Applying Supersymmetry
to
Quantum Mechanics
with a study of
Bound States in the Continuum

Senior Thesis

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Abstract
Techniques from Supersymmetry (a theory in particle physics) are applied to nonrelativistic quantum mechanics to develop a general method of finding ladder operators for exactly solvable potentials. Bound States in the Continuum are studied, and the same techniques from Supersymmetry are used to generate potentials which permit bound states in the continuum.

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Introduction

The following work is a contribution to a better understanding of nonrelativistic quantum mechanics. It will explore techniques for factorizing and solving Schrödinger's equation for various potentials. It will also explore unusual potentials such as potentials leading to bound states within the continuum, and ways that our new techniques for solving Schrödinger's equation can help us create such potentials.

Supersymmetry is a theory in particle physics which is used to predict the masses and various other quantum numbers of the fundamental particles. Techniques similar to those used in supersymmetry can be used to factorize and solve the Schrödinger equation. These techniques are known as Supersymmetric Quantum Mechanics or SUSY QM, and will be explored in detail in the first part of this work.

Localized potentials, such as that of the hydrogen atom, usually have several discrete bound states at negative energies, i.e. below the asymptotic value of the potential, and a continuum of unbound states at positive energies (Ballentine, p.205). However, Wigner and Von Neuman discovered that localized potentials may be created which permit discrete bound states within the continuum and with positive energies. (J. von Neumann). These states are known as bound states in the continuum, or BICs. In the second part of this work we will discuss these potentials, their properties, and how we may use supersymmetric techniques to generate them.

Supersymmetric Quantum Mechanics

Typical solution methods for finding energy eigenvalues of the Schrödinger equation include guess work and expansions in a power series. Goswami for example uses these two methods in his text. Unfortunately the first of these options is not systematic and the second is mechanical and lacks physical insight. However, in some cases, such as the simple harmonic oscillator, an operator method can be used to greatly simplify the work and still reliably yield the energy eigenvalues. If this method of operators could be expanded to all or even most exactly solvable potentials, i.e. potentials for which we can form an exact analytical solution to the Schrödinger equation, it would be very useful. A way to do this can be found through supersymmetry.

The techniques of supersymmetry can be applied profitably to regular nonrelativistic quantum mechanics. These techniques allow us to solve directly for the wave functions of many potential wells by generalizing the operator technique used to solve the problem of the simple harmonic oscillator.

We begin studying supersymmetry by looking at the Schrödinger equation in one dimension. If we take $\hbar = 1$ and $m = \frac{1}{2}$ then the equation reads as

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

for an arbitrary potential $V(x)$ where $\psi(x)$ is any eigenfunction of the equation and E is the energy eigenvalue corresponding to that eigenfunction. If we know any of these eigenfunction we can solve for $V(x)$ in terms of that eigenfunction, and its eigenvalue. We will do this for the ground state eigenfunction, but the method can be applied to any eigenfunction. We will discuss excited states more when we discuss bound states in the continuum (BICs). The potential is then expressed as

$$V(x) = \frac{\psi_0''(x)}{\psi_0(x)} + E_0$$

and the hamitonian can be expressed as

$$H = -\frac{d^2}{dx^2} + \frac{\psi_0''}{\psi_0} + E_0.$$

The eigenvalue equation can then be written as

$$H\psi(x) = E\psi(x)$$

or

$$-\frac{d^2}{dx^2}\psi(x) + \frac{\psi_0''}{\psi_0}\psi(x) = (E - E_0)\psi(x). \tag{1}$$

We wish to look for ways to factorize the left side of this equation. To begin we will choose a simple operator A such that $A\psi_0 = 0$. One simple operator that satisfies this relation is $A = \frac{d}{dx} - \frac{\psi_0'}{\psi_0}$. We can then choose our phase such that $\psi_0(x)$ is real, and write the hermitian conjugate of A as $A^\dagger = -\frac{d}{dx} - \frac{\psi_0'}{\psi_0}$. We multiply A and $A^\dagger$ to get

$$A^\dagger A = -\frac{d^2}{dx^2} + \frac{\psi_0''}{\psi_0}$$

which is just the left side of the equation above. So we will define a new hamiltonian $H^{(-)} = H - E_0$. This is simply a rescaling of the potential energy such that the potential energy of the ground sate is zero. We can now write

$$A^\dagger A = H^{(-)}$$

and

$$A^\dagger A \psi_n^{(-)} = E_n^{(-)}\psi_n^{(-)} = (E_n - E_0)\psi_n^{(-)}.$$

This leads to an interesting observation. If we multiply both sides of the equation by A from the left we get

$$A(A^\dagger A \psi_n^{(-)}(x)) = A(E_n^{(-)}\psi_n^{(-)}(x)).$$

Then by applying the associativity of A on the left and the linearity of A on the right we get

$$AA^\dagger(A\psi_n^{(-)}(x)) = E_n^{(-)}(A\psi_n^{(-)}(x)).$$

In other words we have found a new hamiltonian $H^{(+)} = AA^\dagger$ with the same eigenvalues as $H^{(-)}$ (except for the ground state which has disappeared) but with eigenfunctions modified by the the operator A. This can be expressed in the following manner

$$E_n^{(+)} = E_{n+1}^{(-)}$$

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$$\psi_n^{(+)} \propto A\psi_{n+1}^{(-)}.$$

Our next task is to normalize $\psi_n^{(+)}$ which we do using Dirac ket and bra notation. Assume

$$|\psi_n^{(+)}\rangle = cA|\psi_{n+1}^{(-)}\rangle$$

where c is a complex constant and

$$\langle \psi_n^{(-)}|\psi_n^{(-)}\rangle = 1$$

$$\langle \psi_n^{(+)}|\psi_n^{(+)}\rangle = 1.$$

With a few manipulations we get

$$\langle \psi_n^{(+)}|\psi_n^{(+)}\rangle = \langle cA\psi_{n+1}^{(-)}|cA\psi_{n+1}^{(-)}\rangle = |c|^2 \langle v_{n+1}^{(-)}|A^\dagger A|\psi_{n+1}^{(-)}\rangle = |c|^2 \langle \psi_{n+1}^{(-)}|E_{n+1}^{(-)}|\psi_{n+1}^{(-)}\rangle$$

$$= |c|^2 E_{n+1}^{(-)} = 1.$$

Choosing c to be real and positive we get

$$c = \left(E_{n+1}^{(-)}\right)^{-\frac{1}{2}}.$$

By a similar process we can start with an eigenfunction of the $H^{(+)}$ operator and operate to the left with the $A^\dagger$ operator to get eigenfunctions of the $H^{(-)}$ operator

$$H^{(+)}\psi_n^{(+)}(x) = AA^\dagger \psi_n^{(+)} = E_n^{(+)}\psi_n^{(+)}$$

$$A^\dagger(AA^\dagger)\psi_n^{(+)} = A^\dagger E_n^{(+)}\psi_n^{(+)}$$

$$A^\dagger A(A^\dagger \psi_n^{(+)}) = E_n^{(+)}A^\dagger \psi_n^{(+)}.$$

This implies that

$$\psi_{n+1}^{(-)} \propto A^\dagger \psi^{(+)},$$

which can be normalized as we did above.

So we see that while the A operator changes eigenfunctions of $H^{(-)}$ into eigenfunctions of $H^{(+)}$, $A^\dagger$ changes eigenfunctions of $H^{(+)}$ into eigenfunctions of $H^{(-)}$. Our findings can be summarized in the following equations:

$$E_n^{(+)} = E_{n+1}^{(-)} \tag{2}$$

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$$\psi_n^{(+)} = \left(E_{n+1}^{(-)}\right)^{-\frac{1}{2}} A \psi_{n+1}^{(-)} \tag{3}$$

$$\psi_{n+1}^{(-)} = \left(E_n^{(+)}\right)^{-\frac{1}{2}} A^\dagger \psi_n^{(+)}. \tag{4}$$

Because of this relationship between $H^{(+)}$ and $H^{(-)}$ we refer to them as supersymmetric partners or a supersymmetric pair.

Suppose that we were to do the same thing as before, but this time treating $H^{(+)}$ as our starting point. We would then find another hamiltonian, with one fewer bound state, since the ground state of $H^{(+)}$ will have disappeared, and we could transform its eigenfunctions to the eigenfunctions of $H^{(+)}$ as we did between $H^{(+)}$ and $H^{(-)}$. To illustrate this let us call the first two hamiltonians $H_0^{(-)}$ and $H_0^{(+)}$ then

$$H_0^{(-)} = A_0^\dagger A_0 = -\frac{d^2}{dx^2} + V_0(x) - E_{0,0}$$

$$H_0^{(+)} = A_0 A_0^\dagger = -\frac{d^2}{dx^2} + V_1(x) - E_{0,0}$$

where the first index on the energy represents the potential it is in and the second represents the eigenstate of the hamiltonian corresponding to that potential.

If we start with eigenfunctions of $H_0^{(+)}$ we can show that

$$\frac{\psi_0^{(+)\prime\prime}}{\psi_0^{(+)}} = V_1(x) - E_{1,0}$$

where $E_{1,0} = E_{0,1}$. We can then define

$$A_1 = -\frac{d}{dx} + \frac{\psi_0^{(+)\prime}}{\psi_0^{(+)}}$$

and

$$A_1^\dagger = -\frac{d}{dx} - \frac{\psi_0^{(+)\prime}}{\psi_0^{(+)}}.$$

With these relations we can show that

$$H_1^{(-)} = A_1^\dagger A_1 = -\frac{d^2}{dx^2} + V_1(x) - E_{1,0}.$$

Therefore

$$H_0^{(+)} - H_1^{(-)} = E_{1,0} - E_{0,0}$$

or

$$H_0^+ = H_1^- + (E_{1,0} - E_{0,0})$$

The same thing can be done with the ground state of $H_1^{(+)}$ to get $H_2^{(-)}$ and $H_2^{(+)}$. By recursion we obtain a series of potentials obeying the relationships:

$$H_\ell^{(+)} = H_{\ell+1}^{(-)} + (E_{\ell+1,0} - E_{\ell,0}) \tag{5}$$

$$H_\ell^{(-)} = A_\ell^\dagger A_\ell \tag{6}$$

$$H_\ell^{(+)} = A_\ell A_\ell^\dagger. \tag{7}$$

If the various hamiltonians correspond to potentials with similar forms. i.e. $V_\ell(x) = V(a_\ell, x)$ where a_ℓ is a set of constants depending only on ℓ then the potentials are said to be shape invariant (F. Cooper, p. 289-290). As a result, we can find a general form for A_ℓ and $A_\ell^\dagger$ and use it to find the energy eigenvalues and eigenfunctions of each hamiltonian in the series.

To illustrate this I will introduce some notation. The integer ℓ will be an index to label the potential $V_\ell(x)$ that we are working with, and $H_\ell^{(-)}$ and $H_{\ell-1}^{(+)}$ will both correspond to this potential. The index n will be used to designate which eigenstate in the potential V_ℓ we are concerned with. Thus, hamiltonians, potentials, and the operators A and $A^\dagger$ only need to be labeled with ℓ, but eigenfunctions and eigenvalues must be labeled by both ℓ and n.

The energy scales for the different hamiltonians differ by a constant, because the ground state of each H_ℓ^- is defined to be zero. We can find the constant from equation (5)

$$H_\ell^{(+)} - H_{\ell+1}^{(-)} = E_{\ell+1,0} - E_{\ell,0} = R_{\ell+1}.$$

Where $R_{\ell+1}$ represents the difference between the energy scale of the two hamiltonians corresponding to $V_{\ell+1}$. From this we can find the energy level of the ground state of each potential in terms of the energy scale in the original potential, namely

$$E_{\ell,0} = E_{0,0} + \sum_{i=1}^{\ell} R_i.$$

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In this formula $E_{0,0}$ is the energy level of the ground state of our original potential $V_0(x)$. We can generalize equation (2) and find that $E_{\ell,n} = E_{\ell+1,n-1}$. We can apply this formula recursively to find the energy level for any excited state in terms of the energy of the ground state of another potential, $E_{\ell,n} = E_{\ell+n,0}$. Thus we find that in general

$$E_{\ell,n} = E_{0,0} + \sum_{i=1}^{\ell+n} R_{(\ell)} \qquad (8)$$

We can go through a similar process to find $\psi_{\ell,n}$ for any ℓ or n. We start from

$$A_\ell \psi_{\ell,0} = 0 \qquad (9)$$

which is a first order differential equation that can be solved and normalized rather easily. We also know that given a normalized eigenfunction of a hamiltonian corresponding to V_ℓ we can find the normalized eigenfunctions corresponding to $V_{\ell+1}$ and $V_{\ell-1}$ that have the same energy through the relationships

$$\psi_{\ell+1,n-1} = \left(E_{\ell,n} - E_{\ell,0} \right)^{-\frac{1}{2}} A_\ell \psi_{\ell,n} \qquad (10)$$

$$\psi_{\ell-1,n+1} = \left(E_{\ell,n} - E_{\ell-1,0} \right)^{-\frac{1}{2}} A_{\ell-1}^\dagger \psi_{\ell,n}. \qquad (11)$$

These equations are generalizations of equations (3) and (4). We can use equation (11) recursively to find any eigenfunction $\psi_{\ell,n}$ from the eigenfunction $\psi_{\ell+n,0}$. This yields

$$\psi_{\ell,n} = \left\{ \prod_{i=\ell+n-1}^{\ell} \left[(E_{\ell,n} - E_{i,0}) A_i^\dagger \right] \right\} \psi_{\ell+n,0}. \qquad (12)$$

In this way we can find ladder operators that allow us to find all of the energy levels and eigenfunctions of any shape invariant potential. The index is writen in descending order because the different $A_\ell^\dagger$ operators do not commute and we must operate with those of higher ℓ first.

I will now illustrate the method with three examples: a simple harmonic oscillator, a square box, and a hydrogen atom.

1. Simple Harmonic Oscillator

The simplest illustration of supersymmetric techniques is the solution of the simple harmonic oscillator

$$V_0(x) = \frac{1}{2}kx^2$$

The constant k determines only the width of the potential, and is not particularly interesting in this example so we will choose $k = 2$ so that the hamiltonian becomes

$$H = -\frac{d^2}{dx^2} + x^2.$$

The ground state to this equation can easily be found to be

$$\psi_{0,0}(x) = (\pi)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right)$$

which has an energy of $E_{0,0} = 1$. The operators A_0 and $A_0^\dagger$ can be found to be

$$A_0 = \frac{d}{dx} - x$$

$$A_0^\dagger = -\frac{d}{dx} - x.$$

Then the hamiltonians become

$$H_0^{(-)} = -\frac{d^2}{dx^2} + x^2 - 1$$

$$H_0^{(+)} = -\frac{d^2}{dx^2} + x^2 + 1$$

We notice that both $H_0^{(-)}$ and $H_0^{(+)}$ have exactly the same form, except that they differ by a constant 2. We can then reason by induction and find that all of the potentials are identical except that their zero points have been shifted, and that all of the energy levels differ by a constant $R_\ell = 2$. Thus we find from equation (8) that

$$E_n = E_0 + \sum_{i=1}^{n} R_\ell = (2n + 1)$$

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and from equation (12)

$$\psi_n = \left[\prod_{i=n-1}^{0} \left\{ [(2n+1)-(2i+1)]^{-\frac{1}{2}} A^\dagger \right\} \right] (\pi)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) = [\pi 2^n n!]^{-\frac{1}{2}} A^{\dagger^n} \exp\left(-\frac{x^2}{2}\right)$$

We recover therefore the results obtained from the conventional operator treatment of the simple harmonic oscillator.

2. The Square Box

Another interesting potential to which supersymmetry may be applied is the square box. For the sake of simplicity we will take a box with a potential defined as

$$V_0(x) = \begin{cases} \infty, & \text{if } x < 0; \\ 0, & \text{if } 0 \le x \le \pi; \\ \infty & \text{if } x > \pi. \end{cases}$$

It is then very simple to find the ground state eigenfunction $\psi_{0,0} = \sin(x)$, which we will leave unnormalized for simplicity. Then the operators and hamiltonians become.

$$A_0 = \frac{d}{dx} - \cot(x)$$

$$A_0^\dagger = -\frac{d}{dx} - \cot(x)$$

$$H_0^{(-)} = 0\csc^2(x) - 1$$

$$H_0^{(+)} = 2\csc^2(x) - 1$$

The first feature that we notice is that the supersymmetric technique gives us a good way to visualize a square well. A square well is nothing but a $\csc^2(x)$ function with a zero coefficient multiplying it. Indeed, if we graph $A\csc^2(x)$ and let A go to 0 slowly, the graph does indeed approach a square well (See figure 1).

The second feature that we notice is that $H_0^{(-)}$ and $H_0^{(+)}$ no longer differ by a constant, but by some factor multiplying the $\csc^2(x)$ function. However, both are still $\csc^2(x)$ functions. We need to find operators A_ℓ and $A_\ell^\dagger$ for which

$$A_0^\dagger A_0 = H_0^{(-)}$$

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$$A_0 A_0^\dagger = H_0^{(+)}$$

and from equations (5), (6) and (7)

$$A_\ell A_\ell^\dagger = A_{\ell+1}^\dagger A_{\ell+1} + R_{\ell+1}. \tag{13}$$

Guessing by the form of A_0 and $A_0^\dagger$ we try

$$A_\ell = \frac{d}{dx} - f(\ell) \cot(x)$$

$$A_\ell^\dagger = -\frac{d}{dx} - f(\ell) \cot(x).$$

Substituting these into equation (13) we get

$$f(\ell)\big[f(\ell)+1\big] \csc^2(x) - f^2(\ell) = \big[f(\ell+1)-1\big]f(\ell+1) \csc^2(x) - f^2(\ell+1) + R_{\ell+1}.$$

This implies that either $f(\ell+1) = -f(\ell)$, or $f(\ell+1) = f(\ell)+1$. The first of these does not give us a ladder so we ignore it. We also know that $f(0) = 1$ so $f(\ell) = \ell + 1$ This implies that

$$V_\ell = \ell(\ell+1) \csc^2(x)$$

$$H_\ell^{(-)} = -\frac{d^2}{dx^2} + V_\ell - (\ell+1)^2$$

$$H_\ell^{(+)} = -\frac{d^2}{dx^2} + V_{\ell+1} - (\ell+1)^2$$

$$R_\ell = H_{\ell-1}^{(+)} - H_\ell^{(-)} = (\ell+1)^2 - \ell^2.$$

Equation (9) then becomes

$$\left[\frac{d}{dx} - (\ell+1) \cot(x)\right] \psi_{\ell,0} = 0$$

which we solve to find that the ground state of the ℓth potential is

$$\psi_{\ell,0} = \sin^{\ell+1}(x).$$

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We can then use equation (12) to find the other eigenfunctions, and from equation (8) we find that the energy spectrum is

$$E_{\ell,n} = 1 + \sum_{i=1}^{\ell+n}\left[(i+1)^2 - i^2\right] = (\ell + n + 1)^2.$$

3. The Hydrogen Atom

The final potential that we will use to illustrate the supersymmetric solution technique is the hydrogen atom. We will start by looking only at the radial part with angular quantum number ℓ equal to zero, and we will use the wave function $u(r) = rR(r)$ thus reducing the problem to one dimension. We will thus get

$$H = -\frac{d^2}{dr^2} - \frac{2}{r}$$

where we have taken $ke^2 = 2$ for simplicity. This differential equation is of exactly the same form as a cartesian one-dimensional Schrödinger equation so we can use the same operators and hamiltonians as we have used before by simply substituting r for x. The unnormalized ground state of this function is $u(r) = r\exp(-r)$. From this we can find the operators.

$$A_0 = \frac{d}{dr} - \frac{1}{r} + 1$$

$$A_0^\dagger = -\frac{d}{dr} - \frac{1}{r} + 1$$

and the hamiltonians

$$H_0^{(-)} = -\frac{d^2}{dr^2} - \frac{2}{r} + 1 \qquad \swarrow$$

$$H_0^{(+)} = -\frac{d^2}{dr^2} + \frac{2}{r^2} - \frac{2}{r} + 1. \qquad \smile$$

We quickly recognize that the new potential $H^{(+)}$ is the same as the effective potential for a Hydrogen atom with a term for centrifugal repulsion. This particular case corresponds to an angular quantum number $\ell = 1$. We can solve for a general form for the ladder operators A_ℓ and $A_\ell^\dagger$ in the same way

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that we did when studying the square box. First, we assume a general form for the ladder operators

$$A_\ell = \frac{d}{dr} - \frac{f(\ell)}{r} + g(\ell)$$

$$A_\ell^\dagger = \frac{d}{dr} - \frac{f(\ell)}{r} + g(\ell).$$

Then we substitute these into equation (13) and use the known values for the $\ell = 0$ case and find that $f(\ell) = \frac{1}{g(\ell)} = (\ell + 1)$. We therefore get

$$A_\ell = \frac{d}{dr} - \frac{\ell + 1}{r} + \frac{1}{\ell + 1}$$

$$A_\ell^\dagger = -\frac{d}{dr} - \frac{\ell + 1}{r} + \frac{1}{\ell + 1}$$

$$V_\ell(x) = \frac{\ell(\ell + 1)}{r^2} - \frac{2}{r}$$

$$R_\ell = \frac{1}{\ell^2} - \frac{1}{(\ell + 1)^2}.$$

We can now see that the potential $V_\ell(x)$ is simply the effective potential for a hydrogen atom with angular quantum number ℓ. This makes sense since each time we increase the angular quantum number by 1 we loose the lowest bound state, but preserve the rest of the the energy spectrum, which is exactly what the supersymmetric transformation does. We will now see that our operators give the correct wave functions and energy spectrum.

Equation (9) becomes

$$\left(\frac{d}{dr} - \frac{\ell + 1}{r} + \frac{1}{\ell + 1} \right) u_{\ell,0}(r) = 0$$

which we can solve to find that

$$u_{\ell,0}(r) = r^{\ell+1} \exp\left(-\frac{r}{\ell + 1}\right).$$

With equation (13) we can find the rest of the wave functions, and from the equation (8) we can find the energy spectrum:

$$E_{\ell,n} = -\frac{1}{(\ell + n + 1)^2}$$

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thus recovering the well known wave functions and energy spectrum for the hydrogen atom. It is interesting to note that although we started with the intent of solving only the case with angular quantum number $\ell = 0$ we ended up finding the radial solutions for all possible angular quantum numbers.

Bound States in the Continuum

Let us suppose that we have a central potential $V_0(r)$ which tends to zero as the radius goes to infinity and which has a continuum of positive energy solutions to the Schrödinger equation. Let us assume that we have a radial S state wave function $\psi_0(r)$ such that $\psi_0(r) = \frac{u_0(r)}{r}$. Then $u_0(r)$ obeys the equation

$$-u_0''(r) + V_0(r)u_0(r) = Eu_0(r)$$

which is simply the Schrödinger equation in dimensionless form for $\ell = 0$. States other than S states may be treated by letting the potential be an effective potential which includes the $\frac{\ell(\ell+1)}{r^2}$ term.

We can therefore express V_0 in terms of the eigenfunction $u_0(r)$ as

$$V_0(r) = E + \frac{u_0''(r)}{u_0(r)}. \tag{14}$$

If we modify $u_0(r)$ in such a way that it becomes a bound state, then we can use equation (14) to determine the necessary modifications to $V(r)$ to accomodate such a wave function.

Since the potential tends to zero at infinity, the wave function $u_0(r)$ tends to $\sin(kr)$ as r becomes very large. The result is that the wave function can not be normalized because it does not tend to zero for large r and therefore

$$\int_o^\infty u_0^2(r)dr = \infty.$$

To make a new wave function $u(r)$ that goes to zero quickly enough to be normalizable we can multiply $u_0(r)$ by some function $f(r)$ which does not diverge near the origin and goes to zero at least as quickly as $r^{-\frac{1}{2}}$ for large values of r. If we assume that the energy of $u_0(r)$ and $u(r)$ are the same and substitute the new wave function $u(r) = u_0(r)f(r)$ into the Schrödinger equation, we get

$$-u_0''(r)f(r) - 2u_0'(r)f'(r) - u_0(r)f''(r) + V(r)u_0(r)f(r) = Eu_0(r)f(r)$$

which yields the following form for the potential

$$V(r) = E + \frac{u_0''(r)}{u_0(r)} + 2\frac{u_0'(r)}{u_0(r)}\frac{f'(r)}{f(r)} + \frac{f''(r)}{f(r)}.$$

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The first two terms are simply $V_0(r)$ so the new potential may be rewriten as the original potential $V_0(r)$ plus some variation on it

$$V(r) = V_0(r) + 2\frac{u_0'(r)}{u_0(r)}\frac{f'(r)}{f(r)} + \frac{f''(r)}{f(r)}. \tag{15}$$

In general we may expect that for large r $\frac{f'(r)}{f(r)} \to 0$ like $\frac{1}{r}$ and $\frac{f''(r)}{f(r)} \to 0$ like $\frac{1}{r^2}$. Therefore the potential still tends to zero for large values of r. We will discuss the behavior of the potential and the wavefunction at infinity in more detail later.

In order for the potential to be well behaved we require $f'(r)$ to go to zero whenever $u_0(r)$ is zero, and we require $f(r)$ to have no roots. The first condition can be met by making $f(r)$ some function of the integral of a monotonic function proportional to $u_0(r)$. If the function is not monotonic then the integral will not increase continuously at large r. As a result one can not use it to construct a function of r that goes to zero like $r^{-\frac{1}{2}}$ without using other functions of r which would prevent $f'(r)$ from going to zero when $u_0(r)$ does. We give two of many possibilities here.

$$f(r) = f(s(r))$$

where

$$s(r) = \int_0^r u_0^2(r')dr'$$

or

$$s(r) = \int_0^r r'u_0^2(r')dr'.$$

The Behavior of the Potential and the Wavefunction at Large Radii

We now wish to explore the behavior of the potential and the wave function as r goes to infinity. Specifically we want to see i) if we can make them both go to zero, and ii) how quickly they will go to zero. To examine this more closely we will make the substitution $f(r) = e^{g(r)}$. This can be done without loss of generality if we allow g(r) to be a complex function. We can then express the wave function and the potential in terms of the new function $g(r)$

$$u(r) = u_0(r) \exp g(r) \tag{16}$$

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$$V = V_0 + 2\frac{u_0'(r)}{u_0(r)}g'(r) + \left(g'(r)\right)^2 + g''(r), \tag{17}$$

and we can extract the behavior of $g(r)$ by studying the desired behavior of $u(r)$ and $V(r)$.

Since $u_0(r)$ oscilates with constant amplitude at large values of r, the wave function $u(r)$ will die off like $e^{g(r)}$. This means that to have a bound state $g(r)$ must go to negative infinity as r goes to infinity.

Next we look at the potential. For the potential to approach zero as r becomes large both $g'(r)$ and $g''(r)$ must approach zero as r goes to infinity. If this is true, then at large values of r the term with $g'(r)$ will dominate.

Let us suppose that $g'(r)$ goes like $r^{-1-\delta}$; $g(r)$ will then go as $r^{-\delta}$. If we want the potential to go to zero more quickly than r^{-1} we must let δ be positive, and therefore $g(r)$ will also go to zero as r becomes large. This implies that $u(r) = u_0(r)\exp g(r)$ behaves like $u_0(r)$ for large values of r. It oscillates between fixed values and does not converge, therefore it is not bound.

If δ is taken to be negative then $g(r)$ goes to a positive or negative infinity for large values of r. This can result in a bound state, but also gives a potential that approaches zero very slowly or else one that does not approach zero at all. This would be a non-local potential.

Therefore the way to generate the most local (one which goes to zero most quickly) potential $V(r)$ that permits a bound state is to let $g'(r)$ go like r^{-1}. This restricts $g(r)$ to go like $n\ln(r)$ or $\ln(r^n)$ where n is any real number.

If we use equation (16) to solve for the wave function when $g(r)$ goes like $n\ln(r)$ we find that $u(r)$ goes like r^n. Therefore, we can make the wave function behave like any power of r including negative powers that allow us to normalize it thus creating a bound state. However, the potentials corresponding to these wavefunctions decay like r^{-1}. We can not generate a potential that decreases more quickly than r^{-1} that also permits a bound state, nor can we have a state which decreases more quickly than by some power of r without allowing for non-local potentials.

We will give a simple illustration starting with an unperturbed solution that is a free particle

$$V_0(r) = 0$$

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$$u_0(r) = \frac{\sin(kr)}{k}$$

$$E = k^2.$$

We will now alter these functions in accordance with our previous discussion by multiplying $u_0(r)$ by $f(r)$ where

$$f(r) = \frac{1}{A + k^3 s(r)}$$

$$s(r) = \int_0^r \frac{\sin^2(kr')}{k^2} dr'.$$

The integral can be worked explicitly to yield

$$u(r) = \frac{4\sin(kr)}{k[4A + 2kr - \sin(2kr)]} \tag{18}$$

$$V(r) = 8k^2 \left\{ \frac{\sin(2kr)}{4A + 2kr - \sin(2kr)} - \frac{4\sin^4(kr)}{[4A + 2kr - \sin(2kr)]^2} \right\} \tag{19}$$

and the energy is the same as before

$$E = k^2.$$

So we see that this simple modification has changed the potential from a null one to a radially decreasing sinusoidal function which for large values of r has the form

$$\frac{4k^2 \sin(2kr)}{kr}.$$

This new potential has a bound state at the positive energy $E = k^2$ which, depending on the choice of A, may even be greater than the largest peak of the potential function, as illustrated in figure 2.

Supersymmetry and BICs

The techniques we developed in the first part of this paper, can be used to find bound states in the continuum. In that part of the paper we said that we could factorize a hamiltonian such that

$$H = A^\dagger A + E_0$$

Where

$$A = \frac{d}{dr} - \frac{u_0'(r)}{u_0(r)}$$

and

$$A^\dagger = -\frac{d}{dr} - \frac{u_0'(r)}{u_0(r)}$$

where $u_0(r)$ represented the ground state of the hamiltonian and E_0 was the ground state energy. We could then find a similar hamiltonian

$$H^+ = AA^\dagger + E_0$$

which had the same eigenspectrum except that the ground state had been erased, and the eigenfunctions $u^+(r)$ were related to the eigenfunctions $u(r)$ of H in the following way:

$$u^+(r) = Au(r).$$

The superscript $-$ and the parentheses arround the superscript $+$ have been dropped for simplicity. The previous statements can be generalized. First, u_0 does not have to be the ground state, but can be any eigenfunction of the original hamiltonian, including continuum states. If any state other than the ground state is used, the new potential will have poles at each of the zeros of the eigenfunction, but the technique can still be applied. Second, the new hamiltonian does indeed have an eigenfunction at the energy E_0, which is the energy of the state we are working with, not necessarily the ground state.

To illustrate the second of these two statements we notice that A and $A^\dagger$ can be written as $\pm\frac{d}{dr} - \frac{d \ln u_0}{dr}$. We can then see that

$$H = -\frac{d^2}{dr^2} + \frac{d^2 \ln u_0}{dr^2} + \left(\frac{d \ln u_0}{dr}\right)^2$$

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$$H^+ = H - 2\frac{d^2 \ln u_0}{dr^2}.$$

We can show by direct substitution that $\frac{1}{u_0(r)}$ is an eigenfunction of H^+ with an eigenvalue of E_0. This eigenfunction was previously ignored, because it has poles at all the roots of $u_0(r)$, but since we are using it as a construct to find bound states in the continuum, we can keep it. It can also be shown by direct substitution that

$$\frac{\int_0^{r'=r} u_0^2(r')dr'}{u_0(r)}$$

is also an eigenfunction of H^+ with the same eigenvalue E_0. If we let

$$I(r) = \int_0^{r'=r} u_0^2(r')dr'$$

then the general form for the wavefunction at the energy level E_0 can be written as

$$u_0^+ = \frac{a + bI(r)}{u_0(r)}.$$

If we treat H^+ as the original hamiltonian and alter it in the same way as H before, we get

$$\hat{H} = H^+ - 2\frac{d^2 \ln u_0^+}{dr^2}.$$

This new hamiltonian has an eigenfunction

$$\hat{u}_0 = \frac{1}{u_0^+} = \frac{u_0}{a + bI(r)}. \tag{20}$$

If neither a nor b is zero then this new eigenfunction $\hat{u}_0(r)$ will represent a bound state even if the original eigenstate $u_0(r)$ was a continuum state. The new hamiltonian is simply

$$\hat{H} = H - 2\frac{d^2 \ln u_0}{dr^2} - 2\frac{d^2 \ln \frac{a+bI(r)}{u_0}}{dr^2} = H - 2\frac{d^2 \ln(a + bI(r))}{dr^2}. \tag{21}$$

The other eigenfunctions of the new hamiltonian can also be found quite readily. We remember that we changed eigenfunctions of H into eigenfunctions of H^+ by applying the operator $A = \frac{d}{dr} - \frac{d \ln u_0}{dr}$. In the same way we

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change from an eigenfunction of H^+ to an eigenfunction of $\hat{H}$ by applying the operator $\hat{A} = \frac{d}{dr} - \frac{d \ln u_0^+}{dr}$. So we get

$$\hat{u} = \hat{A} A u$$

which after some manipulation becomes

$$\hat{u}(r) = \left\{ \frac{b}{a + bI(r)} \left[u u_0' - u_0 u' \right] \right\} u_0 - Eu. \tag{22}$$

As we can see, if we let b equal zero neither the potential nor the wave functions are altered (except for being multiplied by constants), because we are then simply returning to our original potential. If however, $b \neq 0$ then u_0 becomes a BIC and we have a direct method for finding the other eigenfunctions of the new hamiltonian from the eigenfunctions of the old hamiltonian. Furthermore, the alterations to any eigenfuntion other than $u_0(r)$ are localized, therefore, for large values of r the other wavefunction are only slightly affected.

We can now see that the SUSY method of finding bound states in the continuum is not as general as the previous treatment, but it has several advantages. First, it is simple and direct; second, it allows us to find the other eigenfunctions of the new potential; and third, it can easily be repeated on other eigenstates of the hamiltonian to get multiple bound states in the continuum.

Conclusions

Supersymmetric techniques can be used very effectively to find the eigenvalues and eigenfunctions of known potentials. They can also find solutions to problems that were not known to be exactly solvable. For example, when we studied the square well, we not only found the solutions corresponding to the square well potential, but also to several $csc^2(x)$ potentials. However, supersymmetric techniques do have a disadvantage. To use them we must know the ground state eigenfunction and eigenvalue of the original hamiltonian. However, it is often easier to solve for the ground state of a potential than it is to find all of the eigenfunctions and eigenvalues. In both the case of a simple harmonic oscillator and a hydrogen atom, simply studying the assymptotic behavior is enough to determine the ground state. (Goswami, p. 132-136, p. 257-260).

Supersymmetric techniques may also be used to generate potentials which permit bound states in the continuum. Although this method for generating BICs limits the BICs that can be found to a rather restricted class of BICs, we can easily use it to find the other states of the system and to generate potentials which permit multiple BICs.

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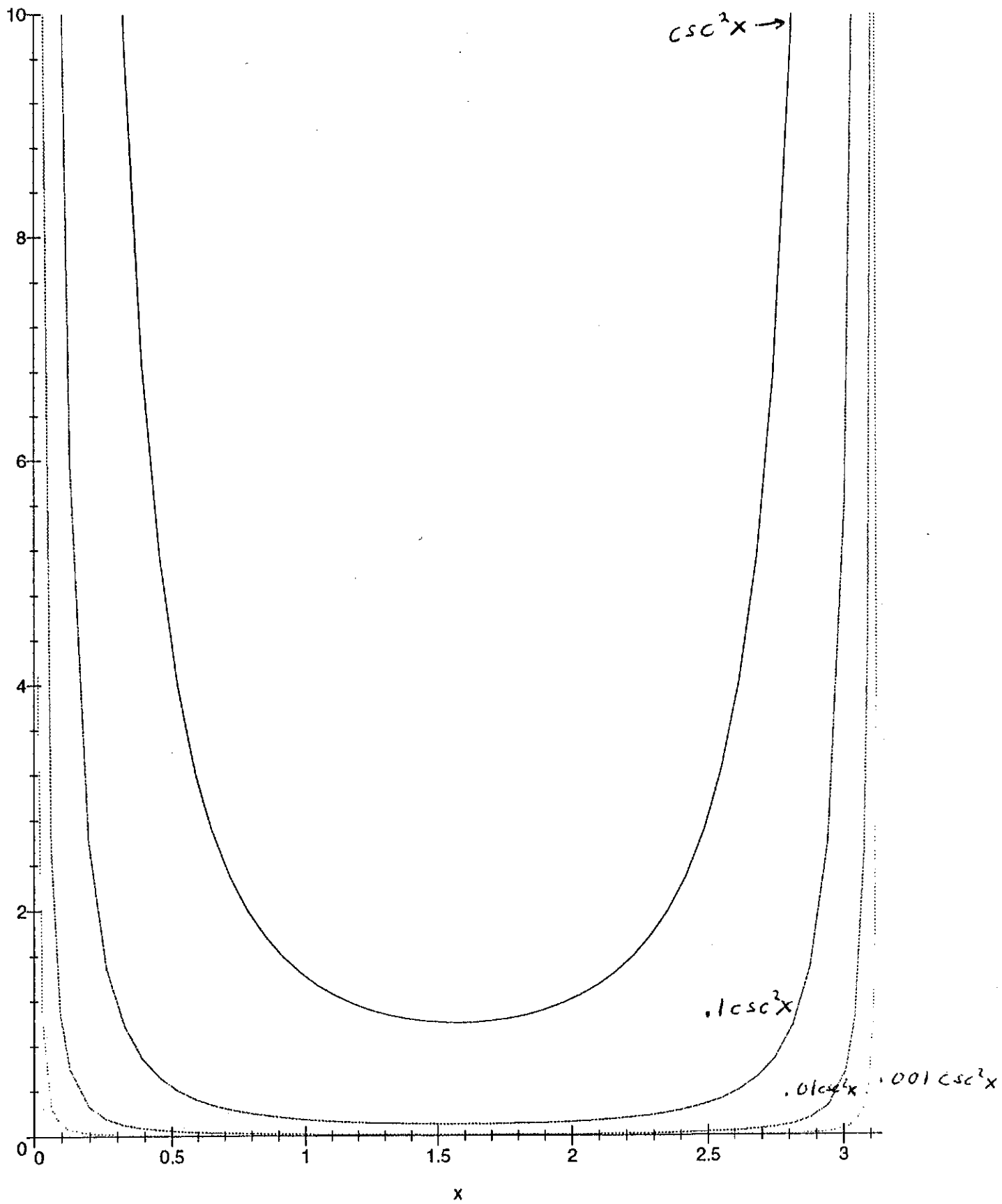
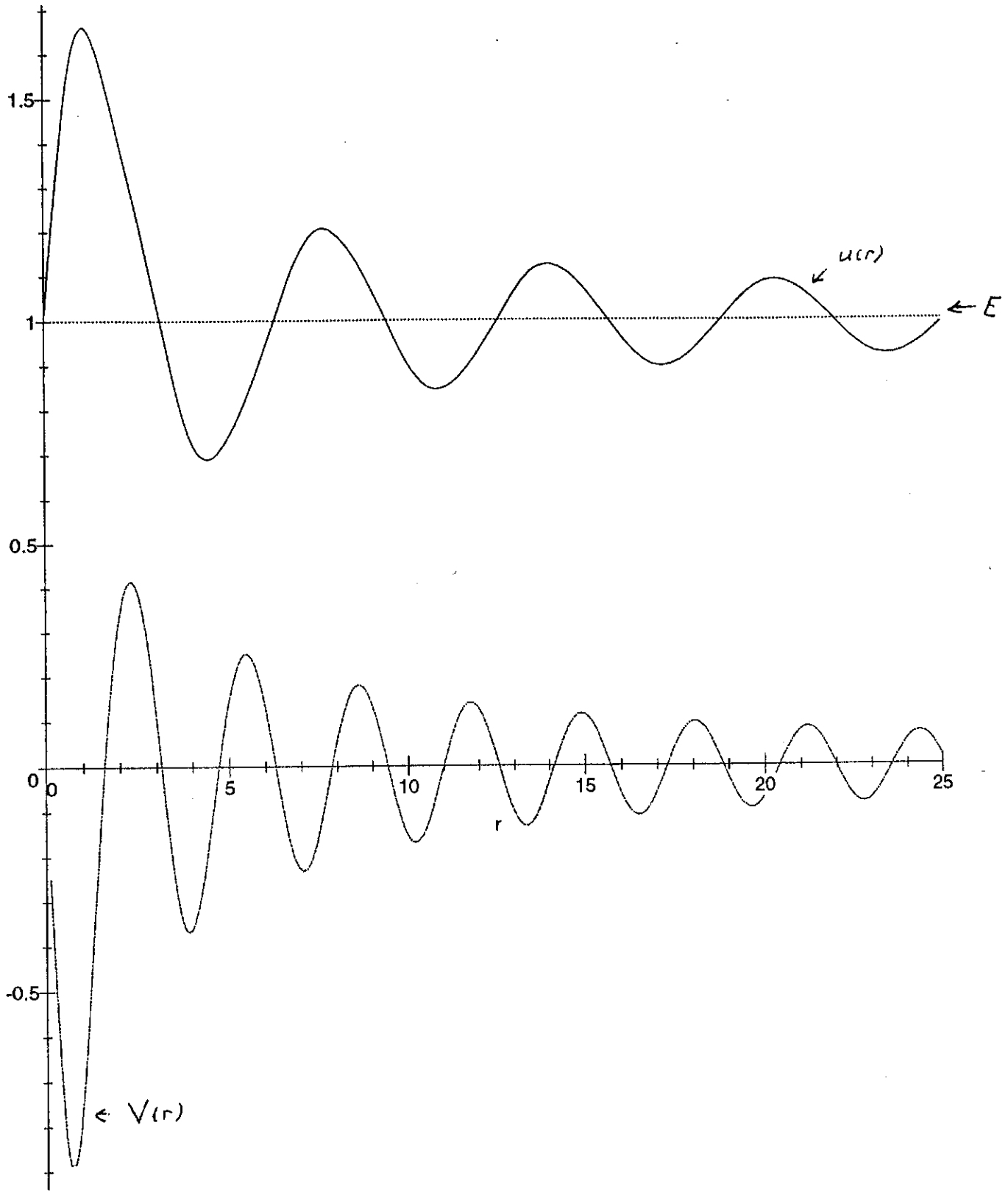J. Von Neumann and E. Wigner, Phys. Z. 30, 465 (1929).

Figure 1

Figure 2