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Complementarity versus Uncertainty

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Introduction

The original intent of this work was to investigate an issue that is currently being debated among various groups of physicist. We hoped to find a new angle of approach that might shed light on the issue. Complementarity is the principle that a system, such as an electron, can be described either in terms of particle or in terms of wave motion. Marlin Scully and his colleagues have proposed theoretical as well as experimental work in which they claim that the principle of complementarity is upheld by some, as yet undefined and poorly understood, information law that is more fundamental than the uncertainty principle. This proposal deviates from the traditional explanation that complementarity is enforced by invoking the uncertainty principle ultimately through some physical momentum kick. Walls and his colleagues argue that Scully's work fails to accomplish its objectives and that physical momentum kicks are indispensable in explaining complementarity. Although it seemed at times that the whole debate was a matter of splitting hairs, we find the related question exciting and very challenging. For details of Scully's and Walls' work we refer the reader to the original papers [1] [2] and a subsequent review [5].

While researching the aforementioned issue, a paper written by Wootters and Zurek [3] caught our attention. In this paper Wootters and Zurek develop a scheme for quantifying complementarity. We saw this paper as a useful tool to study the issue at hand and the field of atom optics in general. We have extended their work to a general case and we relate our findings to the uncertainty relation and the correspondence principle. Parts of this work were presented at the Fifth International Conference on Squeezed States and Uncertainty Relations [7].

The Model

The model that Wootters and Zurek use is a slight variation of one of the thought experiments that Einstein proposed in his attempts to disprove complementarity [6]. Einstein proposed placing a single slit plate in front of a traditional double slit experiment. The convention that we will use in discussing this arrangement is to call the single slit plate, plate 1, the double slit, plate 2, and the detector screen, the screen. Einstein argued that by measuring the momentum of plate 1 one could know which path the photon travelled and keep the interference pattern as well. Bohr shows that this arguement fails because the initial momentum of plate 1 must be known to within a certain accuracy to know the path of the photon. The knowledge of the initial momentum, according to the uncertainty principle, neccessarily introduces an uncertainty in the position of plate 1 that in turn destroys the interference pattern. Thus Einstein fails to disprove complementarity.

Wootters and Zurek use this same experimental configuration. The only modification is that plate 1 is attached to an ideal spring. Thus the position and momentum of plate 1, and thus the single slit itself, are described by the wavefunction of a simple harmonic oscillator (SHO). Wootters and Zurek were not out to disprove complementarity but to develop a quantitative statement of the principle. Whereas the principle is described as either exclusively particle or exclusively wave, their treatment allows for intermediate situations. They treated only the case when the single slit is in the ground state energy level, $n = 0$. We have extended this work to the case of arbitrary n.

The uncertainty relation for a simple harmonic oscillator in the $n = 0$ state reads,

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{1}$$

which satisfies the limiting equality of the general uncertainty relation

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{2}$$

The uncertainty relation for the n th excited state is

$$\Delta x \Delta p = (n + \frac{1}{2})\hbar \tag{3}$$

This uncertainty relation can be derived by evaluating the variance of the position and momentum operators for a system in the nth excited state of the SHO: $(\Delta x)^2 =< x^2 > - < x >^2$ and $(\Delta p)^2 =< p^2 > - < p >^2$. The terms$< x^2 >and< p^2 >$ can be found by explicit integration but it is much easier to use Dirac notation. We will return to this relationship and discuss its importance in the final analysis.

Position Method

There are various ways to derive the interference pattern that will result from this single+double slit configuration. The simplest method is the position method. This technique is to determine the contribution to the pattern that a subensemble of particles, all emitted from a given position, makes and then to weight it according to the probability of the single slit being in that position. This technique is conceptually and mathematically simple but also has the disadvantage of not providing any 'which-path' information. In other words, a measurement of the single slits position when a particle successfully traverses the experiment tells us nothing about which slit of the double slit it passed through.

The standard wavefunction for a simple harmonic oscillator provides the probability distribution of the single slit that is needed for this procedure.

$$\psi_n(x) = 2^{-n/2}(n!)^{-1/2}(\frac{m\omega}{\pi\hbar})^{1/4}H_n((\frac{m\omega}{\hbar})^{1/2}x)e^{-\frac{m\omega x^2}{2\hbar}} \tag{4}$$

By defining $a^2 = \frac{\hbar}{m\omega}$ we simplify the expression a little and we note that the variable 'a' is related to the spring constant of the harmonic oscillator as follows: $a = (\frac{\hbar}{(km)^{1/2}})^{1/2}$. Thus a small 'a' corresponds to a large spring constant k. The expression becomes:

$$\psi_n(x) = 2^{-n/2}(n!)^{-1/2}\pi^{-1/4}a^{-1/2}H_n(\frac{x}{a})e^{-\frac{x^2}{2a^2}} \tag{5}$$

The contribution to the final pattern made by a subensemble of particles, all originating from a given vertical position ξ, is found from the standard optics expression:

$$I_x(\xi) = 1 + \cos(2k_o(\xi + x)) \tag{6}$$

Now to find the total interference pattern we simply sum up the weighted contributions of the subensembles by integrating over all positions along plate 1.

$$\mathcal{F}_n(\xi) = \int_{-\infty}^{\infty} |\psi_n(x)|^2 I_x(\xi) dx \qquad (7)$$

The following trigonometric identity is used to solve the integral.

$$\cos 2k_o(\xi + x) = \cos 2k_o\xi \cos 2k_o x - \sin 2k_o\xi \sin 2k_o x \qquad (8)$$

By using an integral table, Ref [4], and defining $\beta = k_o\sqrt{\frac{2\hbar}{m\omega}}$ we solve the integrals and arrive at the formula for the interference pattern

$$\mathcal{F}_n(\xi) = 1 + e^{-k_o^2 a^2} L_n(2k_o^2 a^2) \cos(2k_o\xi) \qquad (9)$$

where $L_n(x)$ represents the nth Laguerre polynomial.

Momentum Method

The momentum method is another technique that allows us to determine the final interference pattern produced from this experiment. The momentum method is conceptually and mathematically more complex than the position method but it is advantageous because it also provides 'which-path' information. We derive the interference pattern in this section and discuss how the 'which-path' information is obtained in the next.

The momentum method is implemented in the following manner. Because we model plate 1 as a simple harmonic oscillator its momentum is described by the SHO wavefunction in k-space. The wavefunction in k-space is obtained by taking the Fourier Transform of $\psi(x)$.

$$\psi_n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_n(x) e^{-ikx} dx \tag{10}$$

$$\psi_n(k) = \frac{2^{-n/2}(n!)^{-1/2}(m\omega)^{1/4}}{(2\pi)^{1/2}(\pi\hbar)^{1/4}} \int_{-\infty}^{\infty} H_n((\frac{m\omega}{\hbar})^{1/2}x) e^{-\frac{m\omega x^2}{2\hbar}} e^{-ikx} dx \tag{11}$$

By making a change of variable $\sigma = \sqrt{\frac{m\omega}{\hbar}}x$ and defining $y = \sqrt{\frac{\hbar}{m\omega}}k$ we put the integral into a form that can be solved using an integral table, Ref [4].

$$\psi_n(k) = \frac{2^{-n/2}(n!)^{-1/2}(m\omega)^{1/4}}{(2\pi)^{1/2}(\pi\hbar)^{1/4}} \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} H_n(\sigma) e^{-\frac{\sigma^2}{2}} e^{-iy\sigma} d\sigma \tag{12}$$

$$= \frac{i^n 2^{-n/2}(n!)^{-1/2}(\hbar)^{1/4}}{(m\pi\omega)^{1/4}} H_n(\sqrt{\frac{\hbar}{m\omega}}k) e^{-\frac{\hbar k^2}{2m\omega}} \tag{13}$$

$$= i^n 2^{-n/2}(n!)^{-1/2}\pi^{-1/4}a^{1/2}H_n(ak)e^{-\frac{a^2k^2}{2}} \tag{14}$$

If we measure the momentum of plate 1 just after a particle registers on the screen its momentum will consist of two parts. The measured momentum of plate 1 will be the sum of the initial momentum of plate 1, some k specified by $\psi(k)$, and the momentum, $\pm k_o$, imparted to plate 1 by the particle. Because the particle has to go through one of the two slits, the magnitude of k_o is defined by the geometry of the system. Thus the measured momentum κ is:

$$\kappa = k \pm k_o \tag{15}$$

The total distribution $\mathcal{D}(\kappa)$ of wave numbers is the sum of all the partial distributions $D_k(\kappa)$, weighted by $|\psi(k)|^2$. We recall that k_o is fixed by the geometry of the system and that the partial distribution $D_k(\kappa)$ is the distribution of the κ's for a fixed k.

$$D_k(\kappa) = \frac{1}{2}[\delta((\kappa + k_o) - k) + \delta((\kappa - k_o) - k)] \tag{16}$$

$$\mathcal{D}_n(\kappa) = \int |\psi_n(k)|^2 D_k(\kappa) dk \tag{17}$$

$$\mathcal{D}_n(\kappa) = \frac{2^{-n-1}a}{n!\pi^{1/2}}[H_n^2(a(\kappa + k_o))e^{-a^2(\kappa+k_o)^2} + H_n^2(a(\kappa - k_o))e^{-a^2(\kappa-k_o)^2}] \tag{18}$$

The interference pattern generated by a subensemble corresponding to a given κ is:

$$i_\kappa(\xi) = 1 + 2P_A^{1/2}(\kappa)P_B^{1/2}(\kappa)\cos(2k_o\xi) \tag{19}$$

$$P_A(\kappa) = \frac{H_n^2(a(\kappa + k_o))e^{-a^2(\kappa+k_o)^2}}{H_n^2(a(\kappa + k_o))e^{-a^2(\kappa+k_o)^2} + H_n^2(a(\kappa - k_o))e^{-a^2(\kappa-k_o)^2}} \tag{20}$$

$$P_B(\kappa) = \frac{H_n^2(a(\kappa - k_o))e^{-a^2(\kappa-k_o)^2}}{H_n^2(a(\kappa + k_o))e^{-a^2(\kappa+k_o)^2} + H_n^2(a(\kappa - k_o))e^{-a^2(\kappa-k_o)^2}} \tag{21}$$

The final interference pattern is found by summing the partial interference patterns after weighting them with the momentum distribution.

$$\mathcal{F}_n(\xi) = \int \mathcal{D}_n(\kappa) i_\kappa(\xi) d\kappa \tag{22}$$

$$\mathcal{F}_n(\xi) = 1 + e^{-k_o^2 a^2} L_n(2k_o^2 a^2) \cos(2k_o \xi) \tag{23}$$

We see that the position and momentum methods produce the same result. The final pattern is a simple cosine function with a coefficient in front. We see that the coefficient portion depends on k_o , a, and the energy level n. The coefficient can vary between 0 and 1 and thus determines the sharpness of the pattern. We define a sharpness variable accordingly.

$$S = e^{-k_o^2 a^2} L_n(2k_o^2 a^2) \tag{24}$$

Information Approach

As mentioned earlier, the momentum method provides more than just the formula describing the resulting interference pattern. In the derivation of the interference pattern we came across the terms $P_A(\kappa)$ and $P_B(\kappa)$. Those are the probabilities of passing through slit A or slit B given a certain κ. This indicates that we have access to some 'which-path' information. Again we follow the example of Wootters and Zurek and use the Shannon formulation of information to quantify the amount of information we have. The basic formula being:

$$H = -\sum_{i=1}^{N}(P_i ln(P_i)) \tag{25}$$

H in this formula represents the amount of information we lack from a system with N possible outcomes where P_i are the probabilities of the various outcomes. As an example, if there are two slits and it is equally probable that the particle will go through either then $ln(2)$ is the amount of information that is obtained if one discovers which slit was traversed.

$$H_o = -(\frac{1}{2}ln(\frac{1}{2}) + \frac{1}{2}ln(\frac{1}{2})) = ln(2) \tag{26}$$

Wootters and Zurek outline four different variations of this experiment and show that even though the Sharpness, S, in all four cases is the same the amount of 'which-path' information obtained can vary. What this tells us is that the amount of information obtained depends on the cleverness of the experimenter. Wootters and Zurek prove that one of the experiments (not the SHO model) they outline provides the most information theoretically allowed given the constraint of a specified sharpness, S. They develop an inequality,

$H \geq H(S)$, to defined that theoretical limit. The H term represents the amount of information given up in any given experiment. The *H(S)* term is the theoretical limit defining the least amount of information that has to be lost. So the inequality says that nobody can devise an experiment and give up less information than *H(S)*. For the details of this work we refer the reader to Ref [3].

With the SHO experiment the average information we give up per particle is:

$$H = -\int_{-\infty}^{\infty} [P_A(\kappa)ln(P_A(\kappa)) + P_B(\kappa)ln(P_B(\kappa))]\mathcal{D}_n(\kappa)d\kappa \qquad (27)$$

We solved this integral numerically. We note that the sharpness S, and the 'which-path' information H, are similar to the conjugate variables x and p. S is a measure of the wavelike nature and H measures the particle nature.

Analysis

As noted earlier, the uncertainty principle for an object in the ground state of a SHO is the equality $\Delta x \Delta p = \frac{\hbar}{2}$. For any higher energy level n, it becomes $\Delta x \Delta p \geq (n + \frac{1}{2})\hbar$. Figure 1 is a plot of the equality $\Delta x \Delta p = \frac{\hbar}{2}$. We see that the line divides the theoretically forbidden and allowed regions of phase space. Figure 2 shows the inequality developed by Wootters and Zurek. Figure 1 and 2 present the same information but in a different language. We intuitively relate H to the Δx and, knowing DeBroglie's relation $\lambda = \frac{h}{p}$, we relate S to Δp. As demonstrated in the previous section we can now determine $H_n(S)$, for an arbitrary n by numeric integration. Figures 3 and 4 show $H_n(S)$. Note that the curves of Figures 3 and 4 fall within the theoretically allowed region as expected. We see that as n increases the data points fall farther away from the theoretical best curve. Thus we see that the ground state is the best case for giving up as little information as possible.

What remains to be done is to find (if possible) an analytic form for $H_n(S)$ that would then be analogous to $\Delta x \Delta p \geq (n + \frac{1}{2})\hbar$. We briefly looked at the possibility of curve fitting to try and obtain an analytic form from the plots obtained for several values of the parameters. This analytic form still eludes us.

The results do not show any unexpected behavior but they do provide us with the quantitative changes as n increases. We also believe it would be interesting to repeat this work while allowing $\psi(x)$ to be a coherent superposition of states.

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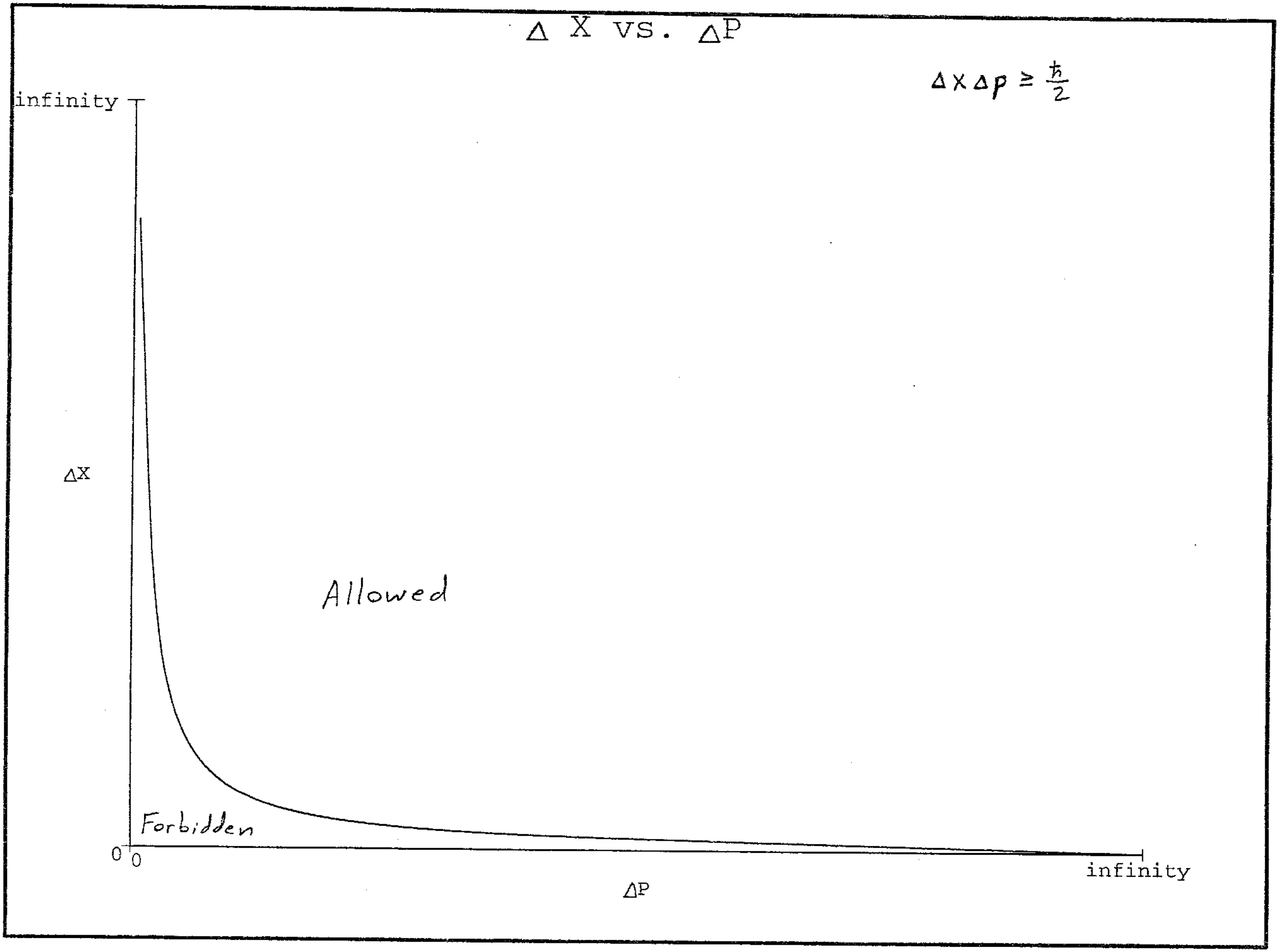
Δ X vs. ΔP
$\Delta x \Delta p \geq \frac{\hbar}{2}$
infinity
ΔX
Allowed
Forbidden
0
0
infinity
ΔP

Figure 1

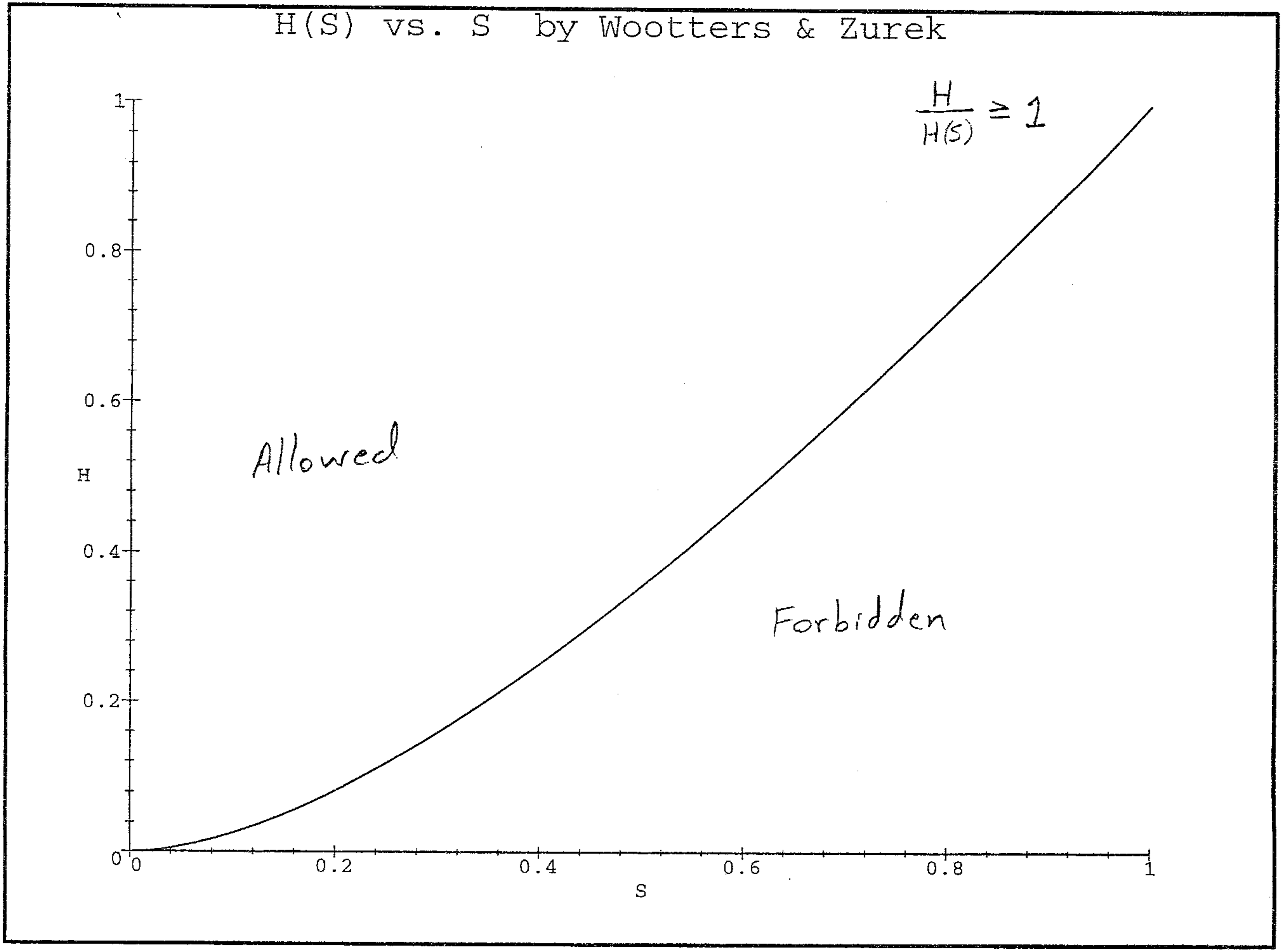
H(S) vs. S by Wootters & Zurek
H / H(S) ≥ 1
Allowed
Forbidden
H
S
0
0.2
0.4
0.6
0.8
1

Figure 2

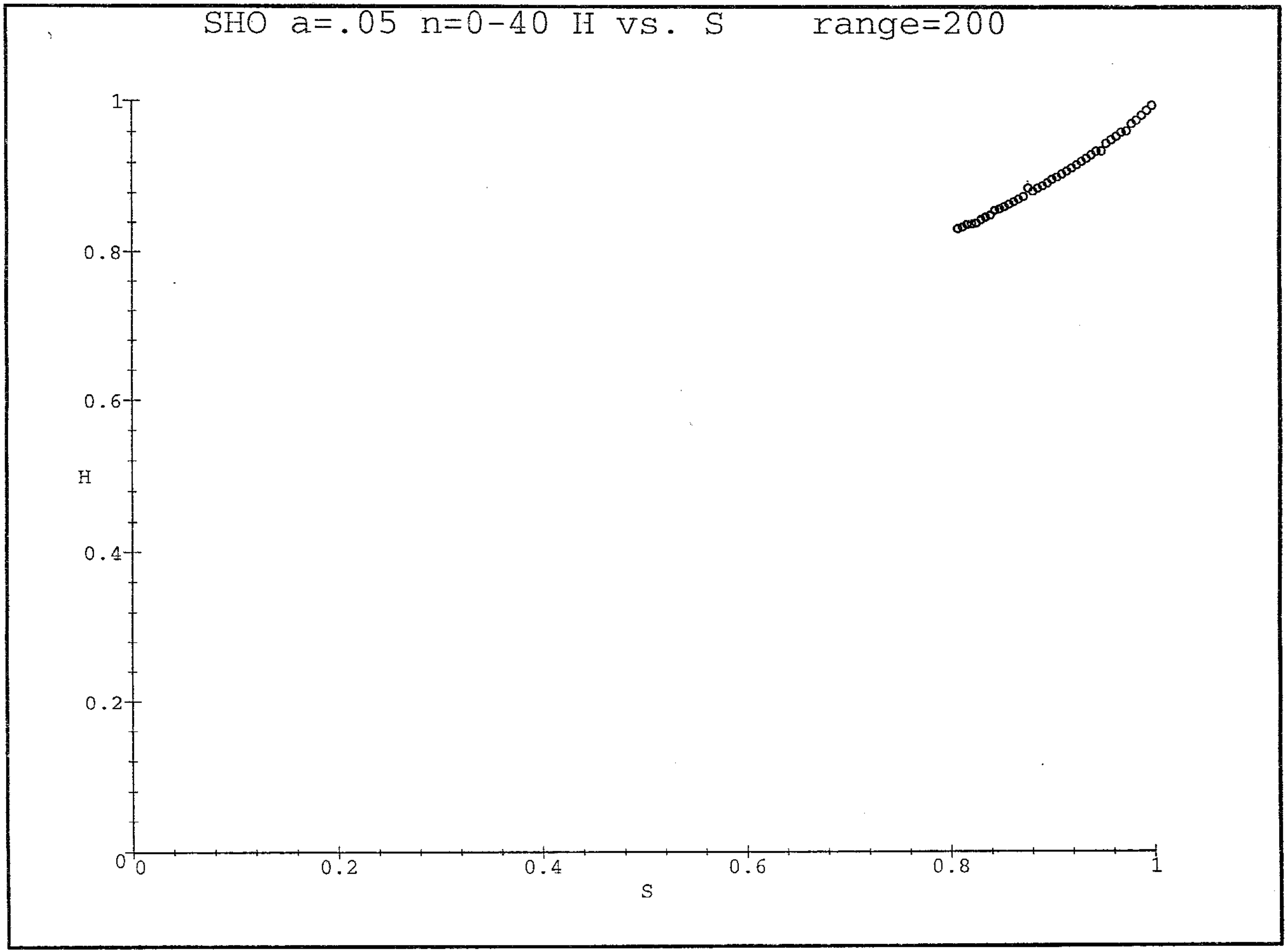

Figure 3

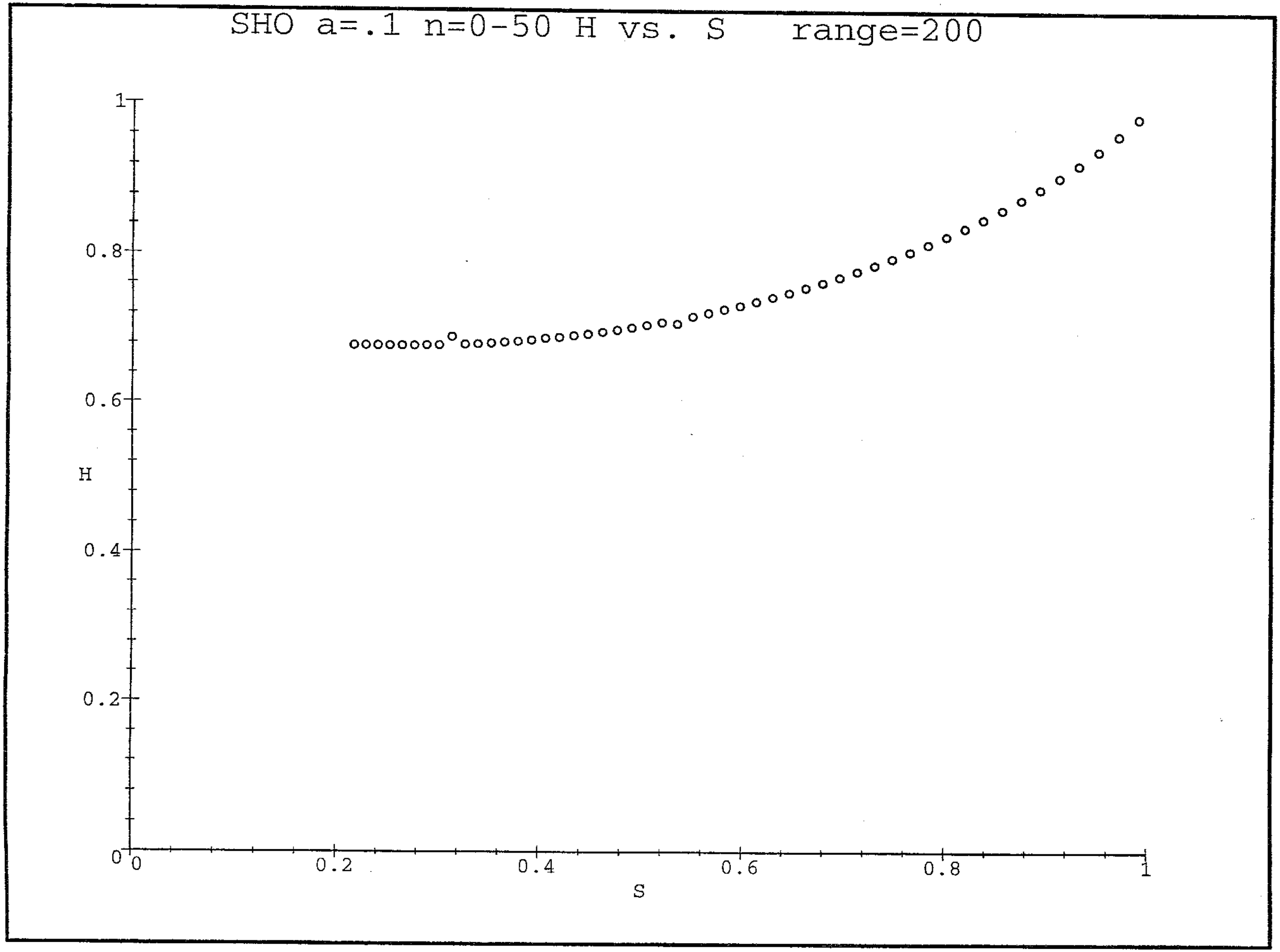

Figure 4