Running of the Magnetic Coupling

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ABSTRACT

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Magnetic monopoles, or particles that emit radial magnetic fields, are as yet undiscovered. However, quantum field theory makes many predictions about them. The prediction most relevant to this paper is that the strength of the interaction between magnetic monopoles and other particles depends on the interaction's energy scale. This phenomenon is well understood for electrically charged particles and is known as the "running of the electric coupling." In this paper, we show a formalism for determining how the magnetic coupling would run and how it could affect the running of the electric coupling as well. We hope to be able to extend these methods to other types of theories.

Keywords: Hypothetical Particles, Running Coupling, Electric Coupling, Magnetic Coupling, Magnetic Monopoles, Quantization Condition

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Chapter 1

Introduction

This thesis is written largely as a companion to the paper Schwinger v. Coleman [1]. It is meant to explain math and concepts needed to understand this paper and others in a way that is accessible to undergraduate students. It is hoped that when another undergraduate comes along who wants to study magnetic monopoles or get a basic introduction to some aspects of Quantum Field Theory, they will be able to read the paper and this thesis, and shorten their floundering time significantly.

The running of the electric and magnetic coupling is a quantum field theoretic phenomenon that two scientists, Julian Schwinger and Sydney Coleman, studied. They got essentially opposite answers to the question: How does the magnetic coupling run relative to the electric? This paper, in conjunction with [1], aims to answer that question and reconcile the two answers by employing a new formalism for calculating each piece.

To get to a point where we can make these calculations, this introduction covers the basics of magnetic monopoles, coupling and charge, the Dirac quantization condition, and the running coupling. Using this background, in the next chapter, we walk through the new formalism to reconcile Schwinger and Coleman's answers. Finally, we conclude that Coleman was correct and Schwinger's error was due to the mishandling of some topological terms. We elaborate further on the implications of these solutions and include ideas for future projects involving this formalism.

1.1 Magnetic Monopoles

A monopole is any one particle that produces a radial field; for example, an electron is an example of an electric monopole - a particle that produces a radial electric field. A magnetic monopole is similar; it is any particle that produces a radial magnetic field. However, as is typically taught in undergraduate electricity and magnetism classes, $\nabla \cdot B = 0$. This means that the magnetic field is divergence-less, or has no sources or sinks. This would indicate that there are no such things as magnetic monopoles, as they are defined as sources of a radial magnetic field. This is not a requirement of Maxwell's equations as a theory; it is simply indicative that we have not yet observed a magnetic monopole.

1.1.1 Quantization Condition

While magnetic monopoles have not yet been discovered, there are some compelling arguments for their existence. One of these is the quantization of electric charges, which is discussed in the introduction of [1]. The idea is fairly simple, imagine we have one magnetic monopole and one electric monopole separated by a distance r as shown in Figure 1.1.

As discussed in [1], we can calculate the angular momentum associated with this system. To do this, we need the Coulomb field produced by the electric charge, which is a standard result given by:

$$\vec{E}(\vec{r}) = \frac{eq}{4\pi}\frac{\vec{r}-\vec{r_e}}{|\vec{r}-\vec{r_e}|^3} \ . \tag{1.1}$$

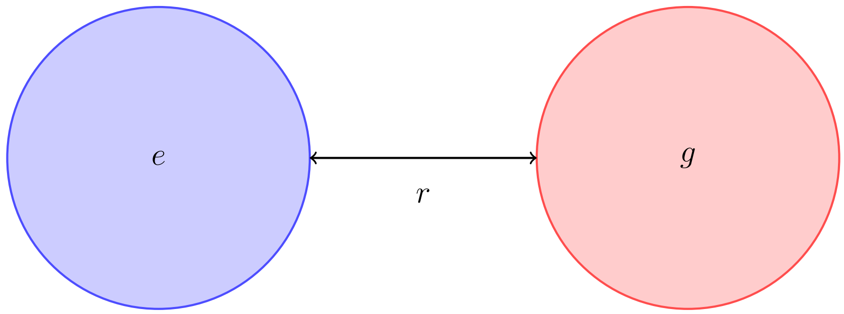

Figure 1.1 An electric monopole e and a magnetic monopole g separated by a distance r.

Here e is the coupling and q is the charge of the electric monopole, the difference between these two values is discussed in Sec. 1.1.2, and $\vec{r}_e$ is the distance from some origin to the electric monopole.

We also need the magnetic field produced by the magnetic charge, which is given by:

$$\vec{B}(\vec{r}) = \frac{bg}{4\pi} \frac{\vec{r} - \vec{r_b}}{|\vec{r} - \vec{r_b}|^3} \ . \tag{1.2}$$

Similar to the electric case, b and g are the coupling and charge, respectively, of the magnetically charged particle, and $\vec{r}_b$ is the distance from the same origin to the magnetic monopole. In essence, the magnetic monopole functions in the same way as the electric monopole; it is just magnetic instead of electric.

Using equations (1.1) and (1.2), we can calculate the angular momentum for this system. In 1904 J.J. Thomson [2] found this to be

$$\vec{L} = \frac{eqbg}{4\pi} \widehat{r} \ . \tag{1.3}$$

Here, $\widehat{r}$ is a unit vector pointing from the electric monopole to the magnetic one. This result is particularly interesting because it does not depend on the distance between the charges ($\widehat{r}$ is a unit vector). Quantum mechanics tells us that angular momentum is quantized, which means we can also write that $L = n\hbar$, to get

$$\begin{aligned} |L| = n\hbar = \frac{eqbg}{4\pi} \ , \\ eqbg = & 4\pi n\hbar \ . \end{aligned} \tag{1.4}$$

It is at this point that physicists often assume that

$$b = \frac{4\pi}{e} \ . \tag{1.5}$$

It is important to note that this assumption is non-trivial. So much so that one of the main goals of [1] was to show that Eq. (1.5) is justified by quantum field theory. The reasoning for the assumption is explained in Sec. 1.3. This assumption leads to the aforementioned charge quantization condition, which is calculated below.

When we plug Eq. (1.5) into Eq. (1.4), we get

$$\begin{aligned} eq\left(\frac{4\pi}{e}\right)g = 4\pi n\hbar \\ qg = n\hbar \ . \end{aligned} \tag{1.6}$$

This gives us the condition between electric and magnetic monopoles that we were looking for. In this equation, we have the electric charge and the magnetic charge multiplied together to create some integer multiple. This is interesting because if g is zero, i.e., there are no magnetic monopoles, then q is unrestrained; it can take any value. However, if g is non-zero, q must take certain quantized values that are integer multiples of each other.

As we can see in Table 1.1, this integer multiple-ness is what we see in our universe. Imagine we have some elementary charge, which we suggestively call $\frac{1}{3}q_d$. Every single fundamental particle that we have observed thus far has had an electric charge that is an integer multiple of $\frac{1}{3}q_d$, as is seen in Table 1.1.

q_d	$-2q_d$	$3q_d$	$0q_d$
down quark strange quark bottom quark	up quark charm quark top quark	electron muon tau W boson	neutrinos Z boson photon gluon Higgs boson

Table 1.1 Electric charge of fundamental particles in terms of the charge of the down quark

While this quantization argument does not guarantee the existence of magnetic monopoles, it does offer an extremely elegant explanation of charge quantization, which warrants further investigation.

1.1.2 Coupling v. Charge

In QFT, two quantities determine the strength of a charged particle's interaction with the electromagnetic field, and they are easy to confuse. The coupling e is a property of the electromagnetic field; it sets the overall strength of the electromagnetic interaction with any given particle. In contrast, the charge q is the label given to a certain particle that specifies how many "units of coupling" that particle carries. We often choose this to be $+1$ for the proton and -1 for the electron and let e contain the actual strength of the interaction. It is important to note that once we select a value for q for any one particle, the rest of the particles' q's are determined. For example, if we choose $q_{electron} = 10$, then q_{proton} better be -10. In reality, it is the combination of q and e: qe that gives the force between particles.

A similar story holds for the magnetic charge and coupling in Eq. (1.2). The coupling b is again a property of the electromagnetic field; it sets the overall strength of the interaction

Figure 1.2 The classical vacuum.

with any given particle. The charge g is again something that we choose to define for a given particle. Though because of the relation we found in Eq. (1.6), once we have chosen a value for q or g, the rest of the particles' q's and g's are determined.

1.2 Running Couplings

To understand the running coupling , we need to understand the vacuum. When we think about a classical vacuum, we normally think about a completely empty space, as in Fig. 1.2. However, the theory that we are using to describe the universe is not classical; it is called quantum field theory (QFT), and it makes some different predictions about the vacuum. It predicts what we call vacuum fluctuations in each quantum field, such as the electron field and the photon field. We often think of these fluctuations as virtual particle-antiparticle pairs that are popping in and out of existence, as in Fig. 1.3. This means that if we were to observe a particle in a vacuum, we would observe it, along with the soup of particle-antiparticle pairs it is sitting in and interacting with. For example, consider an electron in this soupy vacuum

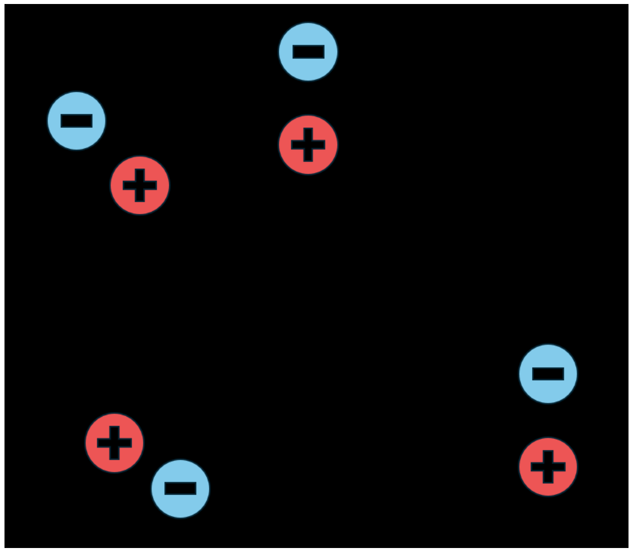

Figure 1.3 The QFT vacuum. Many particle-antiparticle pairs are popping in and out of existence.

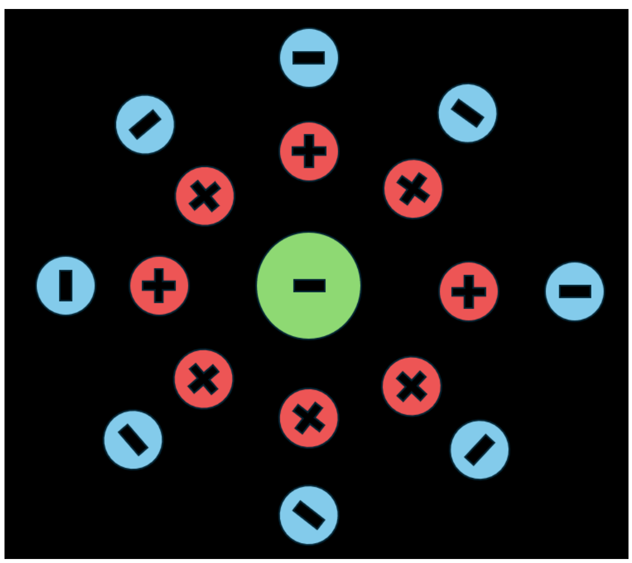

Figure 1.4 Particle-antiparticle pairs shielding an electron in the QED vacuum

as in Fig. 1.4. The particle-antiparticle pairs act as expected, the positive end nears the electron, and the negative end moves away. This creates a "cloud" of virtual particles that causes a shielding effect. This shielding effect is what causes the running of the coupling. We can think of measuring the electron in this state as probing it with various wavelengths of

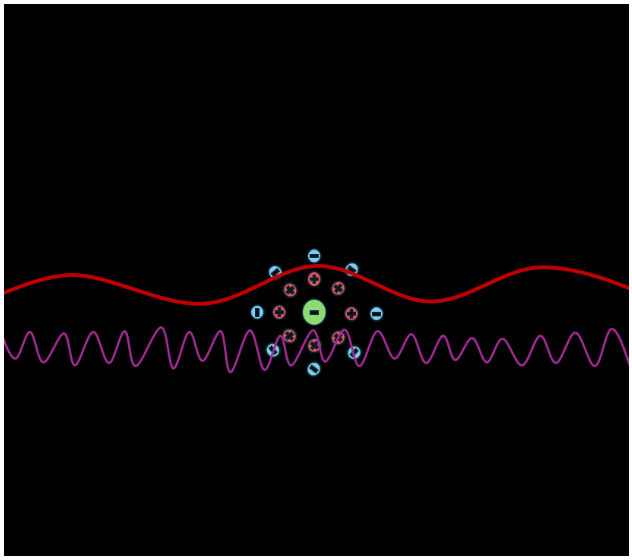

Figure 1.5 Measuring the electron in the QED vacuum with a high frequency light (violet) and a low frequency light (red).

light, as in Fig. 1.5. The red, low-frequency, lower-energy light gets what is essentially a "blurry picture" of the electron. It cannot resolve the difference between the electron and the soup. This means the value it returns for the electric coupling is slightly lower. This makes it seem like the electron interacts more weakly with the field. On the other hand, the violet, high-frequency, high-energy light can get a "clearer picture"; it probes more of the nuance in the system and isn't as blocked by the soup of particles. This higher frequency light returns a value that indicates the electron has a stronger coupling to the electric field. This idea of the measured value of the coupling changing with the energy used to measure it is what we mean by a running coupling. As the energy scale increases, so does the value for how strongly the electron interacts with the field

In Fig. 1.6 we can see data of this phenomenon, which comes from the L3 collaboration. They measured a parameter alpha, which depends on e:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \ . \tag{1.7}$$

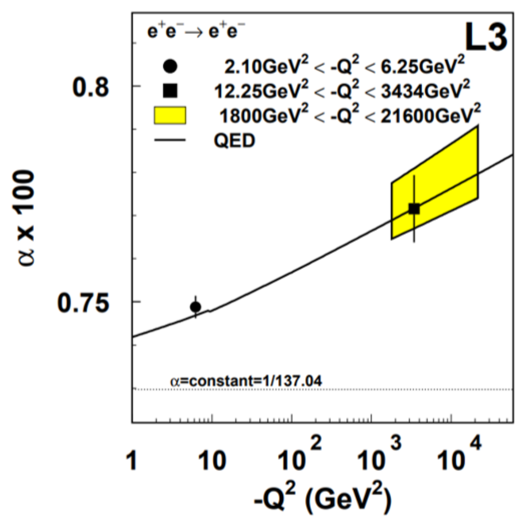

Figure 1.6 From the L3 collaboration [3]. Evolution of the electromagnetic coupling e, the solid line represents predictions from QED, and the solid dots with error bars and yellow region are experimental data from the L3 group; the width of the yellow box represents the error in the measurements.

We can see that as the coupling, e, increases, so does α. Looking at Fig. 1.6, we can see that as the energy scale increases, the measured value of alpha, and thus of the electric coupling, increases. This brings us back to our central problem. We know that the electric coupling works this way, but we don't have any magnetic monopoles to perform this experiment with. Thankfully, the two scientists mentioned earlier, Julian Schwinger and Sydney Coleman, each thought about this problem. Unfortunately, they got different answers; it is our job to reconcile them.

1.3 Historical Background

Sydney Coleman and Julian Schwinger were both interested in the issue of the running of the magnetic coupling; both studied it and obtained drastically different answers to the question of how the magnetic coupling runs. Schwinger found that the electric and magnetic couplings run in the same way [4]. He found that

$$\begin{aligned} e(\mu) &= \sqrt{Z_S(\mu)}e_0, \\ b(\mu) &= \sqrt{Z_S(\mu)}b_0 \ . \end{aligned} \tag{1.8}$$

This means that electric coupling increases with energy scale $\sqrt{Z_S(\mu)}$, which we know from experiment, and that the magnetic coupling does the same thing. As the energy scale increases, the magnetic coupling does as well.

Coleman, on the other hand, found that the magnetic coupling ran inverse to the electric coupling [5]:

$$\begin{aligned} e(\mu) &= \sqrt{Z_S(\mu)}e_0, \\ b(\mu) &= \frac{1}{\sqrt{Z_S(\mu)}}b_0 \ . \end{aligned} \tag{1.9}$$

This means that as the energy scale increases, the amount that the magnetic monopole interacts with the field decreases. This is where we get that big assumption that is Eq. (1.5).

$$b = \frac{4\pi}{e} \ . \tag{1.10}$$

If $e(\mu) = \sqrt{Z_S(\mu)}e_0$ and $b(\mu) = \frac{1}{\sqrt{Z_S(\mu)}}b_0$ then we get this relation exactly.

$$b(\mu) = \frac{1}{\sqrt{Z_S(\mu)}}b_0 = \frac{4\pi}{\sqrt{Z_S(\mu)}e_0} \rightarrow b_0 = \frac{4\pi}{e_0} \ , \tag{1.11}$$

and we maintain the argument for charge quantization. We now have a substantial amount of background and can move on to the specific formalism used to show that this big assumption is, in fact, supported by quantum field theory.

Chapter 2

Mathematical Methods and Results

In this section we develop the formal mathematical methods to calculate the running of the electric and magnetic couplings. We do this perturbatively by building Feynman diagrams for specific processes involving electric and magnetic sources. Looking at charge quantization, the electron charge is inversely related to the magnetic charge, if the electron charge is small, the magnetic charge will be massive. However, as is explained in [1] we can get around this large non-perturbative magnetic charge by using a "dark" magnetic monopole whose charge is "suppressed by a dimensionless kinetic mixing". We start with a single potential formalism in standard QED (no magnetic monopoles yet). This is a well-known result and gives us something to check the rest of our work against. We then calculate the same results using field strengths. This field strength formalism is what allows us to calculate with magnetic monopoles relatively easily. Once we have an established method for the field strength calculation, we apply it to QEMD (QED with magnetic monopoles) to calculate the running of the magnetic coupling.

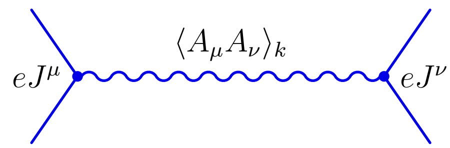

Figure 2.1 One-photon exchange between electric currents.

2.1 Single Potential Formalism for Standard QED

We begin with the standard QED Lagrangian for electric currents [1]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu J^\mu \ , \tag{2.1}$$

We have a term with the coupling, the photon, and the current, which, according to standard Feynman rules , means there is a vertex involving the current and the photon. We also have the kinetic term $F_{\mu\nu}F^{\mu\nu}$ which results in a standard photon propagator [2]:

$$-i\frac{\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2 - i\epsilon} = \langle A_\mu A_\nu \rangle_k \ . \tag{2.2}$$

We start by creating a Feynman diagram for the most basic version of the system described by this Lagrangian: one-photon exchange between electric currents, Fig. 2.1. We can then use what we know about Feynman rules in momentum space to determine the leading order matrix element between two electric currents: $i\mathcal{M}_0^{JJ}$. The Feynman rules for our system are as follows:

- vertex: $-ieJ_e^\mu$
- photon propagator: $-i\dfrac{\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2 - i\epsilon}$

Using these Feynman rules, we get that the zero-order matrix element for photon transfer between two electric currents is:

$$i\mathcal{M}_0^{JJ} = (-ie)J_e^\mu(-i)\frac{\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2 - i\epsilon}(-ie)J_e^\nu \ . \tag{2.3}$$

[1]This corresponds to chapter 4 in Schwinger v. Coleman [1].

[2]A derivation of this can be found in Schwartz [6] section 8.5 “The photon propagator”.

We can use the following identities to simplify this:

- $\eta_{\mu\nu} J_e^\nu = J_{e\mu}$.
- $k_\mu J_e^\mu = 0$, which arises from the fact that electric current is a conserved quantity. In position space, we have $\partial_\mu J^\mu = 0$, but when we do a Fourier transform to momentum space, the ∂_μ becomes ik_μ [6].

Using these two identities, we get:

$$\begin{aligned}
i\mathcal{M}_0^{JJ} &= (-ie)J_e^\mu(-i)\frac{\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2 - i\epsilon}(-ie)J_e^\nu \\
&= \frac{ie^2}{k^2 - i\epsilon}\left(J_e^\mu \eta_{\mu\nu} J_e^\nu - (1-\xi)J_e^\mu \frac{k_\mu k_\nu}{k^2} J_e^\nu\right) \\
&= \frac{ie^2}{k^2 - i\epsilon}\left(J_e^\mu J_{e\mu} - 0\right) \\
&= \frac{ie^2 J_e^\mu J_{e\mu}}{k^2 - i\epsilon}\ .
\end{aligned} \tag{2.4}$$

This matches the standard result in [6].

We can now find the first-order correction to this "correlation function." We again start by making our Feynman diagram. Since we are looking at corrections to the photon propagator, we are going to have essentially the same diagram as Fig. 2.1, but with a loop in the photon propagator that we call $\Pi_J^{\alpha\beta}$, the J indicating it is part of a system that has two electric currents. Our Feynman diagram is Fig. 2.2.

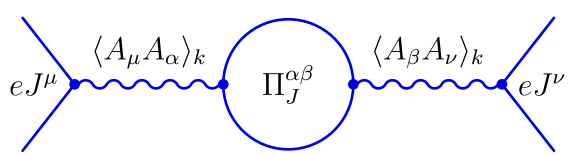

Figure 2.2 First order correction to the photon propagator for one-photon exchange between electric currents.

We can follow a similar process to the first calculation, but now we have an extra piece. We still have $-ieJ_e^\mu$ for the vertices, and $(-i)\frac{\eta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2-i\epsilon}$ for the photon propagators. So our first order correction looks like:

$$i\mathcal{M}_1^{JJ} = (-ie)J_e^\mu(-i)\frac{\eta_{\mu\alpha} - (1-\xi)\frac{k_\mu k_\alpha}{k^2}}{k^2 - i\epsilon} i\Pi_J^{\alpha\beta}(-i)\frac{\eta_{\beta\nu} - (1-\xi)\frac{k_\beta k_\nu}{k^2}}{k^2 - i\epsilon}(-ie)J_e^\nu \ . \tag{2.5}$$

A note: Schwinger v. Coleman [1] drops the $i\epsilon$ terms, for simplicity [1]. From here on out, we do the same thing.

We can now work through simplifying Eq. (2.5), we start by factoring out the is and es expanding it into four terms:

$$\begin{aligned} i\mathcal{M}_1^{JJ} = ie^2 \Bigg(& J_e^\mu \frac{\eta_{\mu\alpha}}{k^2}\Pi_J^{\alpha\beta}\frac{\eta_{\beta\nu}}{k^2}J_e^\nu + J_e^\mu\frac{\eta_{\mu\alpha}}{k^2}\Pi_J^{\alpha\beta}\frac{-(1-\xi)\frac{k_\beta k_\nu}{k^2}}{k^2}J_e^\nu \\ & + J_e^\mu\frac{-(1-\xi)\frac{k_\mu k_\alpha}{k^2}}{k^2}\Pi_J^{\alpha\beta}\frac{\eta_{\beta\nu}}{k^2}J_e^\nu + J_e^\mu\frac{-(1-\xi)\frac{k_\mu k_\alpha}{k^2}}{k^2}\Pi_J^{\alpha\beta}\frac{-(1-\xi)\frac{k_\beta k_\nu}{k^2}}{k^2}J_e^\nu \Bigg) \ . \end{aligned} \tag{2.6}$$

We can see that the second and fourth terms go to zero immediately because $k_\nu J_e^\nu = 0$. In addition, because we are working in momentum space, k_μ is just a number, and thus it commutes with J_e^μ. This leads us to another useful equation $J_e^\mu k_\mu = 0$, which eliminates the third term. We are now left with just the first term, which simplifies to

$$\frac{ie^2}{k^4}J_{e\alpha}\Pi_J^{\alpha\beta}J_{e\beta} \ . \tag{2.7}$$

We now use the Ward Identity to rewrite the propagator $\Pi_J^{\alpha\beta}$ in a slightly more useful form. The Ward Identity states $k_\mu\mathcal{M}^\mu = 0$, where $\mathcal{M}$ is the amplitude for all physical states. As in Eq. (2.6), we can see that $\mathcal{M}$ is proportional to the photon propagator, so to satisfy this, we know our propagator must have the form: $\Pi(k^2)(k^2\eta^{\mu\nu} - k^\mu k^\nu)$. If we act on it with k^μ, we get:

$$\begin{aligned} &= k_\mu(k^2\eta^{\mu\nu} - k^\mu k^\nu) \\ &= k^2k^\nu - k_\mu k^\mu k^\nu \\ &= k^2k^\nu - k^2k^\nu = 0. \end{aligned} \tag{2.8}$$

This satisfies the Ward Identity. This gives us the general form of our photon propagator. We can use this new form to simplify our expression further.

$$\frac{ie^2}{k^4} J_{e\alpha} \Pi_J(k^2)(k^2\eta^{\alpha\beta} - k^\alpha k^\beta) J_{e\beta} \ . \tag{2.9}$$

Again, we know $k^\beta J_{e\beta} = 0$, so the second term goes to zero, and we can evaluate the first term to get:

$$= ie^2 \Pi_J \frac{J_e^\mu J_{e\mu}}{k^2} \ . \tag{2.10}$$

We can now add the first order correction to the tree level value to get:

$$\mathcal{M}_0^{JJ} + \mathcal{M}_{1J}^{JJ} = e^2 \frac{J_e^\mu J_{e\mu}}{k^2}[1 + \Pi_J] \ . \tag{2.11}$$

Remembering that we are essentially creating a perturbation series, we have something of the form: $1 + \epsilon + ...$, which is the Taylor series expansion for $\frac{1}{1-\epsilon}$. We can now rewrite Eq. (2.11), using this idea to get:

$$\mathcal{M}_0^{JJ} + \mathcal{M}_{1J}^{JJ} = \frac{J_e^\mu J_{e\mu}}{k^2} \frac{e^2}{[1 - \Pi_J(k^2)]} = \frac{J_e^\mu J_{e\mu}}{k^2} e_{\mathrm{R}}^2(k^2) \ . \tag{2.12}$$

We can relabel the $\frac{e^2}{[1-\Pi_J(k^2)]}$ as $e_{\mathrm{R}}^2(k^2)$ because e is what is getting renormalized in this equation. The renormalized e depends on energy scale (k^2), and as energy scale increases, e_R^2 increases as well, as expected.

We can also recognize that Eq. (2.12) is the Fourier transform of the Coulomb potential [7, 8], which is another standard result. All of these checks indicate that we are on the right track; we can now move to working in the field strength formalism.

2.2 Field Strength Formalism for Standard QED

While the single potential formalism worked well for QED, it gets complicated fast when we introduce magnetic monopoles. Thus, we develop a formalism that involves the field strengths instead.

The first thing we need to build our matrix element is a way of defining the propagator in terms of field strength. We know $F^{\mu\nu} = k^\mu A^\nu - k^\nu A^\mu$, so we can plug that definition in to:

$$\langle F^{\alpha\beta} F^{\gamma\delta} \rangle, \tag{2.13}$$

to get:

$$\begin{aligned}\langle F^{\alpha\beta} F^{\gamma\delta} \rangle_k &= \langle (k^\alpha A^\beta - k^\beta A^\alpha)(k^\gamma A^\delta - k^\delta A^\gamma) \rangle_k \\ &= k^\alpha k^\gamma \langle A^\beta A^\delta \rangle_k - k^\alpha k^\delta \langle A^\beta A^\gamma \rangle_k - k^\beta k^\gamma \langle A^\alpha A^\delta \rangle_k + k^\beta k^\delta \langle A^\alpha A^\gamma \rangle_k \ .\end{aligned} \tag{2.14}$$

Now the algebra here is non-trivial, and it is not immediately obvious that k commutes with A or can come out of the brackets; a more detailed derivation can be found in [1] Appendix A. We can now write this in terms of our propagator from the last section $-i\frac{\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2 - i\epsilon} = \langle A_\mu A_\nu \rangle_k$. This gives us:

$$\begin{aligned}\langle F^{\alpha\beta} F^{\gamma\delta} \rangle_k &= k^\alpha k^\gamma \left(-i \frac{\eta^{\beta\delta} - (1-\xi)\frac{k^\beta k^\delta}{k^2}}{k^2} \right) - k^\alpha k^\delta \left(-i \frac{\eta^{\beta\gamma} - (1-\xi)\frac{k^\beta k^\gamma}{k^2}}{k^2} \right) \\ &\quad - k^\beta k^\gamma \left(-i \frac{\eta^{\alpha\delta} - (1-\xi)\frac{k^\alpha k^\delta}{k^2}}{k^2} \right) + k^\beta k^\delta \left(-i \frac{\eta^{\alpha\gamma} - (1-\xi)\frac{k^\alpha k^\gamma}{k^2}}{k^2} \right) \\ &= \frac{-i}{k^2} \left(k^\alpha k^\gamma \eta^{\beta\delta} - k^\alpha k^\delta \eta^{\beta\gamma} - k^\beta k^\gamma \eta^{\alpha\delta} + k^\beta k^\delta \eta^{\alpha\gamma} \right) \\ &\quad + \frac{-i(1-\xi)}{k^4} \left(k^\alpha k^\gamma k^\beta k^\delta - k^\alpha k^\delta k^\beta k^\gamma - k^\beta k^\gamma k^\alpha k^\delta + k^\beta k^\delta k^\alpha k^\gamma \right) \\ &= \frac{-i}{k^2} \left(\eta^{\beta\delta} k^\alpha k^\gamma - \eta^{\beta\gamma} k^\alpha k^\delta - \eta^{\alpha\delta} k^\beta k^\gamma + \eta^{\alpha\gamma} k^\beta k^\delta \right) \\ &\quad + \frac{-i(1-\xi)}{k^4} \left(k^\alpha k^\delta k^\beta k^\gamma - k^\alpha k^\delta k^\beta k^\gamma - k^\beta k^\delta k^\alpha k^\gamma + k^\beta k^\delta k^\alpha k^\gamma \right) \\ &= \frac{-i}{k^2} \left(\eta^{\beta\delta} k^\alpha k^\gamma - \eta^{\beta\gamma} k^\alpha k^\delta - \eta^{\alpha\delta} k^\beta k^\gamma + \eta^{\alpha\gamma} k^\beta k^\delta \right) + 0 \ .\end{aligned} \tag{2.15}$$

Where the second term has gone to 0 because the ks commute with each other. We also know from [1] section 3, that the class of operators that map two forms to two forms (like the field strength tensor) is defined by:

$$ {}^{\alpha\beta}_{(q)}P^{\gamma\delta}_{(r)} = \frac{1}{2q \cdot r} (\eta^{\beta\delta} q^\alpha r^\gamma - \eta^{\beta\gamma} q^\alpha r^\delta - \eta^{\alpha\delta} q^\beta r^\gamma + \eta^{\alpha\gamma} q^\beta r^\delta) \ , \tag{2.16}$$

and we can write Eq. (2.15) in terms of this propagator. With $q = r = k$.

$$\langle F^{\alpha\beta} F^{\gamma\delta} \rangle_k = -2 i^{\alpha\beta}_{(k)} P^{\gamma\delta}_{(k)} \ . \tag{2.17}$$

Before we begin calculating he rest of our matrix elements, we need a few identities to make our lives easier. These all came from [1] section 3. We know from this section that:

$$_{\mu\nu}I_{\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}\right), \tag{2.18}$$

so $F_{\mu\nu}$ can be written in terms of this identity element:

$$F_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} + F_{\mu\nu}) = \frac{1}{2}(F_{\mu\nu} - F_{\nu\mu}) = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}\right) F^{\alpha\beta} =_{\mu\nu} I_{\alpha\beta} F^{\alpha\beta}. \tag{2.19}$$

The third equality comes from the fact that $F_{\mu\nu}$ is antisymmetric. We also know from Section 3 of [1], that the identity element can be written as:

$$_{\mu\nu}I_{\alpha\beta} = \ {}^{(k)}_{\mu\nu}P^{(k)}_{\alpha\beta} -^{(k)}_{*\mu\nu} P^{(k)}_{*\alpha\beta} \ . \tag{2.20}$$

Combining (2.19) with (2.20) we get:

$$F_{\mu\nu} = \ \left({}^{(k)}_{\mu\nu}P^{(k)}_{\alpha\beta} -^{(k)}_{*\mu\nu} P^{(k)}_{*\alpha\beta}\right) F^{\alpha\beta}. \tag{2.21}$$

However, we can think of ${}^{(k)}_{\mu\nu}P^{(k)}_{\alpha\beta}$ as a sort of projector onto a space, and ${}^{(k)}_{*\mu\nu}P^{(k)}_{*\alpha\beta}$ as the projector onto the orthogonal space, so they do not touch. The object ${}^{(k)}_{\mu\nu}P^{(k)}_{\alpha\beta}$ projects into the space of objects that are wedge products with k_μ. Since $F_{\mu\nu}$ is of the form $k \wedge A$, it lives in this space and,

$$\left({}^{(k)}_{*\mu\nu}P^{(k)}_{*\alpha\beta}\right) F^{\alpha\beta} = 0 \ . \tag{2.22}$$

So we can write that,

$$F_{\mu\nu} = \ \left({}^{(k)}_{\mu\nu}P^{(k)}_{\alpha\beta}\right) F^{\alpha\beta}. \tag{2.23}$$

We can now turn our attention to the sourceless Lagrangian, which we can write as:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{\alpha\beta} \ {}^{\alpha\beta}I^{\gamma\delta}F_{\gamma\delta} = -\frac{1}{4}F_{\alpha\beta} \ {}^{\alpha\beta}_{(k)}P^{\gamma\delta}_{(k)}F_{\gamma\delta} \ . \tag{2.24}$$

Which is great, but we still need a source (something like our $-eA_\mu J^\mu$ term in our original Lagrangian Eq. (2.1)). This is a little tricky because it needs to be able to couple to this "two-form", or two-index object, so we can't just use J^μ like before, since that only has one index. To find something suitable, we can look to the normal Maxwell equation in momentum space:

$$ik_\nu F^{\mu\nu} = eJ^\mu \ . \tag{2.25}$$

The useful solution to this equation, chosen in [1], is

$$F_J^{\mu\nu} = -ie\frac{J^\mu n^\nu - J^\nu n^\mu}{n \cdot k} \ . \tag{2.26}$$

Where n is a vector that points normal to the surface enclosed by the current. These surfaces enclosed by the currents give rise to a topological term, which is related to the intersections of those surfaces in 4D spacetime. [9, 10]. We can use this definition to rewrite the $-eA_\mu J^\mu$ term in the Lagrangian.

$$-eA_\mu J^\mu \longrightarrow -eA_\mu \partial_\nu F_J^{\mu\nu} \longrightarrow e\partial_\nu(A_\mu)F_J^{\mu\nu} \ . \tag{2.27}$$

In the first arrow, we have used the Maxwell equations in tensor notation $\partial_\nu F_J^{\mu\nu} = J^\mu$. In the second arrow, we have used integration by parts. Because we technically are dealing with a Lagrangian density that is integrated over, we can perform integration by parts to move the derivative from $F_J^{\mu\nu}$ to A_μ.

We know

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{2.28}$$

and

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu \ , \tag{2.29}$$

since $\partial_\mu A_\nu$ is antisymmetric in μ and ν we can flip the indices and multiply by -1, which gives:

$$F_{\nu\mu} = \partial_\nu A_\mu + \partial_\nu A_\mu = 2\partial_\nu A_\mu \tag{2.30}$$

so $\partial_\nu A_\mu = \frac{1}{2}F_{\nu\mu}$.

We can now plug in the definition of $F_J^{\mu\nu}$ and write $\partial_\nu(A_\mu)$ in terms of $F_{\nu\mu}$ to get:

$$e\partial_\nu(A_\mu)F_J^{\mu\nu} = \frac{1}{2}F_{\nu\mu}(-ie)\frac{J^\mu n^\nu - J^\nu n^\mu}{n\cdot k} = \frac{ie}{2}F_{\mu\nu}\frac{J^\mu n^\nu - J^\nu n^\mu}{n\cdot k}\ . \tag{2.31}$$

Where we have again used the anti-symmetry of $F^{\mu\nu}$ to write $F_{\nu\mu} = -F_{\mu\nu}$. We should also note that all of the previous calculations of the source term have been done in momentum space, which gives us the Feynman rules for the field strength generated by electric currents. We can then move to position space to get the associated term in the Lagrangian. Remember that during a Fourier transform $ik_\mu \to \partial_\mu$, so we get:

$$\frac{-e}{2}F_{\mu\nu}\frac{J^\mu n^\nu - J^\nu n^\mu}{n\cdot ik} \to \frac{-e}{2}F_{\mu\nu}\frac{J^\mu n^\nu - J^\nu n^\mu}{n\cdot \partial}\ . \tag{2.32}$$

We can now write our new QED Lagrangian in terms of the field strengths only, with no potentials involved:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{e}{2}F_{\mu\nu}\frac{J^\mu n^\nu - J^\nu n^\mu}{n\cdot \partial}\ . \tag{2.33}$$

Using this Lagrangian and its associated momentum space Feynman rules, we can calculate the amplitude of the tree-level diagram Fig. 2.3.

This leads us to a new set of Feynman rules for this formalism:

- vertex: $(\pm e)\dfrac{J^\alpha n_1^\beta - J^\beta n_1^\alpha}{2n_1\cdot k}$. It is $+e$ when the momentum k^μ is flowing into the vertex, and $-e$ when the momentum is flowing out of the vertex.

- propagator: $(-2i)^{(k)}_{\alpha\beta}P^{(k)}_{\gamma\delta}$

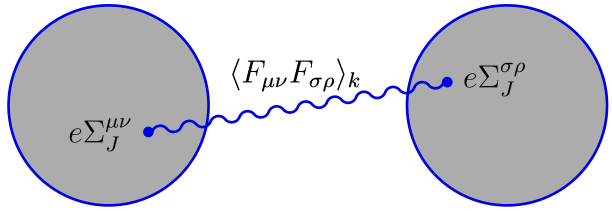

Figure 2.3 One-photon exchange between electric currents (blue) using field strengths. $e\Sigma_J^{\mu\nu}$ is just a label for what the field strength couples to, it could also be called $\frac{J^\alpha n_1^\beta - J^\beta n_1^\alpha}{2n_1 \cdot k}$

.

2.2.1 Tree level calculation

We start by using our new Feynman rules to get the bones of the matrix element down, we get:

$$i\mathcal{M}_0^{JJ} = (-e)\frac{J^\alpha n_1^\beta - J^\beta n_1^\alpha}{2n_1 \cdot k}(-2i)^{(k)}_{\alpha\beta}P^{(k)}_{\gamma\delta}(e)\frac{J^\gamma n_2^\delta - J^\delta n_2^\gamma}{2n_2 \cdot k} \ , \tag{2.34}$$

We can now use the identity given in section 3 of [1], plugging in $q = r = s$:

$$(s^\alpha t^\beta - s^\beta t^\alpha)^{(k)}_{\alpha\beta}P^{(k)}_{\gamma\delta} = \frac{k \cdot t}{k^2}(s_\gamma k_\delta - s_\delta k_\gamma) + \frac{k \cdot s}{k^2}(k_\gamma t_\delta - k_\delta t_\gamma) \ , \tag{2.35}$$

we use this to evaluate the propagator on the left vertex, to get:

$$(J^\alpha n_1^\beta - J^\beta n_1^\alpha)^{(k)}_{\alpha\beta}P^{(k)}_{\gamma\delta} = \frac{k \cdot n_1}{k^2}(J_\gamma k_\delta - J_\delta k_\gamma) + \frac{k \cdot J}{k^2}(k_\gamma n_{1\delta} - k_\delta n_{1\gamma}) \ , \tag{2.36}$$

we know that $k \cdot J = 0$ so the entire second term goes to 0. We can then plug this back in to our original amplitude equation (2.34).

$$i\mathcal{M}_0^{JJ} = (-e)\frac{\frac{k\cdot n_1}{k^2}(J_\gamma k_\delta - J_\delta k_\gamma)}{2n_1 \cdot k}(-2i)(e)\frac{(J^\gamma n_2^\delta - J^\delta n_2^\gamma)}{2n_2 \cdot k} \ . \tag{2.37}$$

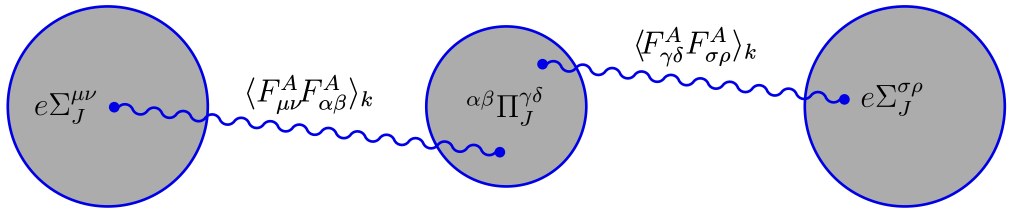

Figure 2.4 One electric loop correction to photon exchange between electric currents using field strengths.

We can now simplify the coefficients and expand the numerator.

$$\begin{aligned}
i\mathcal{M}_0^{JJ} &= \frac{2ie^2(k\cdot n_1)}{4k^2(n_1\cdot k)(n_2\cdot k)}(J_\gamma k_\delta - J_\delta k_\gamma)(J^\gamma n_2^\delta - J^\delta n_2^\gamma) \\
&= \frac{ie^2}{2k^2(n_2\cdot k)}(J_\gamma k_\delta J^\gamma n_2^\delta - J_\delta k_\gamma J^\gamma n_2^\delta - J_\gamma k_\delta J^\delta n_2^\gamma + J_\delta k_\gamma J^\delta n_2^\gamma) \\
&= \frac{ie^2}{2k^2(n_2\cdot k)}(J^2(k\cdot n_2) - 0 - 0 + J^2(k\cdot n_2)) \\
&= \frac{ie^2 2J^2(n_2\cdot k)}{2k^2(n_2\cdot k)} \\
&= \frac{ie^2 J^2}{k^2}\ .
\end{aligned} \tag{2.38}$$

This result is the exact same as the one we got for the single potential calculation, which is a good sign. It is also important to note that there is no n dependence in this result, which means in future calculations we do not differentiate between n_1 and n_2. We can now calculate the one-loop contribution to this process, and again, check to see if it matches the single potential (it should).

2.2.2 One-loop calculation

Looking at Fig.2.4, we can build our matrix element out of our Feynman rules:

$$i\mathcal{M}_1^{JJ} = (-e)\frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n\cdot k}(-2i)^{(k)}_{\alpha\beta}P^{(k)}_{\mu\nu}(i\ {}^{\mu\nu}\Pi^{\sigma\rho})(-2i)^{(k)}_{\sigma\rho}P^{(k)}_{\gamma\delta}(e)\frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n\cdot k}\ . \tag{2.39}$$

We already know how to evaluate most of this, but we need to get Π to a better form so it is easier to evaluate. Looking at the diagram there are two points where the field strength attaches to the loop, we can write these in the way we normally write vertices,

$$\frac{n^\mu\eta^{\nu\alpha}-n^\nu\eta^{\mu\alpha}}{2n\cdot k}\frac{n^\sigma\eta^{\rho\beta}-n^\rho\eta^{\sigma\beta}}{2n\cdot k}\ . \tag{2.40}$$

These tell us the vertices that are on the loop but we still need a rule for the loop itself, which we again get from the Ward Identity, so the full expression for Π is:

$$^{\mu\nu}\Pi^{\sigma\rho}=\frac{n^\mu\eta^{\nu\alpha}-n^\nu\eta^{\mu\alpha}}{2n\cdot k}\frac{n^\sigma\eta^{\rho\beta}-n^\rho\eta^{\sigma\beta}}{2n\cdot k}\Pi(k^2)(k^2\eta_{\alpha\beta}-k_\alpha k_\beta)\ . \tag{2.41}$$

We can now expand the numerator and do some additional algebra to get:

$$\begin{aligned}
^{\mu\nu}\Pi^{\sigma\rho}=&\frac{\Pi(k^2)}{4(n\cdot k)(n\cdot k)}(n^\mu\eta^{\nu\alpha}-n^\nu\eta^{\mu\alpha})(n^\sigma\eta^{\rho\beta}-n^\rho\eta^{\sigma\beta})(k^2\eta_{\alpha\beta}-k_\alpha k_\beta)\\
=&\frac{\Pi(k^2)}{4(n\cdot k)(n\cdot k)}(n^\mu\eta^{\nu\alpha}n^\sigma\eta^{\rho\beta}k^2\eta_{\alpha\beta}-n^\nu\eta^{\mu\alpha}n^\sigma\eta^{\rho\beta}k^2\eta_{\alpha\beta}-n^\mu\eta^{\nu\alpha}n^\rho\eta^{\sigma\beta}k^2\eta_{\alpha\beta}\\
&+n^\nu\eta^{\mu\alpha}n^\rho\eta^{\sigma\beta}k^2\eta_{\alpha\beta}-n^\mu\eta^{\nu\alpha}n^\sigma\eta^{\rho\beta}k_\alpha k_\beta+n^\nu\eta^{\mu\alpha}n^\sigma\eta^{\rho\beta}k_\alpha k_\beta\\
&+n^\mu\eta^{\nu\alpha}n^\rho\eta^{\sigma\beta}k_\alpha k_\beta-n^\nu\eta^{\mu\alpha}n^\rho\eta^{\sigma\beta}k_\alpha k_\beta)\\
=&\frac{\Pi(k^2)}{4(n\cdot k)(n\cdot k)}(n^\mu n^\sigma k^2\eta^{\nu\rho}-n^\nu n^\sigma k^2\eta^{\mu\rho}-n^\mu n^\rho k^2\eta^{\nu\sigma}\\
&+n^\nu n^\rho k^2\eta^{\mu\sigma}-n^\mu n^\sigma k^\nu k^\rho+n^\nu n^\sigma k^\mu k^\rho+n^\mu n^\rho k^\nu k^\sigma-n^\nu n^\rho k^\mu k^\sigma)\\
=&\frac{\Pi(k^2)}{4(n\cdot k)(n\cdot k)}[k^2(n^\mu n^\sigma\eta^{\nu\rho}-n^\nu n^\sigma\eta^{\mu\rho}-n^\mu n^\rho\eta^{\nu\sigma}+n^\nu n^\rho\eta^{\mu\sigma})\\
&-(n^\mu k^\nu-n^\nu k^\mu)(n^\sigma k^\rho-n^\rho k^\sigma)]\ .
\end{aligned} \tag{2.42}$$

We can now recognize the form of $(2n^2)^{\mu\nu}_{(n)}P^{\sigma\rho}_{(n)}$ in the first term to get:

$$\begin{aligned}
^{\mu\nu}\Pi^{\sigma\rho}&=\frac{\Pi(k^2)}{4(n\cdot k)^2}[(2k^2n^2)^{\mu\nu}_{(n)}P^{\sigma\rho}_{(n)}-(n^\mu k^\nu-n^\nu k^\mu)(n^\sigma k^\rho-n^\rho k^\sigma)]\\
&=\frac{1}{2}\Pi(k^2)\left[\frac{(k^2n^2)}{(n\cdot k)^2}{}^{\mu\nu}_{(n)}P^{\sigma\rho}_{(n)}-\frac{1}{2(n\cdot k)^2}(n^\mu k^\nu-n^\nu k^\mu)(n^\sigma k^\rho-n^\rho k^\sigma)\right]\ ,
\end{aligned} \tag{2.43}$$

and use equation 38 from [1]:

$$^{(q)}_{\mu\nu}P^{\gamma\delta}_{(r)}\,{}^{(r)}_{\gamma\delta}P^{(q)}_{\sigma\rho}=\frac{q^2r^2}{(q\cdot r)^2}\,{}^{(q)}_{\mu\nu}P^{(q)}_{\sigma\rho}-\frac{1}{2(q\cdot r)^2}\left(q_\mu r_\nu-q_\nu r_\mu\right)\left(q_\sigma r_\rho-q_\rho r_\sigma\right)\ , \tag{2.44}$$

plugging in n and k for q and r and inserting an identity to raise the indices we get:

$$ {}^{\mu\nu}_{(n)}P^{\gamma\delta}_{(k)}{}_{\gamma\delta}I_{\alpha\beta}{}^{\alpha\beta}_{(k)}P^{\sigma\rho}_{(n)} = \frac{n^2k^2}{(n\cdot k)^2}{}^{\mu\nu}_{(n)}P^{\sigma\rho}_{(n)} - \frac{1}{2(n\cdot k)^2}\left(n^\mu k^\nu - n^\nu k^\mu\right)\left(n^\sigma k^\rho - n^\rho k^\sigma\right) \ , \tag{2.45} $$

we can now see how we can pull this out of Eq. (2.43) to get:

$$ {}^{\mu\nu}\Pi^{\sigma\rho} = \frac{1}{2}\Pi(k^2){}^{\mu\nu}_{(n)}P^{\gamma\delta}_{(k)}{}_{\gamma\delta}I_{\alpha\beta}{}^{\alpha\beta}_{(k)}P^{\sigma\rho}_{(n)} \ . \tag{2.46} $$

Looking back at Eq. (2.39) we can see that ${}^{\mu\nu}\Pi^{\sigma\rho}$ is sandwiched between two propagators:

$$ {}^{(k)}_{\alpha\beta}P^{(k)\mu\nu}_{\mu\nu\ (n)}P^{\theta\phi}_{(k)}{}_{\theta\phi}I_{\xi\zeta}{}^{\xi\zeta}_{(k)}P^{\sigma\rho\,(k)}_{(n)\sigma\rho}P^{(k)}_{\gamma\delta} \ . \tag{2.47} $$

We have changed the indices on the identity so as to not have more than two of a kind in one term, they are just dummy indices so we can do this. Looking to Eq. (39) and (48) from [1], which are written below, we can simplify Eq. (2.47):

$$ {}^{(q)}_{\alpha\beta}P^{\mu\nu}_{(q)}{}^{(r)}_{\mu\nu}P^{(q)}_{\gamma\delta} = {}^{(q)}_{\alpha\beta}P^{\mu\nu}_{(r)}{}^{(q)}_{\mu\nu}P^{(q)}_{\gamma\delta} = {}^{(q)}_{\alpha\beta}P^{(q)}_{\gamma\delta} \ , \tag{2.48} $$

and

$$ {}^{(k)}_{\alpha\beta}P^{\mu\nu}_{(k)}{}^{(k)}_{\mu\nu}P^{(k)}_{\gamma\delta} = {}^{(k)}_{\alpha\beta}P^{(k)}_{\gamma\delta} \ . \tag{2.49} $$

Using Eq. (2.48) twice we can simplify the two sets of Ps on each end to get

$$ {}^{(k)}_{\alpha\beta}P^{\theta\phi}_{(k)\theta\phi}I_{\xi\zeta}{}^{\xi\zeta}_{(k)}P^{(k)}_{\gamma\delta} \ . \tag{2.50} $$

We can now evaluate the identity and use Eq. (2.49) to get:

$$ {}^{(k)}_{\alpha\beta}P^{\theta\phi}_{(k)\theta\phi}I_{\xi\zeta}{}^{\xi\zeta}_{(k)}P^{(k)}_{\gamma\delta} = {}^{(k)}_{\alpha\beta}\ P^{\theta\phi\,(k)}_{(k)\theta\phi}P^{(k)}_{\gamma\delta} = {}^{(k)}_{\alpha\beta}P^{(k)}_{\gamma\delta} \ . \tag{2.51} $$

Plugging this back into our matrix element Eq. (2.39), we get:

$$ \begin{aligned} i\mathcal{M}_1^{JJ} &= (-e)\frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n\cdot k}(-2i)(i)(-2i)\frac{1}{2}\Pi(k^2){}^{(k)}_{\alpha\beta}P^{(k)}_{\gamma\delta}e\frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n\cdot k} \\ &= 2ie^2\Pi(k^2)\frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n\cdot k}{}^{(k)}_{\alpha\beta}P^{(k)}_{\gamma\delta}\frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n\cdot k} \ , \end{aligned} \tag{2.52} $$

which is identical to the tree-level calculation (Eq. (2.34)), so we can use that result to get:

$$i\mathcal{M}_1^{JJ} = ie^2\Pi(k^2)\frac{J^2}{k^2} \ . \tag{2.53}$$

This matches what we got for the single potential formalism exactly! Now we can use this field strength formalism to deal with magnetic charges as well.

2.3 Field Strength Formalism in QEMD

We can now begin to include magnetic monopoles; we can assume that they are also sourced by a magnetic current K^μ [1]. We start by assuming that $F_{\mu\nu}$ is unchanged by the addition of magnetic monopoles to our theory.

We define a source term in the same way $ib^*F_{\mu\nu}\frac{K^\mu n^\nu - K^\nu n^\mu}{2n\cdot k}$, which assumes that the relationship between ${}^*F_{\mu\nu}$ and K^μ is the same as the relationship between $F_{\mu\nu}$ and J^μ. If we were to do this and calculate the tree level amplitude of the photon exchange between two magnetic currents we would get:

$$\begin{aligned} i\mathcal{M}_0^{KK} &= (-b)\frac{K^\alpha n^\beta - K^\beta n^\alpha}{2n\cdot k}(-2i)^{(k)}_{*\alpha\beta}P^{(k)}_{*\gamma\delta}(b)\frac{K^\gamma n^\delta - K^\delta n^\gamma}{2n\cdot k} \\ &= (2ib^2)\frac{K^\alpha n^\beta - K^\beta n^\alpha}{2n\cdot k}\left({}^{(k)}_{*\alpha\beta}P^{(k)}_{*\gamma\delta} - {}_{\alpha\beta}I_{\gamma\delta}\right)\frac{K^\gamma n^\delta - K^\delta n^\gamma}{2n\cdot k} \\ &= ib^2\frac{K^2}{k^2} - ib^2\frac{K^2n^2 - (K\cdot n)^2}{2(n\cdot k)^2} \ . \end{aligned} \tag{2.54}$$

Where we have used Eq. (43) from [1] in the second equality.

However, this doesn't match our previous assumption, that magnetic-magnetic scattering works in the same way as electric-electric scattering. We have an extra term, that [1] calls a "contact term." In order to deal with this we have to take note of our assumptions we have made thus far, we are assuming that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, however, this leads to ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$, when we take a derivative of this, we get a rather unfortunate answer:

$$\begin{aligned}
\partial_\mu {}^*F^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\
&= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(\partial_\mu\partial_\alpha A_\beta - \partial_\mu\partial_\beta A_\alpha) \\
&= \frac{1}{2}(\epsilon^{\mu\nu\alpha\beta}\partial_\alpha\partial_\mu A_\beta - \epsilon^{\mu\nu\alpha\beta}\partial_\beta\partial_\mu A_\alpha) \\
&= \frac{1}{2}(\epsilon^{\mu\nu\alpha\beta}\partial_\alpha\partial_\mu A_\beta - \epsilon^{\mu\nu\beta\alpha}\partial_\alpha\partial_\mu A_\beta) \\
&= \frac{1}{2}(\epsilon^{\mu\nu\alpha\beta}\partial_\alpha\partial_\mu A_\beta + \epsilon^{\mu\nu\alpha\beta}\partial_\alpha\partial_\mu A_\beta) \\
&= \epsilon^{\mu\nu\alpha\beta}\partial_\alpha\partial_\mu A_\beta = 0 \ .
\end{aligned} \tag{2.55}$$

In the last line, we have something antisymmetric times something symmetric, which is always 0, so with our current definition $\partial_\mu {}^*F^{\mu\nu} = 0$. If we want to include magnetic monopoles, we need to have something that is sourcing them. This could have the form $\partial_\mu {}^*F^{\mu\nu} = bK^\nu$.

After some trial and error we can define

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{b}{2}\epsilon_{\mu\nu\alpha\beta}\frac{n^\alpha K^\beta - n^\beta K^\alpha}{n\cdot\partial} \ , \tag{2.56}$$

so that

$${}^*F^{\mu\nu} = b\frac{n^\mu K^\nu - n^\nu K^\mu}{n\cdot\partial} + \epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta \ , \tag{2.57}$$

which gives

$$\partial_\mu {}^*F^{\mu\nu} = bK^\nu \ . \tag{2.58}$$

which is exactly what we want.

Now that we have a new definition of $F_{\mu\nu}$ we can plug it back in to our original Lagrangian for QED $\mathcal{L} = F_{\mu\nu}F^{\mu\nu} - eA_\mu J^\mu$ to get:

$$\mathcal{L} = -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - eA_\mu J^\mu + b\partial_\beta(A_\mu)\frac{1}{n\cdot\partial}\epsilon^{\mu\nu\alpha\beta}K_\nu n_\alpha + \frac{b^2}{2(n\cdot\partial)^2}[n^2K^2 - (n\cdot K)^2] \ . \tag{2.59}$$

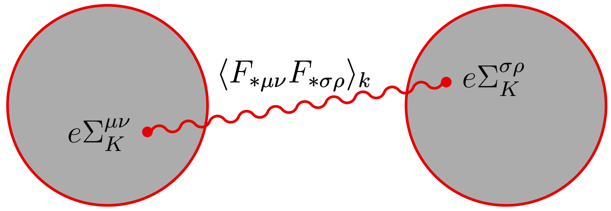

Figure 2.5 One-photon exchange between magnetic currents using field strengths. $e\Sigma_K^{\mu\nu}$ is just a label for what the field strength couples to, it could also be called $\frac{K^\alpha n_1^\beta - K\beta n_1^\alpha}{2n_1 \cdot k}$

.

Or with the field strengths instead of the vector potential we get:

$$\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} + eF^A_{\mu\nu} \frac{J^\mu n^\nu - J^\nu n^\mu}{2n \cdot \partial} + b^* F^A_{\mu\nu} \frac{K^\mu n^\nu - K^\nu n^\mu}{2n \cdot \partial} + b^2 \frac{n^2 K^2 - (n \cdot K)^2}{2(n \cdot \partial)^2} \ . \quad (2.60)$$

We can now use Fourier transforms to get the Feynman rules for this system:

- vertex: $ib \dfrac{K_e^\alpha n^\beta - K_e^\beta n^\alpha}{2n \cdot k}$
- propagator: $-2i^{(k)}_{*\alpha\beta} P^{(k)}_{*\gamma\delta}$
- contact term: $ib^2 \dfrac{K_e^2 n^2 - (K_e \cdot n)^2}{(n \cdot k)^2}$

These, along with our Feynman diagram (Fig. 2.5), allow us to build our tree-level matrix element for magnetic-magnetic photon transfer.

$$\begin{aligned} i\mathcal{M}_0^{KK} &= ib \frac{K_e^\alpha n^\beta - K_e^\beta n^\alpha}{2n \cdot k} (-2i)^{(k)}_{*\alpha\beta} P^{(k)}_{*\gamma\delta} (ib) \frac{K_e^\gamma n^\delta - K_e^\delta n^\gamma}{2n \cdot k} + ib^2 \frac{K_e^2 n^2 - (K_e \cdot n)^2}{(n \cdot k)^2} \\ &= 2ib^2 \frac{K_e^\alpha n^\beta - K_e^\beta n^\alpha}{2n \cdot k} \left({}^{\alpha\beta}_{(k)} P^{(k)}_{\gamma\delta} - {}_{\alpha\beta} I_{\gamma\delta} \right) \frac{K_e^\gamma n^\delta - K_e^\delta n^\gamma}{2n \cdot k} + ib^2 \frac{K_e^2 n^2 - (K_e \cdot n)^2}{(n \cdot k)^2} \\ &= ib^2 \frac{K_e^2}{k^2} - ib^2 \frac{K_e^2 n^2 - (K_e \cdot n)^2}{(n \cdot k)^2} + ib^2 \frac{K_e^2 n^2 - (K_e \cdot n)^2}{(n \cdot k)^2} = ib^2 \frac{K_e^\mu K_{e\mu}}{k^2} \ . \end{aligned} \quad (2.61)$$

This matches what we would expect to get if magnetic-magnetic works the same as electric-electric at tree level, see Eq. (2.38).

2.3.1 One-loop calculation

There are four different one-loop diagrams:

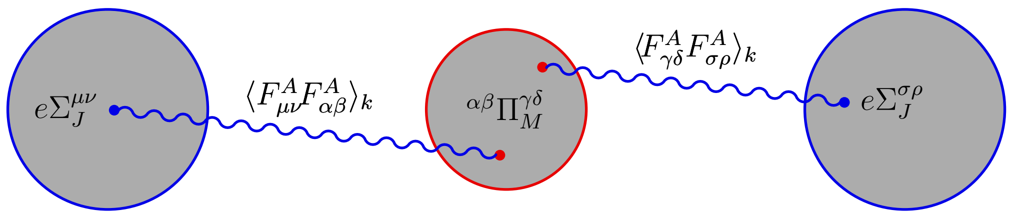

Figure 2.6 One magnetic loop correction to photon exchange between electric currents using field strengths.

- Magnetic currents with a magnetic loop
- Magnetic currents with an electric loop
- Electric currents with an electric loop (Fig. 2.4)
- Electric currents with a magnetic loop (Fig. 2.6)

We have already calculated the one-loop diagram for electric currents with an electric loop, so we are now calculating the one-magnetic-loop correction to electric-electric interactions as in Fig. 2.6. This gives us the full picture of what is happening for the running of the electric coupling

$$i\mathcal{M}_{1K}^{JJ} = (-e)\frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n\cdot k}(-2i)^{(k)}_{\alpha\beta}P^{(k)}_{*\mu\nu}i^{\mu\nu}\Pi^{\sigma\rho}(-2i)^{(k)}_{*\sigma\rho}P^{(k)}_{\gamma\delta}(e)\frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n\cdot k}\ . \tag{2.62}$$

Since we already solved for ${}^{\mu\nu}\Pi^{\sigma\rho}$ we can plug in Eq. (2.46) to get:

$$i\mathcal{M}_{1K}^{JJ} = 2ie^2\Pi_K\frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n\cdot k}{}^{(k)}_{\alpha\beta}P^{(k)\ \mu\nu}_{*\mu\nu\ (n)}P^{\gamma\delta}_{(k)\ \gamma\delta}I_{\alpha\beta}{}^{\alpha\beta}_{(k)}P^{\sigma\rho\,(k)}_{(n)*\sigma\rho}P^{(k)}_{\gamma\delta}\frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n\cdot k}\ . \tag{2.63}$$

Now, for the tricky part, there are a couple of equivalent ways we can write the combination of propagators surrounding the identity. In [1] Eq. (47) states:

$${}^{(q)}_{\alpha\beta}P^{\mu\nu}_{(q)}{}^{(r)}_{*\mu\nu}P^{(q)}_{\gamma\delta} = {}^{(q)}_{\alpha\beta}P^{*\gamma\delta}_{(q)} - {}^{(q)}_{\alpha\beta}P^{*\gamma\delta}_{(r)} = -{}^{*\alpha\beta}_{(q)}P^{(q)}_{\gamma\delta} + {}^{*\alpha\beta}_{(r)}P^{(q)}_{\gamma\delta}\ . \tag{2.64}$$

Or adapting it to what we see in our term:

$${}^{(k)}_{\alpha\beta}P^{(k)\ \mu\nu}_{*\mu\nu\ (n)}P^{\gamma\delta}_{(k)} = {}^{(k)}_{\alpha\beta}P^{*\gamma\delta}_{(k)} - {}^{(k)}_{\alpha\beta}P^{*\gamma\delta}_{(n)} = -{}^{(k)}_{*\alpha\beta}P^{\gamma\delta}_{(k)} + {}^{(n)}_{*\alpha\beta}P^{\gamma\delta}_{(k)}\ . \tag{2.65}$$

Looking at these two ways of expressing the propagator, we can see that the second term on the right in each case is where the n^μ dependence lies. That is the topological term we wish to remove. We wish to remove this because the topological terms are not physical. The question asked in [1] is which definition we should use, since they are equal. They explain that if we want to correctly remove the topological dependence we need to understand how the J terms interact with ${}^{(k)}_{*\alpha\beta}P^{\gamma\delta}_{(k)}$ vs ${}^{(k)}_{\alpha\beta}P^{*\gamma\delta}_{(k)}$.

We know that J exists in a mathematical space. The propagators that pull from the dual to this mathematical space returns 0 when acting on J because no part of J is in the dual space. This means that:

$$\frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n \cdot k} {}^{(n)}_{*\alpha\beta}P^{\gamma\delta}_{(k)} = 0 \ . \tag{2.66}$$

As is stated in [1], while this seems convenient, this goes to zero regardless of what we do, so it is essentially equal to the other option. If we really want to get rid of the topological terms, we need to use the first equality in Eq. (2.65) so that there is a non-zero term we eliminate. This means we should write

$$i\mathcal{M}^{JJ}_{1K} = 2ie^2\Pi_K \frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n \cdot k} \left({}^{(k)}_{\alpha\beta}P^{*\gamma\delta}_{(k)} - {}^{(k)}_{\alpha\beta}P^{*\gamma\delta}_{(n)} \right) {}_{\gamma\delta}I_{\alpha\beta} {}^{\alpha\beta}_{(k)}P^{\sigma\rho\,(k)}_{(n)*\sigma\rho}P^{(k)}_{\gamma\delta} \frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n \cdot k} \ , \tag{2.67}$$

removing the topological term, we get:

$$i\mathcal{M}^{JJ}_{1K} = 2ie^2\Pi_K \frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n \cdot k} \left({}^{(k)}_{\alpha\beta}P^{*\gamma\delta}_{(k)} \right) {}_{\gamma\delta}I_{\alpha\beta} {}^{\alpha\beta}_{(k)}P^{\sigma\rho\,(k)}_{(n)*\sigma\rho}P^{(k)}_{\gamma\delta} \frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n \cdot k} \ . \tag{2.68}$$

Looking at the propagators on the right, we can follow a similar logic, though this time we choose the other definition in Eq. (2.65), because the propagator is acting to the right in this one. In [1] there is a typo in Eq. (106), and Eq. (107); they are all off by an overall minus sign; we have fixed it here. We can use

$${}^{\alpha\beta}_{(k)}P^{\sigma\rho\,(k)}_{(n)*\sigma\rho}P^{(k)}_{\gamma\delta} = -{}^{\alpha\beta}_{(k)}P^{(k)}_{*\gamma\delta} + {}^{\alpha\beta}_{(k)}P^{(n)}_{*\gamma\delta} = {}^{*\alpha\beta}_{(k)}P^{(k)}_{\gamma\delta} - {}^{*\alpha\beta}_{(n)}P^{(k)}_{\gamma\delta} \ , \tag{2.69}$$

to get

$$i\mathcal{M}^{JJ}_{1K} = 2ie^2\Pi_K \frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n\cdot k} \left({}^{(k)}_{\alpha\beta}P^{*\gamma\delta}_{(k)}\right)_{\gamma\delta} I_{\alpha\beta} \left({}^{*\alpha\beta}_{(k)}P^{(k)}_{\gamma\delta} - {}^{*\alpha\beta}_{(n)}P^{(k)}_{\gamma\delta}\right) \frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n\cdot k} \ . \tag{2.70}$$

Removing the topological term:

$$\begin{aligned} i\mathcal{M}^{JJ}_{1K} &= 2ie^2\Pi_K \frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n\cdot k} \left({}^{(k)}_{\alpha\beta}P^{*\gamma\delta}_{(k)}\right)_{\gamma\delta} I_{\alpha\beta} \left({}^{*\alpha\beta}_{(k)}P^{(k)}_{\gamma\delta}\right) \frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n\cdot k} \\ &= 2ie^2\Pi_K \frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n\cdot k} \left({}^{(k)}_{\alpha\beta}P^{*\sigma\rho}_{(k)}\right) \left({}^{(k)}_{*\sigma\rho}P^{(k)}_{\gamma\delta}\right) \frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n\cdot k} \\ &= -2ie^2\Pi_K \frac{J^\alpha n^\beta - J^\beta n^\alpha}{2n\cdot k} \left({}^{(k)}_{\alpha\beta}P^{(k)}_{\gamma\delta}\right) \frac{J^\gamma n^\delta - J^\delta n^\gamma}{2n\cdot k} \\ &= -ie^2\Pi_K \frac{J^2}{k^2} \ . \end{aligned} \tag{2.71}$$

This is significant; there is no extra "contact term", and it has almost the exact same form as the one-electric loop diagram. Adding up all of the diagrams for electric-electric thus far, we get:

$$\mathcal{M}^{JJ}_0 + \mathcal{M}^{JJ}_{1J} + \mathcal{M}^{JJ}_{1K} = e^2\frac{J^2}{k^2}[1 + \Pi_J - \Pi_K] \ . \tag{2.72}$$

Which we can again treat as a Taylor Expansion to get:

$$\mathcal{M}^{JJ}_0 + \mathcal{M}^{JJ}_{1J} + \mathcal{M}^{JJ}_{1K} = e^2\frac{J^2}{k^2}[1 + \Pi_J - \Pi_K] \approx J^2\frac{e^2}{k^2[1 - \Pi_J + \Pi_K]} \ . \tag{2.73}$$

We can repeat this calculation for the instance of a magnetic current with a magnetic loop and a magnetic current with an electric loop. We find that the total contribution is:

$$\mathcal{M}^{KK}_0 + \mathcal{M}^{KK}_{1J} + \mathcal{M}^{KK}_{1K} = b^2\frac{K^2}{k^2}[1 - \Pi_J + \Pi_K] \ . \tag{2.74}$$

We can then see that

$$e_R = \frac{e^2}{[1 - \Pi_J + \Pi_K]} \ , \tag{2.75}$$

and

$$b_R = b^2[1 - \Pi_J + \Pi_K] \ . \tag{2.76}$$

Which is exactly in agreement with Coleman's answer. The electric and magnetic couplings run inversely to one another. In fact if we set $Z_S(\mu) = [1 - \Pi_J + \Pi_K]$ we get Eq. (1.9) exactly.

Chapter 3

Conclusion and Final Remarks

Schwinger and Coleman reached opposite conclusions about how the magnetic coupling runs relative to the electric coupling. We explored a new formalism that reconciles their disagreement. We first calculated the running of the electric coupling using the vector potential as our source for the field. This reproduced standard results and served as a check for future calculations. We then repeated the calculation using the field strength itself. This formalism extends well when we add magnetic monopoles. Using this second method, we calculated the running of the magnetic coupling, careful of the topological terms that arose. Doing this, we found that the electric coupling runs proportionally to energy (k^2):

$$e_R = \frac{e^2}{[1 - \Pi_J(k^2) + \Pi_K(k^2)]} \; , \tag{3.1}$$

and magnetic coupling runs exactly inverse to it:

$$b_R = b^2[1 - \Pi_J(k^2) + \Pi_K(k^2)] \; . \tag{3.2}$$

This is the relationship Coleman predicted, and the opposite of what Schwinger thought. These results are also in agreement with the Dirac quantization condition; $b_0 = \frac{4\pi}{e_0}$ is only true if e and b run inversely. In the future, applying this field strength formalism to a

Lagrangian with a theta term could be insightful. The theta term gives rise to the Witten Effect [11], which is the theoretical phenomenon where magnetic monopoles gain some amount of radial electric charge in addition to their magnetic charge. This electric charge would be proportional to the theta in the Lagrangian. Using this formalism could reveal how theta runs with energy.

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