging, but fortunately we have already found $\int \mathrm{d}\theta\, r J_r$, so we just need to multiply it by $\omega = -\dfrac{a(2mR-q^2)}{\rho}$ and divide by two. Expanding out the combined functions and putting our integrals in the simplest terms available, we get

$$\begin{aligned}\frac{1}{2}\int \mathrm{d}\theta\,\omega r J_r = a^2(2mR-q^2)\Bigg\{ & m\int_0^\pi \mathrm{d}\theta\,\frac{\sin^2\theta}{\Sigma^2} - 2R(2mR-q^2)(R^2+a^2)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^3\theta}{\rho\Sigma^2} \\ & + a^2(2mR-q^2)(R-m)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^5\theta}{\rho\Sigma^2} \\ & + 2q^2R(3R^2+a^2)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^3\theta}{\rho\Sigma^2} - 8q^2R^3(R^2+a^2)^2\int_0^\pi \mathrm{d}\theta\,\frac{\sin^3\theta}{\rho^2\Sigma^2} \\ & + 4a^2q^2R^2(R^2+a^2)(R-m)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^5\theta}{\rho^2\Sigma^2} + 4a^2q^2mR^2\int_0^\pi \mathrm{d}\theta\,\frac{\sin^5\theta}{\rho\Sigma^3} \\ & - 8a^2q^2R^3(2mR-q^2)(R^2+a^2)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^5\theta}{\rho^2\Sigma^3} \\ & + 4a^4q^2R^2(2mR-q^2)(R-m)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^7\theta}{\rho^2\Sigma^3}\Bigg\}\ . \end{aligned} \tag{4.71}$$

This integral is, somehow, even more grotesque than any previous. We defer the precise determination of this quantity until everything is collected and give the answer all at once.

The third term of the complete mass integral is just the substitution of a number of quantities we have already found, giving us

$$\begin{aligned}\int \mathrm{d}\theta\,\frac{rs}{Q}E_rQ^2A_T &= r\int_0^\pi \mathrm{d}\theta\,\frac{s}{Q}Q^2A_T\frac{\sqrt{\Delta}}{r}\left(\partial_R Q^2A_T + s^2A_Y\partial_R\omega\right) \\ &= q^2R(R^2+a^2)(3R^2+a^2)\int_0^\pi \mathrm{d}\theta\,\frac{\sin\theta}{\rho\Sigma} - 4q^2R^3(R^2+a^2)^3\int_0^\pi \mathrm{d}\theta\,\frac{\sin\theta}{\rho^2\Sigma} \\ &\quad + 2q^2a^2R^2(R^2+a^2)^2(R-m)\int_0^\pi \mathrm{d}\theta\,\frac{\sin^3\theta}{\rho^2\Sigma}\ . \end{aligned} \tag{4.72}$$

This expression may look inordinately simple compared to the previous expression, but it is still quite nontrivial. We again do not evaluate it in isolation, but wait until we have all quantities expanded and quote the result there.

To calculate the last integral, we need the derivative of a quantity we have not yet taken,

$$s^2A_Y = aqR\frac{\sin^2\theta}{\Sigma}\ . \tag{4.73}$$

Most of these do not depend upon R, so directly computing the derivative of what does depend on R gives us

$$\partial_R \frac{R}{\Sigma} = \frac{1}{\Sigma} - \frac{R}{\Sigma^2}\partial_R \Sigma = \frac{1}{\Sigma} - \frac{2R^2}{\Sigma^2}\ . \tag{4.74}$$

We may now plug this (and the other relevant values) into the final term of integration, which becomes

$$\int \mathrm{d}\theta\, rsQA_Y\partial_r s^2 A_Y = a^2q^2R\Delta \int_0^\pi \mathrm{d}\theta\, \frac{\sin^2\theta}{\rho\Sigma} - 2a^2q^2R^3\Delta \int_0^\pi \mathrm{d}\theta\, \frac{\sin^3\theta}{\rho\Sigma^2}\ . \tag{4.75}$$

With all of our individual terms expanded and computed, we can quote the result of the mass integral. Some dedicated time shows

$$\int_0^\pi \mathrm{d}\theta\, rM_r = 2m. \tag{4.76}$$

Thus we obtain our final conserved quantity,

$$\mathcal{M} = -\frac{1}{2}\int_0^\pi \mathrm{d}\theta\, rM_r = -m\ . \tag{4.77}$$

Reassuringly, we get our desired conserved quantity, and it follows the same pattern that the other values have formed. It is negative, which suggests that if we find a discrepancy in convention the solution should propagate through to fix all three to be their positive values.

Before moving on, we should stop, zoom out, and examine what we have done. Simply by defining a few divergence-free combinations of fields and vectors using some clever manipulations of the Einstein-Maxwell equations, we have been able to recover the exact conserved quantities (up to a minus sign) of a black hole and its associated spacetime. This adumbrates the significance of these seemingly-arbitrary combinations. Something about putting them together in the way that we have provides a deep connection to the real physical system we are attempting to describe. All of this was grown from a specific assumption of symmetry, which itself is inherited from the stability of black holes.

Chapter 5

Conclusion

As we see, by leveraging the symmetries of an ideal black hole system, we are able to recast the Einstein-Maxwell equations, 10 coupled nonlinear partial differential equations plus four linear coupled partial differential equations, as a set of five divergence-free equations in two variables. This is a substantial decrease in complexity. Furthermore, these five equations are closely related to the conserved quantities of the system.

We see many various connections between seemingly disconnected values throughout this derivation, such as the way that the vector potential vanishes if confined to the two-manifold. This enforces the way that this approach emphasizes the physics over the mathematics, demonstrating that there are likely more fundamental connections that are still to be made in the realm of black hole physics.

We have not been able to recover the metric from this formulation as of yet, but we believe it is possible. This is an excellent place for further research: minimal information has been lost in the conversion of the Einstein-Maxwell equations into our set (3.65), so the metric should be able to be completely reconstructed. This does not mean this formulation is without drawbacks.

As shown above, this method actually produces the negative conserved quantities of the system. While that is just an odd quirk for the values computed, there is no guarantee that further calculation will produce the correct quantities. They could have an additional prefactor that throws off the result. Additionally, this construction only uses half of the Maxwell equations, meaning it is possible that there is some piece of information that we have dropped from the field construction. Finally, this entire analysis relies entirely upon the stationary and axisymmetric structure of the system. While it has been proven that a perturbed slowly-rotating black hole will settle down to the Kerr-Newman solution [26], it is not yet generally shown that any perturbation to a given four-dimensional spacetime will eventually converge back to a Kerr-Newman black hole. This analysis would not work for such a case, because the symmetry used as the foundation of this construction would be broken.

Despite these limitations, this approach potentially has the power to prove the uniqueness of the Kerr-Newman metric as the only stationary, axisymmetric, electrovac, asymptotically flat solution to a black hole with both angular momentum and electric charge, and may be able to determine such a metric for the case where an object has nonzero magnetic charge with minimal modification. Further efforts could bear more fruit in the physical implications of mathematical results. We hope that such investigation will yield further insight into the fundamental nature of black hole spacetime.

Appendix A

Scaled Vectors

We want to define a pair of vectors orthogonal to each other and also orthogonal to the submanifold. We immediately see that both ϕ^α and t^α are orthogonal to the submanifold, but they are not orthogonal to each other. Furthermore, they do not necessarily have unit length. We first normalize the ϕ^α vector, but we want the resulting vector Y^α to be normalized with respect to the original vector ϕ^α, not necessarily to itself. We do this because we really care about the Killing vector ϕ^α, and would like to find something that is dual to this Killing vector instead of something dual to itself. Thus we must find a such that

$$\phi^\alpha Y_\alpha = a\phi^\alpha\phi_\alpha = as^2 = 1 \ . \tag{A.1}$$

This shows us $a = \dfrac{1}{s^2}$, so

$$Y^\alpha = \frac{1}{s^2}\phi^\alpha \ . \tag{A.2}$$

Now we construct a vector that is orthogonal to Y^α using the Gram-Schmidt process. We define $\phi_t = \phi_\alpha t^\alpha$, the inner product of our two initial vectors. The projection of t^α onto ϕ^α is therefore

$$\frac{\phi_\alpha t^\alpha}{\phi_\alpha\phi^\alpha}\phi^\alpha = \frac{\phi_t}{s^2}\phi^\alpha = \phi_t Y^\alpha \ . \tag{A.3}$$

Subtracting that projection from t^α, we have

$$t^\alpha - \phi_t Y^\alpha \ , \tag{A.4}$$

which still needs to be normalized. Again, we want this new vector T^α to be normalized with respect to the inner product with the original vector t^α. This inner product should take the value -1 due to the signature of the metric we are using, as discussed in Sec. 3.1.1. We thus need to find b such that

$$t^\alpha T_\alpha = bt^\alpha(t_\alpha - \phi_t Y_\alpha) = -1 \ . \tag{A.5}$$

Expanding this out, we find

$$b(t^\alpha t_\alpha - \phi_t t^\alpha Y_\alpha) = b\left(-c^2 - \frac{\phi_t^2}{s^2}\right) = -1 \ , \tag{A.6}$$

so $b = \left(c^2 + \frac{\phi_t^2}{s^2}\right)^{-1}$. We give b the alternate identifier $\frac{1}{Q^2}$, so $Q^2 = c^2 + \frac{\phi_t^2}{s^2}$. Ergo

$$T^\alpha = \frac{1}{Q^2}(t^\alpha - \phi_t Y^\alpha) \ . \tag{A.7}$$

We may on occasion write this not with the Y^α vector, but with the ϕ^α vector, introducing an extra factor of s^2:

$$T^\alpha = \frac{1}{Q^2}\left(t^\alpha - \frac{\phi_t}{s^2}\phi_\alpha\right) = \frac{1}{Q^2}(t^\alpha - \omega\phi_\alpha) \ . \tag{A.8}$$

We have constructed these vectors to be orthogonal, but it is a nice sanity check to ensure this bears out in explicit computation. We find

$$Y^\alpha T_\alpha = \frac{1}{s^2 Q^2}\phi^\alpha(t_\alpha - \phi_t Y_\alpha) = \frac{1}{s^2 Q^2}(\phi_t - \phi_t) = 0 \ , \tag{A.9}$$

as desired.

Since these vectors are orthogonal, we can use them as part of a basis for the full four-dimensional spacetime, meaning our metric looks like

$$g_{\alpha\beta} = \sigma_{\alpha\beta} + AY_\alpha Y_\beta + BT_\alpha T_\beta \ , \tag{A.10}$$

where $\sigma_{\alpha\beta}$ spans the last two orthogonal directions. We don't yet know the values of A and B, but there are several ways to compute them. One way is by leveraging this orthogonality that we have built and contracting the equation with our upper-index vectors. For Y^β, we find

$$g_{\alpha\beta}Y^\beta = \sigma_{\alpha\beta}Y^\beta + AY_\alpha Y_\beta Y^\beta + BT_\alpha T_\beta Y^\beta = AY_\alpha \frac{1}{s^2} \ . \tag{A.11}$$

And, since $g_{\alpha\beta}Y^\beta = Y_\alpha$, it must follow that $A = s^2$.

Doing the same thing for T^β, we get

$$g_{\alpha\beta}T^\beta = \sigma_{\alpha\beta}T^\beta + AY_\alpha Y_\beta T^\beta + BT_\alpha T_\beta T^\beta = -BT_\alpha \frac{1}{Q^2} \ . \tag{A.12}$$

Again, $g_{\alpha\beta}T^\beta = T_\alpha$, so $B = -Q^2$. This gives us the full form of our metric,

$$g_{\alpha\beta} = \sigma_{\alpha\beta} + s^2 Y_\alpha Y_\beta - Q^2 T_\alpha T_\beta \ . \tag{A.13}$$

Appendix B

Finding the Connection

The metric for our case is Eq. (3.4),

$$g_{\alpha\beta} = \sigma_{\alpha\beta} + s^2 Y_\alpha Y_\beta - Q^2 T_\alpha T_\beta \ . \tag{B.1}$$

Plugging this in to the definition of the connection, we get

$$\begin{aligned}
\Gamma^\alpha{}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\delta}(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}) \\
&= \frac{1}{2}(\sigma^{\alpha\delta} + s^2 Y^\alpha Y^\delta - Q^2 T^\alpha T^\delta)\Big[\partial_\beta(\sigma_{\gamma\delta} + s^2 Y_\gamma Y_\delta - Q^2 T_\gamma T_\delta) + \partial_\gamma(\sigma_{\beta\delta} + s^2 Y_\beta Y_\delta - Q^2 T_\beta T_\delta) \\
&\qquad -\partial_\delta(\sigma_{\beta\gamma} + s^2 Y_\beta Y_\gamma - Q^2 T_\beta T_\gamma)\Big] \\
&= \frac{1}{2}(\sigma^{\alpha\delta} + s^2 Y^\alpha Y^\delta - Q^2 T^\alpha T^\delta)\Big[(\partial_\beta \sigma_{\gamma\delta} + \partial_\gamma \sigma_{\beta\delta} - \partial_\delta \sigma_{\beta\gamma}) + \partial_\beta(s^2 Y_\gamma Y_\delta - Q^2 T_\gamma T_\delta) \\
&\qquad +\partial_\gamma(s^2 Y_\beta Y_\delta - Q^2 T_\beta T_\delta) - \partial_\delta(s^2 Y_\beta Y_\gamma - Q^2 T_\beta T_\gamma)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sigma^{\alpha\delta}(\partial_\beta\sigma_{\gamma\delta} + \partial_\gamma\sigma_{\beta\delta} - \partial_\delta\sigma_{\beta\gamma}) \\
&\quad + \frac{1}{2}\sigma^{\alpha\delta}\left[\partial_\beta(s^2Y_\gamma Y_\delta - Q^2T_\gamma T_\delta) + \partial_\gamma(s^2Y_\beta Y_\delta - Q^2T_\beta T_\delta) - \partial_\delta(s^2Y_\beta Y_\gamma - Q^2T_\beta T_\gamma)\right] \\
&\quad + \frac{1}{2}s^2Y^\alpha Y^\delta\left[(\partial_\beta\sigma_{\gamma\delta} + \partial_\gamma\sigma_{\beta\delta} - \partial_\delta\sigma_{\beta\gamma}) + \partial_\beta(s^2Y_\gamma Y_\delta - Q^2T_\gamma T_\delta)\right. \\
&\qquad\qquad\qquad \left. + \partial_\gamma(s^2Y_\beta Y_\delta - Q^2T_\beta T_\delta) - \partial_\delta(s^2Y_\beta Y_\gamma - Q^2T_\beta T_\gamma)\right] \\
&\quad - \frac{1}{2}Q^2T^\alpha T^\delta\left[(\partial_\beta\sigma_{\gamma\delta} + \partial_\gamma\sigma_{\beta\delta} - \partial_\delta\sigma_{\beta\gamma}) + \partial_\beta(s^2Y_\gamma Y_\delta - Q^2T_\gamma T_\delta)\right. \\
&\qquad\qquad\qquad \left. + \partial_\gamma(s^2Y_\beta Y_\delta - Q^2T_\beta T_\delta) - \partial_\delta(s^2Y_\beta Y_\gamma - Q^2T_\beta T_\gamma)\right] \; .
\end{aligned}
\tag{B.2}
$$

We take these terms one at a time.

The first term is the definition of the connection, but using the metric of the submanifold instead of the full metric. We therefore call it ${}^{(2)}\Gamma^\alpha{}_{\beta\gamma}$, the connection of the submanifold.

We now work on the second term. Using the derivatives of our scaled vectors Eq. (3.16) and Eq. (3.17), we can crunch through the calculation. Since both of the scaled vectors are orthogonal to the submanifold, any contractions with the scaled vectors and the submanifold metric will go to zero, considerably simplifying the full expression. Applying the product rule, we have

$$
\begin{aligned}
&\frac{1}{2}\sigma^{\alpha\delta}\left[\partial_\beta(s^2Y_\gamma Y_\delta - Q^2T_\gamma T_\delta) + \partial_\gamma(s^2Y_\beta Y_\delta - Q^2T_\beta T_\delta) - \partial_\delta(s^2Y_\beta Y_\gamma - Q^2T_\beta T_\gamma)\right] \\
&\quad = \frac{1}{2}\sigma^{\alpha\delta}(s^2Y_\gamma\partial_\beta Y_\delta + s^2Y_\delta\partial_\beta Y_\gamma + Y_\gamma Y_\delta\partial_\beta s^2 - T_\gamma T_\delta\partial_\beta Q^2 + s^2Y_\beta\partial_\gamma Y_\delta + s^2Y_\delta\partial_\gamma Y_\beta \\
&\qquad\qquad + Y_\beta Y_\delta\partial_\gamma s^2 - T_\beta T_\delta\partial_\gamma Q^2 - s^2Y_\beta\partial_\delta Y_\gamma - s^2Y_\gamma\partial_\delta Y_\beta - Y_\beta Y_\gamma\partial_\delta s^2 + T_\beta T_\gamma\partial_\delta Q^2) \\
&\quad = \frac{1}{2}\sigma^{\alpha\delta}(s^2Y_\gamma\partial_\beta Y_\delta + s^2Y_\beta\partial_\gamma Y_\delta - s^2Y_\beta\partial_\delta Y_\gamma - s^2Y_\gamma\partial_\delta Y_\beta - Y_\beta Y_\gamma\partial_\delta s^2 + T_\beta T_\gamma\partial_\delta Q^2) \\
&\quad = \frac{1}{2}\sigma^{\alpha\delta}(-s^2Y_\gamma T_\delta\partial_\beta\omega - s^2Y_\beta T_\delta\partial_\gamma\omega + s^2Y_\beta T_\gamma\partial_\delta\omega + s^2Y_\gamma T_\beta\partial_\delta\omega - Y_\beta Y_\gamma\partial_\delta s^2 + T_\beta T_\gamma\partial_\delta Q^2) \\
&\quad = \frac{1}{2}\sigma^{\alpha\delta}\left(s^2(Y_\beta T_\gamma + Y_\gamma T_\beta)\partial_\delta\omega - Y_\beta Y_\gamma\partial_\delta s^2 + T_\beta T_\gamma\partial_\delta Q^2\right) \\
&\quad = \frac{1}{2}\sigma^{\alpha\delta}\left(s^2(Y_\beta T_\gamma + Y_\gamma T_\beta)\Delta_\delta\omega - Y_\beta Y_\gamma\Delta_\delta s^2 + T_\beta T_\gamma\Delta_\delta Q^2\right) \; .
\end{aligned}
\tag{B.3}
$$

Moving from the penultimate to the final line, we recognize that the derivatives are only on scalar functions and we are contracting with the submanifold, so the partial derivative can be replaced with the submanifold covariant derivative.

For the next term, we break up the scaled vector $Y^\delta = \frac{1}{s^2}\phi^\delta$, which allows us to pass the Killing vector ϕ^δ inside the partial derivatives. As ϕ^δ is orthogonal to the submanifold, contraction with the submanifold metric again gives zero. Using the inner product values of our vectors, we obtain

$$\begin{aligned}
&\frac{1}{2}s^2 Y^\alpha Y^\delta \Big[(\partial_\beta \sigma_{\gamma\delta} + \partial_\gamma \sigma_{\beta\delta} - \partial_\delta \sigma_{\beta\gamma}) + \partial_\beta(s^2 Y_\gamma Y_\delta - Q^2 T_\gamma T_\delta) + \partial_\gamma(s^2 Y_\beta Y_\delta - Q^2 T_\beta T_\delta) \\
&\qquad\qquad\qquad - \partial_\delta(s^2 Y_\beta Y_\gamma - Q^2 T_\beta T_\gamma)\Big] \\
&= \frac{1}{2}Y^\alpha \Big[(\partial_\beta \phi^\delta \sigma_{\gamma\delta} + \partial_\gamma \phi^\delta \sigma_{\beta\delta} - \phi^\delta \partial_\delta \sigma_{\beta\gamma}) + \partial_\beta(s^2 Y_\gamma \phi^\delta Y_\delta - Q^2 T_\gamma \phi^\delta T_\delta) \\
&\qquad\qquad + \partial_\gamma(s^2 Y_\beta \phi^\delta Y_\delta - Q^2 T_\beta \phi^\delta T_\delta) - \phi^\delta \partial_\delta(s^2 Y_\beta Y_\gamma - Q^2 T_\beta T_\gamma)\Big] \\
&= \frac{1}{2}Y^\alpha \Big[-\phi^\delta \partial_\delta \sigma_{\beta\gamma} + \partial_\beta(s^2 Y_\gamma) + \partial_\gamma(s^2 Y_\beta) - \phi^\delta \partial_\delta(s^2 Y_\beta Y_\gamma - Q^2 T_\beta T_\gamma)\Big] \\
&= \frac{1}{2}Y^\alpha \Big[\partial_\beta(s^2 Y_\gamma) + \partial_\gamma(s^2 Y_\beta) - \phi^\delta \partial_\delta g_{\beta\gamma}\Big] \ . \qquad \text{(B.4)}
\end{aligned}$$

Since the metric has no ϕ dependence and the only nonzero component of ϕ^α is ϕ, the derivative of the metric contracted with the Killing vector is zero, so this term becomes

$$\frac{1}{2}Y^\alpha \Big[\partial_\beta(s^2 Y_\gamma) + \partial_\gamma(s^2 Y_\beta)\Big] \ . \qquad \text{(B.5)}$$

The final term does not employ any new techniques, but is a somewhat lengthy computation given the form of the T^α vector. Going through everything, we find

$$
\begin{aligned}
&\frac{1}{2}Q^2T^\alpha T^\delta\Big[(\partial_\beta\sigma_{\gamma\delta}+\partial_\gamma\sigma_{\beta\delta}-\partial_\delta\sigma_{\beta\gamma})+\partial_\beta(s^2Y_\gamma Y_\delta-Q^2T_\gamma T_\delta)\\
&\qquad\qquad+\partial_\gamma(s^2Y_\beta Y_\delta-Q^2T_\beta T_\delta)-\partial_\delta(s^2Y_\beta Y_\gamma-Q^2T_\beta T_\gamma)\Big]\\
&=\frac{1}{2}T^\alpha(t^\delta-\omega\phi^\delta)\Big[\partial_\beta\sigma_{\gamma\delta}+\partial_\gamma\sigma_{\beta\delta}-\partial_\delta\sigma_{\beta\gamma}+s^2Y_\gamma\partial_\beta Y_\delta+s^2Y_\delta\partial_\beta Y_\gamma+Y_\gamma Y_\delta\partial_\beta s^2\\
&\quad-T_\gamma T_\delta\partial_\beta Q^2+s^2Y_\beta\partial_\gamma Y_\delta+s^2Y_\delta\partial_\gamma Y_\beta+Y_\beta Y_\delta\partial_\gamma s^2-T_\beta T_\delta\partial_\gamma Q^2-\partial_\delta(s^2Y_\beta Y_\gamma-Q^2T_\beta T_\gamma)\Big]\\
&=\frac{1}{2}T^\alpha\Big[\partial_\beta t^\delta\sigma_{\gamma\delta}+\partial_\gamma t^\delta\sigma_{\beta\delta}-t^\delta\partial_\delta\sigma_{\beta\gamma}-s^2Y_\gamma t^\delta T_\delta\partial_\beta\omega+s^2Y_\delta t^\delta\partial_\beta T_\gamma+Y_\gamma t^\delta Y_\delta\partial_\beta s^2-T_\gamma t^\delta T_\delta\partial_\beta Q^2\\
&\qquad-s^2Y_\beta t^\delta T_\delta\partial_\gamma\omega+s^2t^\delta Y_\delta\partial_\gamma Y_\beta+Y_\beta t^\delta Y_\delta\partial_\gamma s^2-T_\beta t^\delta T_\delta\partial_\gamma Q^2-t^\delta\partial_\delta(s^2Y_\beta Y_\gamma-Q^2T_\beta T_\gamma)\Big]\\
&\quad-\frac{1}{2}\omega T^\alpha\Big[\partial_\beta\phi^\delta\sigma_{\gamma\delta}+\partial_\gamma\phi^\delta\sigma_{\beta\delta}-\phi^\delta\partial_\delta\sigma_{\beta\gamma}-s^2Y_\gamma\phi^\delta T_\delta\partial_\beta\omega+s^2Y_\delta\phi^\delta\partial_\beta T_\gamma+Y_\gamma\phi^\delta Y_\delta\partial_\beta s^2\\
&\qquad-T_\gamma\phi^\delta T_\delta\partial_\beta Q^2-s^2Y_\beta\phi^\delta T_\delta\partial_\gamma\omega+s^2\phi^\delta Y_\delta\partial_\gamma Y_\beta+Y_\beta\phi^\delta Y_\delta\partial_\gamma s^2-T_\beta\phi^\delta T_\delta\partial_\gamma Q^2\\
&\qquad\quad-\phi^\delta\partial_\delta(s^2Y_\beta Y_\gamma-Q^2T_\beta T_\gamma)\Big]\\
&=\frac{1}{2}T^\alpha\Big(s^2Y_\gamma\partial_\beta\omega-s^2\omega T_\gamma\partial_\beta\omega+\omega Y_\gamma\partial_\beta s^2+T_\gamma\partial_\beta Q^2+s^2Y_\beta\partial_\gamma Q^2+s^2Y_\beta\partial_\gamma\omega\\
&\qquad-s^2\omega T_\beta\partial_\gamma\omega+\omega Y_\beta\partial_\gamma s^2+T_\beta\partial_\gamma Q^2-t^\delta\partial_\delta g_{\beta\gamma}+s^2\omega T_\gamma\partial_\beta\omega-\omega Y_\gamma\partial_\beta s^2\\
&\qquad+s^2\omega T_\beta\partial_\gamma\omega-\omega Y_\beta\partial_\gamma s^2+\omega\phi^\delta\partial_\delta g_{\beta\gamma}\Big)\\
&=\frac{1}{2}T^\alpha\Big(\partial_\beta(Q^2T_\gamma)+\partial_\gamma(Q^2T_\beta)+s^2(Y_\beta\Delta_\gamma\omega+Y_\gamma\Delta_\beta\omega)\Big)\ .
\end{aligned}
\tag{B.6}
$$

For the last step, we are able to go from $\partial_\gamma\omega$ to $\Delta_\gamma\omega$ because ω is a scalar function of only r and θ.

Pulling all of our quantities together, we get

$$
\begin{aligned}
\Gamma^\alpha{}_{\beta\gamma}={}&^{(2)}\Gamma^\alpha{}_{\beta\gamma}+\frac{1}{2}\sigma^{\alpha\delta}\Big[s^2(Y_\beta T_\gamma+Y_\gamma T_\beta)\Delta_\delta\omega-Y_\beta Y_\gamma\Delta_\delta s^2+T_\beta T_\gamma\Delta_\delta Q^2\Big]\\
&+\frac{1}{2}Y^\alpha\Big[\partial_\beta(s^2Y_\gamma)+\partial_\gamma(s^2Y_\beta)\Big]\\
&-\frac{1}{2}T^\alpha\Big[\partial_\beta(Q^2T_\gamma)+\partial_\gamma(Q^2T_\beta)+s^2\left(Y_\beta\Delta_\gamma\omega+Y_\gamma\Delta_\beta\omega\right)\Big]\ .
\end{aligned}
\tag{B.7}
$$

Appendix C

Derivation of Ricci Tensor

We wish to compute the Ricci tensor for our decomposition which, as a reminder, is given by Eq. (3.10):

$$R_{\alpha\beta} = \partial_\gamma \Gamma^\gamma{}_{\alpha\beta} - \partial_\beta \Gamma^\gamma{}_{\gamma\alpha} + \Gamma^\delta{}_{\alpha\beta}\Gamma^\gamma{}_{\gamma\delta} - \Gamma^\delta{}_{\alpha\gamma}\Gamma^\gamma{}_{\beta\delta} \ . \tag{C.1}$$

We also define several helper functions for the sake of brevity,

$$\Xi_{\alpha\beta\gamma} = s^2(Y_\alpha T_\beta + Y_\beta T_\alpha)\Delta_\gamma \omega - Y_\alpha Y_\beta \Delta_\gamma s^2 + T_\alpha T_\beta \Delta_\gamma Q^2 \ , \tag{C.2}$$

$$\lambda_{\alpha\beta} = \partial_\alpha(s^2 Y_\beta) + \partial_\beta(s^2 Y_\alpha) \ , \tag{C.3}$$

$$\tau_{\alpha\beta} = \partial_\alpha(Q^2 T_\beta) + \partial_\beta(Q^2 T_\alpha) + s^2(Y_\alpha \Delta_\beta \omega + Y_\beta \Delta_\alpha \omega) \ . \tag{C.4}$$

In this shorthand, the connection takes the form

$$\Gamma^\alpha{}_{\beta\gamma} = {}^{(2)}\Gamma^\alpha{}_{\beta\gamma} + \frac{1}{2}\sigma^{\alpha\delta}\Xi_{\beta\gamma\delta} + \frac{1}{2}Y^\alpha \lambda_{\beta\gamma} - \frac{1}{2}T^\alpha \tau_{\beta\gamma} \ . \tag{C.5}$$

The first term we consider is $\partial_\beta \Gamma^\gamma{}_{\gamma\alpha}$. Using the party trick from Eq. (2.11), we find

$$\Gamma^\gamma{}_{\gamma\alpha} = \frac{1}{\sqrt{\sigma}}\partial_\alpha\sqrt{\sigma} + \frac{1}{sQ}\partial_\alpha sQ = {}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} + \frac{1}{sQ}\partial_\alpha sQ \ . \tag{C.6}$$

Thus this full term is

$$\partial_\beta \Gamma^\gamma{}_{\gamma\alpha} = \partial_\beta\left({}^{(2)}\Gamma^\gamma{}_{\gamma\alpha} + \frac{1}{sQ}\partial_\alpha sQ\right) \ . \tag{C.7}$$

The next term is also fairly straightforward. Writing it out, we get

$$\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\gamma}{}_{\gamma\delta} = \left({}^{(2)}\Gamma^{\delta}{}_{\alpha\beta} + \frac{1}{2}\sigma^{\delta\epsilon}\Xi_{\alpha\beta\epsilon} + \frac{1}{2}Y^{\delta}\lambda_{\alpha\beta} - \frac{1}{2}T^{\delta}\tau_{\alpha\beta}\right)\left({}^{(2)}\Gamma^{\gamma}{}_{\gamma\delta} + \frac{1}{sQ}\partial_{\delta}sQ\right) \ . \tag{C.8}$$

In the second term, the δ is restricted to r and θ, since sQ is only a function of r and θ. Thus the Y^{δ} and T^{δ} contractions go to zero, so this term is equal to

$$\begin{aligned}\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\gamma}{}_{\gamma\delta} &= {}^{(2)}\Gamma^{\delta}{}_{\alpha\beta}{}^{(2)}\Gamma^{\gamma}{}_{\gamma\delta} + \frac{1}{sQ}{}^{(2)}\Gamma^{\delta}{}_{\alpha\beta}\partial_{\delta}sQ + \frac{1}{2}{}^{(2)}\Gamma^{\gamma}{}_{\gamma\delta}\sigma^{\delta\epsilon}\Xi_{\alpha\beta\epsilon} + \frac{1}{2sQ}\sigma^{\delta\epsilon}\Xi_{\alpha\beta\epsilon}\partial_{\delta}sQ \\ &= {}^{(2)}\Gamma^{\delta}{}_{\alpha\beta}{}^{(2)}\Gamma^{\gamma}{}_{\gamma\delta} + \frac{1}{sQ}{}^{(2)}\Gamma^{\delta}{}_{\alpha\beta}\partial_{\delta}sQ + \frac{1}{2}{}^{(2)}\Gamma^{\gamma}{}_{\gamma\delta}\Xi_{\alpha\beta}{}^{\delta} + \frac{1}{2sQ}\Xi_{\alpha\beta}{}^{\delta}\partial_{\delta}sQ \ .\end{aligned} \tag{C.9}$$

The next two terms are not quite so friendly, but are merely tedious. We start with the first term in the Ricci tensor, $\partial_{\gamma}\Gamma^{\gamma}{}_{\alpha\beta}$. Expanding this, it becomes

$$\partial_{\gamma}\Gamma^{\gamma}{}_{\alpha\beta} = \partial_{\gamma}{}^{(2)}\Gamma^{\gamma}{}_{\alpha\beta} + \frac{1}{2}\partial_{\gamma}\sigma^{\gamma\delta}\Xi_{\alpha\beta\delta} + \frac{1}{2}\partial_{\gamma}Y^{\gamma}\lambda_{\alpha\beta} - \frac{1}{2}\partial_{\gamma}T^{\gamma}\tau_{\alpha\beta} \ . \tag{C.10}$$

We take these terms one at a time, starting from the last one. Before doing this, we recall the form of the derivatives of the scaled vectors from Eqs. (3.48) and (3.49). With the derivative contracted with the vector, these become

$$\partial_{\gamma}Y^{\gamma} = \frac{2}{s}Y^{\gamma}\Delta_{\gamma}s = 0 \ , \tag{C.11}$$

$$\partial_{\gamma}T^{\gamma} = -\frac{s^2}{Q^2}Y^{\gamma}\Delta_{\gamma}\omega - \frac{2}{Q}T^{\gamma}\Delta_{\gamma}Q = 0 \ , \tag{C.12}$$

because the scaled vectors are orthogonal to the submanifold. Now returning to the last term in (C.10), we find

$$\begin{aligned}\partial_{\gamma}T^{\gamma}\tau_{\alpha\beta} &= T^{\gamma}\partial_{\gamma}\tau_{\alpha\beta} = T^{\gamma}\partial_{\gamma}\left[\partial_{\alpha}(Q^2T_{\beta}) + \partial_{\beta}(Q^2T_{\alpha}) + s^2(Y_{\alpha}\Delta_{\beta}\omega + Y_{\beta}\Delta_{\alpha}\omega\right] \\ &= T^{\gamma}\left[T_{\beta}\partial_{\gamma}\Delta_{\alpha}Q^2 + T_{\alpha}\partial_{\gamma}\Delta_{\beta}Q^2 + (\Delta_{\gamma}s^2)(Y_{\alpha}\Delta_{\beta}\omega + Y_{\beta}\Delta_{\alpha}\omega) + s^2\partial_{\gamma}(Y_{\alpha}\Delta_{\beta}\omega + Y_{\beta}\Delta_{\alpha}\omega)\right] \\ &= s^2T^{\gamma}\left[(\partial_{\gamma}Y_{\alpha})\Delta_{\beta}\omega + Y_{\alpha}\partial_{\gamma}\Delta_{\beta}\omega + (\partial_{\gamma}Y_{\beta})\Delta_{\alpha}\omega + Y_{\beta}\partial_{\gamma}\Delta_{\alpha}\omega\right] \\ &= s^2T^{\gamma}\left[-T_{\alpha}(\Delta_{\gamma})\Delta_{\beta}\omega - T_{\beta})\Delta_{\gamma}\omega)\Delta_{\alpha}\omega\right] \\ &= 0 \ .\end{aligned} \tag{C.13}$$

Similarly, the Y^γ term is found to be

$$\begin{aligned}
\partial_\gamma Y^\gamma \lambda_{\alpha\beta} &= Y^\gamma \partial_\gamma \lambda_{\alpha\beta} = Y^\gamma \partial_\gamma \left[\partial_\alpha (s^2 Y_\beta) + \partial_\beta (s^2 Y_\alpha)\right] \\
&= Y^\gamma \partial_\gamma (Y_\beta \partial_\alpha s^2 - s^2 T_\beta \partial_\alpha \omega + Y_\alpha \partial_\beta s^2 - s^2 T_\alpha \partial_\beta \omega) \\
&= Y^\gamma \Big[(\partial_\gamma Y_\beta)\partial_\alpha s^2 + Y_\beta \partial_\gamma \partial_\alpha s^2 - (\partial_\gamma s^2) T_\beta \partial_\alpha \omega - s^2 T_\beta \partial_\gamma \partial_\alpha \omega \\
&\qquad + (\partial_\gamma Y_\alpha)\partial_\beta s^2 + Y_\alpha \partial_\gamma \partial_\beta s^2 - (\partial_\gamma s^2) T_\alpha \partial_\beta \omega - s^2 T_\alpha \partial_\gamma \partial_\beta \omega\Big] \\
&= Y^\gamma \left[-T_\beta (\partial_\gamma \omega)\partial_\alpha s^2 - T_\alpha (\partial_\gamma \omega) \partial_\beta s^2\right] \\
&= 0 \; .
\end{aligned} \tag{C.14}$$

To spell it out more precisely, these terms cancel because any time we have a derivative of a scalar function contracted with the vector that inner product is zero, since the scalar functions are only functions of r and θ. Thus what we are left with is

$$\partial_\gamma \Gamma^\gamma{}_{\alpha\beta} = \partial_\gamma {}^{(2)}\Gamma^\gamma{}_{\alpha\beta} + \frac{1}{2} \partial_\gamma \sigma^{\gamma\delta} \Xi_{\alpha\beta\delta} \; . \tag{C.15}$$

We leave the first term alone, since it is already in the form that is most useful to us. For the second, we find

$$\begin{aligned}
\partial_\gamma \sigma^{\gamma\delta} \Xi_{\alpha\beta\delta} &= \partial_\gamma \Xi_{\alpha\beta}{}^\gamma = \partial_\gamma \left[s^2 (Y_\alpha T_\beta + Y_\beta T_\alpha) \Delta^\gamma \omega - Y_\alpha Y_\beta \Delta^\gamma s^2 + T_\alpha T_\beta \Delta^\gamma Q^2\right] \\
&= \partial_\gamma s^2 Y_\alpha T_\beta \Delta^\gamma \omega + \partial_\gamma s^2 Y_\beta T_\alpha \Delta^\gamma \omega - \partial_\gamma Y_\alpha Y_\beta \Delta^\gamma s^2 + \partial_\gamma T_\alpha T_\beta \Delta^\gamma Q^2 \\
&= T_\beta \left[s^2 Y_\alpha \partial_\gamma \Delta^\gamma \omega + s^2 (\Delta^\gamma \omega) \partial_\gamma Y_\alpha + Y_\alpha (\Delta^\gamma \omega) \partial_\gamma s^2\right] \\
&\quad + T_\alpha \left[s^2 Y_\beta \partial_\gamma \Delta^\gamma \omega + s^2 (\Delta^\gamma \omega) \partial_\gamma Y_\beta + Y_\beta (\Delta^\gamma \omega) \partial_\gamma s^2\right] \\
&\quad - Y_\alpha Y_\beta \partial_\gamma \Delta^\gamma s^2 - Y_\alpha (\Delta^\gamma s^2) \partial_\gamma Y_\beta - Y_\beta (\Delta^\gamma s^2) \partial_\gamma Y_\alpha + T_\alpha T_\beta \partial_\gamma \Delta^\gamma Q^2 \\
&= T_\beta \left[s^2 Y_\alpha \partial_\gamma \Delta^\gamma \omega - s^2 (\Delta^\gamma \omega) T_\alpha \Delta_\gamma \omega + Y_\alpha (\Delta^\gamma \omega) \partial_\gamma s^2\right] \\
&\quad + T_\alpha \left[s^2 Y_\beta \partial_\gamma \Delta^\gamma \omega - s^2 (\Delta^\gamma \omega) T_\beta \Delta_\gamma \omega + Y_\beta (\Delta^\gamma \omega) \partial_\gamma s^2\right] \\
&\quad - Y_\alpha Y_\beta \partial_\gamma \Delta^\gamma s^2 + Y_\alpha T_\beta (\Delta^\gamma s^2) \Delta_\gamma \omega + Y_\beta T_\alpha (\Delta^\gamma s^2) \Delta_\gamma \omega + T_\alpha T_\beta \partial_\gamma \Delta^\gamma Q^2
\end{aligned}$$

$$= (Y_\alpha T_\beta + Y_\beta T_\alpha)\left[s^2\partial_\gamma\Delta^\gamma\omega + 2(\Delta^\gamma\omega)\Delta_\gamma s^2\right] - Y_\alpha Y_\beta\partial_\gamma\Delta^\gamma s^2$$
$$- T_\alpha T_\beta\left[2s^2(\Delta^\gamma\omega)\Delta_\gamma\omega + \partial_\gamma\Delta^\gamma Q^2\right] \ . \tag{C.16}$$

Overall, this full term is

$$\partial_\gamma\Gamma^\gamma{}_{\alpha\beta} = \partial_\gamma{}^{(2)}\Gamma^\gamma{}_{\alpha\beta} - \frac{1}{2}Y_\alpha Y_\beta\partial_\gamma\Delta^\gamma s^2 + \frac{1}{2}(Y_\alpha T_\beta + Y_\beta T_\alpha)\left[s^2\partial_\gamma\Delta^\gamma\omega + 2(\Delta^\gamma\omega)\Delta_\gamma s^2\right]$$
$$- \frac{1}{2}T_\alpha T_\beta\left[2s^2(\Delta^\gamma\omega)\Delta_\gamma\omega + \partial_\gamma\Delta^\gamma Q^2\right] \tag{C.17}$$

Our final term, $\Gamma^\delta{}_{\alpha\gamma}\Gamma^\gamma{}_{\beta\delta}$, involves a lot of distributing. To facilitate this, we compute a few paired quantities here so that they may be plugged in to simplify the expression. We find

$$\Xi_{\alpha\beta\gamma}Y^\beta = Y^\beta\left[s^2(Y_\alpha T_\beta + Y_\beta T_\alpha)\Delta_\gamma\omega - Y_\alpha Y_\beta\Delta_\gamma s^2 + T_\alpha T_\beta\Delta_\gamma Q^2\right]$$
$$= T_\alpha\Delta_\gamma\omega - \frac{1}{s^2}Y_\alpha\Delta_\gamma s^2 \ , \tag{C.18}$$

$$\Xi_{\alpha\beta\gamma}T^\beta = T^\beta\left[s^2(Y_\alpha T_\beta + Y_\beta T_\alpha)\Delta_\gamma\omega - Y_\alpha Y_\beta\Delta_\gamma s^2 + T_\alpha T_\beta\Delta_\gamma Q^2\right]$$
$$= -\frac{s^2}{Q^2}Y_\alpha\Delta_\gamma\omega - \frac{1}{Q^2}T_\alpha\Delta_\gamma Q^2 \ , \tag{C.19}$$

$${}^{(2)}\Gamma^\alpha{}_{\beta\gamma}\Xi_{\delta\alpha\epsilon} = {}^{(2)}\Gamma^\alpha{}_{\beta\gamma}\left[s^2(Y_\delta T_\alpha + Y_\alpha T_\delta)\Delta_\epsilon\omega - Y_\delta Y_\alpha\Delta_\epsilon s^2 + T_\delta T_\alpha\Delta_\epsilon Q^2\right] = 0 \ , \tag{C.20}$$

$$\sigma^{\alpha\beta}\Xi_{\gamma\alpha\delta} = \sigma^{\alpha\beta}\left[s^2(Y_\gamma T_\alpha + Y_\alpha T_\gamma) - Y_\gamma Y_\alpha\Delta_\delta s^2 + T_\gamma T_\alpha\Delta_\delta Q^2\right] = 0 \ , \tag{C.21}$$

$$Y^\alpha\tau_{\beta\alpha} = Y^\alpha\left[\partial_\beta(Q^2T_\alpha) + \partial_\alpha(Q^2T_\beta) + s^2(Y_\beta\Delta_\alpha\omega + Y_\alpha\Delta_\beta\omega\right]$$
$$= Y^\alpha\left[\partial_\beta(Q^2T_\alpha) + \partial_\alpha(Q^2T_\beta)\right] + \Delta_\beta\omega \ , \tag{C.22}$$

$$T^\alpha\tau_{\beta\alpha} = T^\alpha\left[\partial_\beta(Q^2T_\alpha) + \partial_\alpha(Q^2T_\beta) + s^2(Y_\beta\Delta_\alpha\omega + Y_\alpha\Delta_\beta\omega\right]$$
$$= T^\alpha\left[\partial_\beta(Q^2T_\alpha) + \partial_\alpha(Q^2T_\beta)\right] \ . \tag{C.23}$$

We also note that the first two indices of $\Xi_{\alpha\beta\gamma}$ are scaled vectors in a symmetric manner, so any contraction between either of those and something entirely contained to the submanifold will be zero, such as the specific cases of (C.20) and (C.21) above. Performing the distribution, we obtain

$$
\begin{aligned}
\Gamma^{\delta}{}_{\alpha\gamma}\Gamma^{\gamma}{}_{\beta\delta} = &\left({}^{(2)}\Gamma^{\delta}{}_{\alpha\gamma} + \frac{1}{2}\sigma^{\delta\epsilon}\Xi_{\alpha\gamma\epsilon} + \frac{1}{2}Y^{\delta}\lambda_{\alpha\gamma} - \frac{1}{2}T^{\delta}\tau_{\alpha\gamma}\right)\left({}^{(2)}\Gamma^{\gamma}{}_{\beta\delta} + \frac{1}{2}\sigma^{\gamma\zeta}\Xi_{\beta\delta\zeta}\right. \\
&\left. + \frac{1}{2}Y^{\gamma}\lambda_{\beta\delta} - \frac{1}{2}T^{\gamma}\tau_{\beta\delta}\right) \\
= &\ {}^{(2)}\Gamma^{\delta}{}_{\alpha\gamma}{}^{(2)}\Gamma^{\gamma}{}_{\beta\delta} + \frac{1}{2}{}^{(2)}\Gamma^{\gamma}{}_{\beta\delta}\sigma^{\delta\epsilon}\Xi_{\alpha\gamma\epsilon} + \frac{1}{2}{}^{(2)}\Gamma^{\delta}{}_{\alpha\gamma}\sigma^{\gamma\zeta}\Xi_{\beta\delta\zeta} + \frac{1}{4}\sigma^{\gamma\zeta}\Xi_{\beta\delta\zeta}\sigma^{\delta\epsilon}\Xi_{\alpha\gamma\epsilon} \\
&+ \frac{1}{4}\sigma^{\gamma\zeta}\Xi_{\beta\delta\zeta}Y^{\delta}\lambda_{\alpha\gamma} - \frac{1}{4}\sigma^{\gamma\zeta}\Xi_{\beta\delta\zeta}T^{\delta}\tau_{\alpha\gamma} + \frac{1}{4}\sigma^{\delta\epsilon}\Xi_{\alpha\gamma\epsilon}Y^{\gamma}\lambda_{\beta\delta} + \frac{1}{4}Y^{\delta}Y^{\gamma}\lambda_{\alpha\gamma}\lambda_{\beta\delta} \\
&- \frac{1}{4}Y^{\gamma}\lambda_{\beta\delta}T^{\delta}\tau_{\alpha\gamma} - \frac{1}{4}\sigma^{\delta\epsilon}\Xi_{\alpha\gamma\epsilon}T^{\gamma}\tau_{\beta\delta} - \frac{1}{4}T^{\gamma}\tau_{\beta\delta}Y^{\delta}\lambda_{\alpha\gamma} + \frac{1}{4}T^{\gamma}\tau_{\beta\delta}T^{\delta}\tau_{\alpha\gamma} \\
= &\ {}^{(2)}\Gamma^{\delta}{}_{\alpha\gamma}{}^{(2)}\Gamma^{\gamma}{}_{\beta\delta} + \frac{1}{4}\sigma^{\gamma\zeta}\left(T_{\beta}\Delta_{\zeta}\omega - \frac{1}{s^2}Y_{\beta}\Delta_{\zeta}s^2\right)\lambda_{\alpha\gamma} \\
&+ \frac{1}{4}\sigma^{\gamma\zeta}\left(\frac{s^2}{Q^2}Y_{\beta}\Delta_{\zeta}\omega + \frac{1}{Q^2}T_{\beta}\Delta_{\zeta}Q^2\right)\tau_{\alpha\gamma} + \frac{1}{4}\sigma^{\delta\epsilon}\left(T_{\alpha}\Delta_{\epsilon}\omega - \frac{1}{s^2}Y_{\alpha}\Delta_{\epsilon}s^2\right) \\
&+ \frac{1}{4}Y^{\delta}Y^{\gamma}\lambda_{\alpha\gamma}\lambda_{\beta\delta} - \frac{1}{4}T^{\delta}\lambda_{\beta\delta}\left\{Y^{\gamma}\left[\partial_{\alpha}(Q^2T_{\gamma}) + \partial_{\gamma}(Q^2T_{\alpha})\right] + \Delta_{\alpha}\omega\right\} \\
&+ \frac{1}{4}\sigma^{\delta\epsilon}\tau_{\beta\delta}\left(\frac{s^2}{Q^2}Y_{\alpha}\Delta_{\epsilon}\omega + \frac{1}{Q^2}T_{\alpha}\Delta_{\epsilon}Q^2\right) - \frac{1}{4}T^{\gamma}\lambda_{\alpha\gamma}\left\{Y^{\delta}\left[\partial_{\beta}(Q^2T_{\delta}) + \partial_{\delta}(Q^2T_{\beta})\right] + \Delta_{\beta}\omega\right\} \\
&+ \frac{1}{4}T^{\gamma}T^{\delta}\left[\partial_{\alpha}(Q^2T_{\gamma}) + \partial_{\gamma}(Q^2T_{\alpha})\right]\left[\partial_{\beta}(Q^2T_{\delta}) + \partial_{\delta}(Q^2T_{\beta})\right] \\
= &\ {}^{(2)}\Gamma^{\delta}{}_{\alpha\gamma}{}^{(2)}\Gamma^{\gamma}{}_{\beta\delta} + \frac{1}{4}\left(T_{\beta}\Delta^{\gamma}\omega - \frac{1}{s^2}Y_{\beta}\Delta^{\gamma}s^2\right)\left(Y_{\alpha}\Delta_{\gamma}s^2 - s^2T_{\alpha}\Delta_{\gamma}\omega\right) \\
&+ \frac{1}{4}\left(T_{\beta}\Delta^{\gamma}\omega - \frac{1}{s^2}Y_{\beta}\Delta^{\gamma}s^2\right)\left(Y_{\alpha}\Delta_{\gamma}s^2 - s^2T_{\alpha}\Delta_{\gamma}\omega\right) \\
&+ \frac{1}{4}\left(\frac{s^2}{Q^2}Y_{\alpha}\Delta^{\delta}\omega + \frac{1}{Q^2}T_{\alpha}\Delta^{\delta}Q^2\right)\left(T_{\beta}\Delta_{\delta}Q^2 + s^2Y_{\beta}\Delta_{\delta}\omega\right) \\
&+ \frac{1}{4}\left(\frac{s^2}{Q^2}Y_{\beta}\Delta^{\gamma}\omega + \frac{1}{Q^2}T_{\beta}\Delta^{\gamma}Q^2\right)\left(T_{\alpha}\Delta_{\gamma}Q^2 + s^2Y_{\alpha}\Delta_{\gamma}\omega\right) \\
&- \frac{1}{4}T^{\gamma}(\Delta_{\beta}\omega)\left[Y_{\gamma}\Delta_{\alpha}s^2 - s^2T_{\gamma}\Delta_{\alpha}\omega\right] - \frac{1}{4}T^{\delta}(\Delta_{\alpha}\omega)\left[Y_{\delta}\Delta_{\beta}s^2 - s^2T_{\delta}\Delta_{\beta}\omega\right] \\
&+ \frac{1}{4}Y^{\gamma}\left(Y_{\gamma}\Delta_{\alpha}s^2 - s^2T_{\gamma}\Delta_{\alpha}\omega\right)Y^{\delta}\left(Y_{\delta}\Delta_{\beta}s^2 - s^2T_{\delta}\Delta_{\beta}\omega\right) + \frac{1}{4}T^{\gamma}T_{\gamma}(\Delta_{\alpha}Q^2)T^{\delta}T_{\delta}\Delta_{\beta}Q^2\ .
\end{aligned}
\tag{C.24}
$$

Simplifying and collecting like terms, this becomes

$$\begin{aligned}\Gamma^{\delta}{}_{\alpha\gamma}\Gamma^{\gamma}{}_{\beta\delta} = {}^{(2)}\Gamma^{\delta}{}_{\alpha\gamma}{}^{(2)}\Gamma^{\gamma}{}_{\beta\delta} &+ \frac{1}{s^2}(\Delta_\alpha s)\Delta_\beta s + \frac{1}{Q^2}(\Delta_\alpha Q)\Delta_\beta Q - \frac{s^2}{2Q^2}(\Delta_\alpha \omega)\Delta_\beta \omega \\ &+ (Y_\alpha T_\beta + Y_\beta T_\alpha)\left[\frac{1}{2}(\Delta^\gamma \omega)\Delta_\gamma s^2 + \frac{s^2}{2Q^2}(\Delta^\gamma\omega)\Delta_\gamma Q^2\right] \\ &+ Y_\alpha Y_\beta\left[\frac{s^4}{2Q^2}(\Delta^\gamma\omega)\Delta_\gamma\omega - \frac{1}{2s^2}(\Delta^\gamma s^2)\Delta_\gamma s^2\right] \\ &+ T_\alpha T_\beta\left[\frac{1}{2Q^2}(\Delta^\gamma Q^2)\Delta_\gamma Q^2 - \frac{s^2}{2}(\Delta^\gamma\omega)\Delta_\gamma\omega\right] \ . \end{aligned} \tag{C.25}$$

Now we may finally assemble our Ricci tensor. After shrinking the font so that each term can fit on one line, the entire expression yields

$$\begin{aligned} R_{\alpha\beta} = {}& \partial_\gamma {}^{(2)}\Gamma^{\gamma}{}_{\alpha\beta} - \partial_\beta {}^{(2)}\Gamma^{\gamma}{}_{\alpha\gamma} + {}^{(2)}\Gamma^{\delta}{}_{\alpha\beta}{}^{(2)}\Gamma^{\gamma}{}_{\gamma\delta} - {}^{(2)}\Gamma^{\delta}{}_{\alpha\gamma}{}^{(2)}\Gamma^{\gamma}{}_{\beta\delta} \\ & - \partial_\beta\left(\frac{1}{sQ}\partial_\alpha sQ\right) + {}^{(2)}\Gamma^{\delta}{}_{\alpha\beta}\frac{1}{sQ}\partial_\delta sQ - \frac{1}{s^2}(\Delta_\alpha s)\Delta_\beta s - \frac{1}{Q^2}(\Delta_\alpha Q)\Delta_\beta Q + \frac{s^2}{2Q^2}(\Delta_\alpha\omega)\Delta_\beta\omega \\ & - Y_\alpha Y_\beta\left[\frac{1}{2}\left(\partial_\gamma\Delta^\gamma s^2 + {}^{(2)}\Gamma^{\gamma}{}_{\gamma\delta}\Delta^\delta s^2\right) - \frac{1}{2s^2}(\Delta^\gamma s^2)\Delta_\gamma s^2 + \frac{s^4}{2Q^2}(\Delta^\gamma\omega)\Delta_\gamma\omega + \frac{1}{2sQ}(\Delta^\delta s^2)\Delta_\delta sQ\right] \\ & + (Y_\alpha T_\beta + Y_\beta T_\alpha)\left[\frac{s^2}{2}\left(\partial_\gamma\Delta^\gamma\omega + {}^{(2)}\Gamma^{\gamma}{}_{\gamma\delta}\Delta^\delta\omega\right) + \frac{1}{2}(\Delta^\gamma\omega)\Delta_\gamma s^2 - \frac{s^2}{2Q^2}(\Delta^\gamma\omega)\Delta_\gamma Q^2 + \frac{s}{2Q}(\Delta^\gamma\omega)\Delta_\gamma sQ\right] \\ & + T_\alpha T_\beta\left[\frac{1}{2}\left(\partial_\gamma\Delta^\gamma Q^2 + {}^{(2)}\Gamma^{\gamma}{}_{\gamma\delta}\Delta^\delta Q^2\right) - \frac{s^2}{2}(\Delta^\gamma\omega)\Delta_\gamma\omega - \frac{1}{2Q^2}(\Delta^\gamma Q^2)\Delta_\gamma Q^2 + \frac{1}{2sQ}(\Delta^\gamma Q^2)\Delta_\gamma sQ\right] \ . \end{aligned} \tag{C.26}$$

We take one line at a time.

The first line is manifestly the definition of the Ricci tensor, but using the connection on the submanifold instead of the full connection. We may therefore reasonably call that collection of four terms ${}^{(2)}R_{\alpha\beta}$.

For the second line, we require a new piece of knowledge. When acting on a covector field, a field with index down, the definition of the covariant derivative has a relative minus sign:

$$\partial_\alpha V_\beta - \Gamma^{\gamma}{}_{\alpha\beta}V_\gamma = \nabla_\alpha V_\alpha \ . \tag{C.27}$$

This means that our first two terms in the second line can combine, giving us

$$\begin{aligned}\partial_\beta\left(\frac{1}{sQ}\ \partial_\alpha sQ\right) - {}^{(2)}\Gamma^\delta{}_{\alpha\beta}\frac{1}{sQ}\partial_\delta sQ &= \Delta_\beta\left(\frac{1}{sQ}\Delta_\alpha sQ\right)\\ &= \frac{1}{Q}\Delta_\alpha\Delta_\beta Q + \frac{1}{s}\Delta_\alpha\Delta_\beta s - \frac{1}{Q^2}(\Delta_\alpha Q)\Delta_\beta Q - \frac{1}{s^2}(\Delta_\alpha s)\Delta_\beta s\ .\end{aligned} \tag{C.28}$$

Inserting this result, our first two lines become

$${}^{(2)}R_{\alpha\beta} - \frac{1}{s}\Delta_\alpha\Delta_\beta s - \frac{1}{Q}\Delta_\alpha\Delta_\beta Q + \frac{s^2}{2Q^2}(\Delta_\alpha\omega)\Delta_\beta\omega\ . \tag{C.29}$$

Upon some careful inspection, we may notice that everything here lies in the submanifold. It would be nice if we could show that at a glance. We therefore leverage the Kronecker delta only on the submanifold, $\sigma^\alpha{}_\beta = {}^{(2)}\delta^\alpha{}_\beta$, to obtain

$$\sigma^\gamma{}_\alpha\sigma^\delta{}_\beta\left[{}^{(2)}R_{\gamma\delta} - \frac{1}{s}\Delta_\gamma\Delta_\delta s - \frac{1}{Q}\Delta_\gamma\Delta_\delta Q + \frac{s^2}{2Q^2}(\Delta_\gamma\omega)\Delta_\delta\omega\right]\ . \tag{C.30}$$

The third line in the full Ricci tensor, the $Y_\alpha Y_\beta$ term, cam be simplified when we recognize the covariant derivative of $\Delta_\gamma\Delta^\gamma s^2$ from the term in the parentheses. Putting in this definition and grouping it with two of the three other terms in that line, we find

$$\begin{aligned}&\frac{1}{2}\Delta_\gamma\Delta^\gamma s^2 - \frac{1}{2s^2}(\Delta^\gamma s^2)\Delta_\gamma s^2 + \frac{1}{2sQ}(\Delta^\gamma s^2)\Delta_\gamma sQ\\ &\qquad = s\Delta_\gamma\Delta^\gamma s + (\Delta^\gamma s)\Delta_\gamma s - 2(\Delta^\gamma s)\Delta_\gamma s + \frac{1}{Q}(\Delta^\gamma s)(Q\Delta_\gamma s + s\Delta_\gamma Q)\\ &\qquad = s\Delta_\gamma\Delta^\gamma s + \frac{s}{Q}(\Delta_\gamma s)\Delta^\gamma Q\\ &\qquad = \frac{s}{Q}\Delta_\gamma(Q\Delta^\gamma s)\ .\end{aligned} \tag{C.31}$$

This line may thus be written

$$-Y_\alpha Y_\beta\left[\frac{s}{Q}\Delta_\gamma(Q\Delta^\gamma s) + \frac{s^4}{2Q^2}(\Delta_\gamma\omega)\Delta^\gamma\omega\right]\ . \tag{C.32}$$

The fourth line similarly contains a covariant derivative in the parentheses. Simplifying all of the terms, we get

$$
\begin{aligned}
\frac{s^2}{2}\Delta_\gamma\Delta^\gamma\omega &+ \frac{1}{2}(\Delta^\gamma\omega)\Delta_\gamma s^2 - \frac{s^2}{2Q^2}(\Delta^\gamma\omega)\Delta_\gamma Q^2 + \frac{s}{2Q}(\Delta^\gamma\omega)\Delta_\gamma sQ \\
&= \frac{s^2}{2}\Delta_\gamma\Delta^\gamma\omega + s(\Delta^\gamma\omega)\Delta_\gamma s - \frac{s^2}{Q}(\Delta^\gamma\omega)\Delta_\gamma Q + \frac{s^2}{2Q}(\Delta^\gamma\omega)\Delta_\gamma Q + \frac{s}{2}(\Delta^\gamma\omega)\Delta_\gamma s \\
&= \frac{s^2}{2}\Delta_\gamma\Delta^\gamma\omega - \frac{s^2}{2Q}(\Delta^\gamma\omega)\Delta_\gamma Q + \frac{3}{2}s(\Delta^\gamma\omega)\Delta_\gamma s \\
&= \frac{Q}{2s}\left[\frac{s^3}{Q}\Delta_\gamma\Delta^\gamma\omega + \left(\Delta_\gamma\frac{s^3}{Q}\right)\Delta^\gamma\omega\right] \\
&= \frac{Q}{2s}\Delta_\gamma\left(\frac{s^3}{Q}\Delta^\gamma\omega\right) \ .
\end{aligned}
\tag{C.33}
$$

This fully simplifies the fourth line.

For the fifth line, we may see the pattern. There is once again a hidden covariant derivative. Grabbing all of the terms except the second one, we may simplify:

$$
\begin{aligned}
\frac{1}{2}\Delta_\gamma\Delta^\gamma Q^2 &- \frac{1}{2Q^2}(\Delta^\gamma Q^2)\Delta_\gamma Q^2 + \frac{1}{2sQ}(\Delta^\gamma Q^2)\Delta_\gamma sQ \\
&= \Delta_\gamma(Q\Delta^\gamma Q) - 2(\Delta^\gamma Q)\Delta_\gamma Q + \frac{1}{s}(\Delta^\gamma Q)\Delta_\gamma sQ \\
&= Q\Delta_\gamma\Delta^\gamma Q + \frac{Q}{s}(\Delta^\gamma Q)\Delta_\gamma s \\
&= \frac{Q}{s}\Delta_\gamma(s\Delta^\gamma Q) \ .
\end{aligned}
\tag{C.34}
$$

This means the final line may be rewritten as

$$
T_\alpha T_\beta\left[\frac{Q}{s}\Delta_\gamma(s\Delta^\gamma Q) - \frac{s^2}{2}(\Delta_\gamma\omega)\Delta^\gamma\omega\right] \ . \tag{C.35}
$$

We have, at last, reached the final step. Reassembling all of our terms above, we find the same quantity as was given in Eq. (3.11):

$$\begin{aligned} R_{\alpha\beta} =&\sigma^{\gamma}{}_{\alpha}\sigma^{\delta}{}_{\beta}\left[{}^{(2)}R_{\gamma\delta} - \frac{1}{s}\Delta_\gamma\Delta_\delta s - \frac{1}{Q}\Delta_\gamma\Delta_\delta Q + \frac{s^2}{2Q^2}(\Delta_\gamma\omega)\Delta_\delta\omega\right] \\ &- Y_\alpha Y_\beta\left[\frac{s}{Q}\Delta_\gamma(Q\Delta^\gamma s) + \frac{s^4}{2Q^2}(\Delta_\gamma\omega)\Delta^\gamma\omega\right] + (Y_\alpha T_\beta + Y_\beta T_\alpha)\left[\frac{Q}{2s}\Delta_\gamma\left(\frac{s^3}{Q}\Delta^\gamma\omega\right)\right] \\ &+ T_\alpha T_\beta\left[\frac{Q}{s}\Delta_\gamma(s\Delta^\gamma Q) - \frac{s^2}{2}(\Delta_\gamma\omega)\Delta^\gamma\omega\right]\ . \end{aligned} \tag{C.36}$$

Appendix D

Field Strength Factorization

Starting with the claimed expression Eq. (3.20),

$$F_{\alpha\beta} = \frac{1}{sQ}\left(\phi_\beta B_\alpha - \phi_\alpha B_\beta - \frac{s}{Q}(t_\beta E_\alpha - t_\alpha E_\beta)\right) \ , \tag{D.1}$$

and inserting the definitions of E_α and B_α from Eq. (3.21)

$$E_\alpha = \Delta_\alpha Q^2 A_T + s^2 A_Y \Delta_\alpha \omega \qquad \text{and} \qquad B_\alpha = \frac{Q}{s}\Delta_\alpha s^2 A_Y + \frac{s}{Q}\omega E_\alpha \ , \tag{D.2}$$

we show that the claimed expression is equivalent to the derived expression Eq. (3.19). Expanding this out, we get

$$\begin{aligned} F_{\alpha\beta} &= \frac{1}{sQ}\left(\phi_\beta \frac{Q}{s}\Delta_\alpha s^2 A_Y + \phi_\beta \frac{s}{Q}\omega E_\alpha - \phi_\alpha \frac{Q}{s}\Delta_\beta s^2 A_Y - \phi_\alpha \frac{s}{Q}\omega E_\beta - \frac{s}{Q} t_\beta E_\alpha + \frac{s}{Q} t_\alpha E_\beta\right) \\ &= \frac{\phi_\beta}{s^2}\Delta_\alpha s^2 A_Y + \frac{1}{Q^2}\omega\phi_\beta(\Delta_\alpha Q^2 A_T + s^2 A_Y \Delta_\alpha\omega) - \frac{1}{Q^2}\omega\phi_\alpha(\Delta_\beta Q^2 A_T + s^2 A_Y \Delta_\beta\omega) \\ &\quad - \frac{\phi_\alpha}{s^2}\Delta_\beta s^2 A_Y - \frac{1}{Q^2} t_\beta(\Delta_\alpha Q^2 A_T + s^2 A_Y \Delta_\alpha\omega) + \frac{1}{Q^2} t_\alpha(\Delta_\beta Q^2 A_T + s^2 A_Y \Delta_\beta\omega) \ . \end{aligned} \tag{D.3}$$

Rearranging a bit and converting all of the $\dfrac{\phi_\alpha}{s^2}$ into Y_α, this becomes

$$F_{\alpha\beta} = (Y_\beta\Delta_\alpha - Y_\alpha\Delta_\beta)s^2A_Y - \frac{1}{Q^2}(t_\beta - \omega\phi_\beta)(\Delta_\alpha Q^2A_T + s^2A_Y\Delta_\alpha\omega) + \frac{1}{Q^2}(t_\alpha - \omega\phi_\alpha)(\Delta_\beta Q^2A_T + s^2A_Y\Delta_\beta\omega) \ . \tag{D.4}$$

Finally recognizing the form of the vector T_α, we can rewrite it one more time into its simplest form. Additionally, since the covariant derivatives are all acting on purely scalar functions, they may be rewritten as ordinary derivatives, so

$$F_{\alpha\beta} = (Y_\beta\partial_\alpha - Y_\alpha\partial_\beta)s^2A_Y - (T_\beta\partial_\alpha - T_\alpha\partial_\beta)Q^2A_T - s^2A_Y(T_\beta\partial_\alpha - T_\alpha\partial_\beta)\omega \ , \tag{D.5}$$

exactly the same as Eq. (3.19).

Appendix E

Computing the Stress-Energy Tensor

In a source-free region, the stress-energy tensor is given by Eq. (3.22), which is

$$4\pi T_{\alpha\beta} = g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \ . \tag{E.1}$$

The form of the field strength tensor that is easiest to work with and to make match with the Ricci tensor form is Eq. (3.50),

$$F_{\alpha\beta} = (Y_\beta \Delta_\alpha - Y_\alpha \Delta_\beta) s^2 A_Y - (T_\beta E_\alpha - T_\alpha E_\beta) \ . \tag{E.2}$$

We start by calculating $F_{\gamma\delta} F^{\gamma\delta}$. We obtain

$$\begin{aligned}
F_{\gamma\delta} F^{\gamma\delta} &= \Big[(Y_\delta \Delta_\gamma - Y_\gamma \Delta_\delta) s^2 A_Y - (T_\delta E_\gamma - T_\gamma E_\delta)\Big] \Big[(Y^\delta \Delta^\gamma - Y^\gamma \Delta^\delta) s^2 A_Y - (T^\delta E^\gamma - T^\gamma E^\delta)\Big] \\
&= Y_\delta Y^\delta (\Delta_\gamma s^2 A_Y)(\Delta^\gamma s^2 A_Y) - Y_\delta (\Delta^\delta s^2 A_Y) Y^\gamma (\Delta_\gamma s^2 A_Y) - Y_\delta T^\delta E^\gamma \Delta_\gamma s^2 A_Y + Y_\delta E^\delta T^\gamma \Delta_\gamma s^2 A_Y \\
&- Y_\gamma (\Delta^\gamma s^2 A_Y) Y^\delta (\Delta_\delta s^2 A_Y) - Y_\gamma Y^\gamma (\Delta_\delta s^2 A_Y)(\Delta^\delta s^2 A_Y) - Y_\gamma E^\gamma T^\delta \Delta_\delta s^2 A_Y + Y_\gamma T^\gamma E^\delta \Delta_\delta s^2 A_Y \\
&- T_\delta Y^\delta E_\gamma \Delta^\gamma s^2 A_Y + T_\delta (\Delta^\delta s^2 A_Y) Y^\gamma E_\gamma + T_\delta T^\delta E_\gamma E^\gamma - T_\delta E^\delta T^\gamma E_\gamma \\
&+ T_\gamma (\Delta^\gamma s^2 A_Y) Y^\delta E_\delta - T_\gamma Y^\gamma E_\delta \Delta^\delta s^2 A_Y - T_\gamma E^\gamma T^\delta E_\delta + T_\gamma T^\gamma E_\delta E^\delta \ .
\end{aligned} \tag{E.3}$$

We may recall that Y^α and T^α are orthogonal. Furthermore, the covariant derivative of the submanifold only has components in the r and θ directions, which are also orthogonal to our scaled vectors. The same is true for E^α. We can thus set any terms with these contractions to zero. This leaves only one term per line being nonzero, or

$$\begin{aligned} F_{\gamma\delta}F^{\gamma\delta} &= \frac{1}{s^2}(\Delta_\gamma s^2 A_Y)(\Delta^\gamma s^2 A_Y) + \frac{1}{s^2}(\Delta_\delta s^2 A_Y)(\Delta^\delta s^2 A_Y) - \frac{1}{Q^2}E_\gamma E^\gamma - \frac{1}{Q^2}E_\delta E^\delta \\ &= \frac{2}{s^2}(\Delta_\gamma s^2 A_Y)(\Delta^\gamma s^2 A_Y) - \frac{2}{Q^2}E_\gamma E^\gamma \ . \end{aligned} \tag{E.4}$$

Doing this step first simplifies the next step quite a lot, when we multiply the quantity (E.4) with the metric:

$$\begin{aligned} \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} &= \left(\sigma_{\alpha\beta} + s^2 Y_\alpha Y_\beta - Q^2 T_\alpha T_\beta\right)\left(\frac{1}{2s^2}(\Delta_\gamma s^2 A_Y)(\Delta^\gamma s^2 A_Y) - \frac{1}{2Q^2}E_\gamma E^\gamma\right) \\ &= \frac{1}{2s^2}\sigma_{\alpha\beta}(\Delta_\gamma s^2 A_Y)(\Delta^\gamma s^2 A_Y) - \frac{1}{2Q^2}\sigma_{\alpha\beta}E_\gamma E^\gamma \\ &\qquad + \frac{1}{2}Y_\alpha Y_\beta\left[(\Delta_\gamma s^2 A_Y)(\Delta^\gamma s^2 A_Y) - \frac{s^2}{Q^2}E_\gamma E^\gamma\right] \\ &\qquad - \frac{1}{2}T_\alpha T_\beta\left[\frac{Q^2}{s^2}(\Delta_\gamma s^2 A_Y)(\Delta^\gamma s^2 A_Y) - E_\gamma E^\gamma\right] \ . \end{aligned} \tag{E.5}$$

We can now move to the other term in the stress-energy definition and compute $g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta}$. Again using the orthogonality of the scaled vectors and the covariant derivative or electric field, we find

$$\begin{aligned} & g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta} = F_\alpha^\delta F_{\beta\delta} \\ &= \left(Y^\delta\Delta_\alpha s^2 A_Y - Y_\alpha\Delta^\delta s^2 A_Y - T^\delta E_\alpha + T_\alpha E^\delta\right)\left(Y_\delta\Delta_\beta s^2 A_Y - Y_\beta\Delta_\delta s^2 A_Y - T_\delta E_\beta + T_\beta E_\delta\right) \\ &= Y^\delta Y_\delta(\Delta_\alpha s^2 A_Y)\Delta_\beta s^2 A_Y - Y^\delta(\Delta_\delta s^2 A_Y)Y_\beta\Delta_\alpha s^2 A_Y - Y^\delta T_\delta E_\beta\Delta_\alpha s^2 A_Y + Y^\delta E_\delta T_\beta\Delta_\alpha s^2 A_Y \\ &\quad - Y_\alpha Y_\delta(\Delta^\delta s^2 A_Y)\Delta_\beta s^2 A_Y + Y_\alpha Y_\beta(\Delta^\delta s^2 A_Y)\Delta_\delta s^2 A_Y + Y_\alpha E_\beta T_\delta\Delta^\delta s^2 A_Y - Y_\alpha T_\beta E_\delta\Delta^\delta s^2 A_Y \\ &\quad - T^\delta Y_\delta E_\alpha\Delta_\beta s^2 A_Y + T^\delta(\Delta_\delta s^2 A_Y)Y_\beta E_\alpha + T^\delta T_\delta E_\alpha E_\beta - T^\delta E_\delta T_\beta E_\alpha \\ &\quad + T_\alpha E^\delta Y_\delta\Delta_\beta s^2 A_Y - T_\alpha Y_\beta E^\delta\Delta_\delta s^2 A_Y - T_\alpha T_\delta E^\delta E_\beta + T_\alpha T_\beta E^\delta E_\delta \ , \end{aligned} \tag{E.6}$$

which reduces to

$$
\begin{aligned}
g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta} = {} & \frac{1}{s^2}(\Delta_\alpha s^2 A_Y)\Delta_\beta s^2 A_Y - \frac{1}{Q^2}E_\alpha E_\beta \\
& + Y_\alpha Y_\beta(\Delta^\delta s^2 A_Y)\Delta_\delta s^2 A_Y - (Y_\alpha T_\beta + Y_\beta T_\alpha)E^\delta \Delta_\delta s^2 A_Y \\
& + T_\alpha T_\beta E^\delta E_\delta \ .
\end{aligned}
\tag{E.7}
$$

Now subtracting (E.5) from (E.7), we find

$$
\begin{aligned}
4\pi T_{\alpha\beta} = {} & \frac{1}{s^2}\left((\Delta_\alpha s^2 A_Y)\Delta_\beta s^2 A_Y - \frac{1}{2}\sigma_{\alpha\beta}(\Delta_\gamma s^2 A_Y)\Delta^\gamma s^2 A_Y)\right) - \frac{1}{Q^2}\left(E_\alpha E_\beta - \frac{1}{2}\sigma_{\alpha\beta}E_\gamma E^\gamma\right) \\
& + \frac{1}{2}Y_\alpha Y_\beta\left[(\Delta^\delta s^2 A_Y)\Delta_\delta s^2 A_Y + \frac{s^2}{Q^2}E_\gamma E^\gamma\right] - (Y_\alpha T_\beta + Y_\beta T_\alpha)E^\delta \Delta_\delta s^2 A_Y \\
& + \frac{1}{2}T_\alpha T_\beta\left[\frac{Q^2}{s^2}(\Delta_\gamma s^2 A_Y)\Delta^\gamma s^2 A_Y + E^\delta E_\delta\right] \ .
\end{aligned}
\tag{E.8}
$$

After some examination, we may see that the first line of the above expression lies entirely in the submanifold. As was done with the Ricci tensor, it would be nice to ensure that this is seen explicitly. Thus, since $\sigma^\beta{}_\alpha = {}^{(2)}\delta^\beta{}_\alpha$, we may write this as

$$
\begin{aligned}
4\pi T_{\alpha\beta} = {} & \sigma^\gamma{}_\alpha \sigma^\delta{}_\beta\left[\frac{1}{s^2}\left((\Delta_\gamma s^2 A_Y)\Delta_\delta s^2 A_Y - \frac{1}{2}\sigma_{\gamma\delta}(\Delta_\epsilon s^2 A_Y)\Delta^\epsilon s^2 A_Y\right) - \frac{1}{Q^2}\left(E_\gamma E_\delta - \frac{1}{2}\sigma_{\gamma\delta}E_\epsilon E^\epsilon\right)\right] \\
& + \frac{1}{2}Y_\alpha Y_\beta\left[(\Delta^\gamma s^2 A_Y)\Delta_\gamma s^2 A_Y + \frac{s^2}{Q^2}E_\gamma E^\gamma\right] - (Y_\alpha T_\beta + Y_\beta T_\alpha)[E^\gamma \Delta_\gamma s^2 A_Y] \\
& + \frac{1}{2}T_\alpha T_\beta\left[\frac{Q^2}{s^2}(\Delta_\gamma s^2 A_Y)\Delta^\gamma s^2 A_Y + E^\gamma E_\gamma\right] \ ,
\end{aligned}
\tag{E.9}
$$

which is exactly what we have in Eq. (3.23).

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