Explorations in Quantum Channels

John G. Gardiner

A senior thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Bachelor of Science

Jean-François S. Van Huele, Advisor

Department of Physics and Astronomy

Brigham Young University

April 2015

ABSTRACT

Explorations in Quantum Channels

John G. Gardiner
Department of Physics and Astronomy, BYU
Bachelor of Science

The idea of separating the universe into a system to be studied and an environment to be ignored is important to much of physics. We explore this idea in the context of quantum systems, where it is captured by the concept of a completely positive trace-preserving map, or quantum channel. We demonstrate the usefulness of the quantum channel formalism by using it to solve the problem of optimally cloning an unknown qubit, given only its purity. We find that knowing just the purity of a qubit does not give any advantage in cloning it. The simplest quantum channels are those whose input and output are single qubits. We introduce a set of these single-qubit quantum channels that are particularly simple and have straightforward physical interpretations. The question arises whether all single-qubit channels are a composition of channels from this set. We show that, in general, they are not, suggesting the complexity of the evolution of even the smallest quantum systems.

Keywords: qubit, quantum information, quantum channel, quantum cloning, Choi-Jamiołkowski isomorphism

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Quantum Information

While working in Bell Labs in 1948, the mathematician Claude Shannon invented a framework for talking about information in a mathematical way. His ideas gave a way to precisely quantify information, allowing information to be viewed as "stuff" with its own unit of measurement, bits. This was the birth of information theory, a field that has grown to have applications not just in communications engineering, but in many other fields, including physics.

As it did for other subjects, like mechanics and field theory, quantum physics has profound consequences for information theory. From the merging of ideas in information theory and quantum physics has come the field of quantum information. Microscopically, *matter* seems to behave much differently from what we are used to from our macroscopic experience. Similarly, *information* on the quantum level behaves differently from the classical information we are used to dealing with. For instance, a fact about information we often take for granted is that it can be duplicated. We duplicate information every time we send an email, or tell a story, or read a book. Quantum information, on the other hand, is impossible to duplicate [1]. Think of a book whose words disappear as you read them or a story that you forget as soon as you tell it. This is analogous to how quantum information works. The impossibility of copying quantum information is called the no-cloning theorem and is important in this thesis.

Other peculiarities of the quantum are related to novel ways of processing information, like so-called quantum teleportation, and novel algorithms, like Shor's algorithm for factoring integers. These are only a few of the new insights into our universe that quantum information offers. For a short and accessible introduction to quantum information in general, see [2]. For a more thorough introduction, see [3].

The generalization of information theory to quantum systems is not just a curiosity. Nature is fundamentally quantum, so information at the deepest levels is encoded in quantum states. To describe this information, quantum information theory is a necessity. Quantum information has come of age as a field; it now has many applications, both direct ones, like quantum computing and cryptography, and indirect ones, like applications to black hole physics [4].

1.2 Outline of Thesis

Often a physicist's object of study is divided into two parts, one designated as the *system* and the other as the *environment*, the environment consisting of those degrees of freedom that are unknown or not under direct consideration, and the system being the degrees of freedom that are under study. A recurring problem in physics is to effectively describe a system, even given the unknown nature of its environment.

For instance, thermodynamics is a description of certain macroscopic properties (the system) where the microscopic details (the environment) are unknown and unimportant. For another example, consider the flipping of a coin. In all detail this is a rather complicated thing, but when describing this situation we simply say the coin will land heads up with probability 1/2. We do this when, in fact, a precise accounting of the movements of the tossing hand, and all the particles of air, and all the particles in the surface the coin lands on, and so on, should, at least in principle, give us an exact prediction of the outcome. In this situation the system is the orientation of the coin while the environment is everything else. In these cases certain important properties are given an effective description, while the unimportant, unknown, or uninteresting aspects, i.e. the environment, are "traced over" or "integrated out."

The system/environment idea is the motivation behind the topic of this thesis: completely positive trace-preserving maps, also known as quantum channels. Quantum channels, the focus of Chapter 2, provide a description of the evolution of a quantum system. In quantum physics the evolution of a closed system, one which does not interact with its environment, is given by a unitary operator. Quantum channels are similar to unitary operators but generalized to also include the more complicated evolution of open systems. As we will see in Chapter 2, the concept of a quantum channel is the natural consequence of the system/environment distinction applied to quantum physics. Chapter 2 begins with the definition of a quantum channel and explains some of its useful properties. Specifically Sections 2.1 and 2.2 contain background information necessary for understanding the results of the thesis, which are obtained in the subsequent sections. In Section 2.3 we use these to solve the problem of optimizing a specific quantum information processing task, while in Section 2.4 we explore the possibility of decomposing quantum channels into the composition of a handful of simpler channels, an exploration that will give some insight into their nature.

Chapter 3 restates the original results from Chapter 2 and discusses possible interpretations of the results.

1.3 Preliminary Concept: the Qubit

We end this introduction with a brief explanation of some concepts used throughout. A reader with prior exposure to quantum information might skip this section.

We will use Dirac notation throughout this work. For an elementary explanation of Dirac notation, see [2].

Analogous to the bit in classical information theory, the fundamental unit of quantum information is the *qubit*. A qubit, like a bit, can be one or zero. But a qubit, being a quantum state, can also be in a superposition of 1 and 0. An arbitrary qubit is of the form $\alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers. Of course, as qubits are quantum states, they are distinguished only up to a norm and a phase. The space of distinct qubits is topologically a sphere. One can thus visualize all the possible qubits by imagining a unit sphere where each point represents a different qubit state and points opposite to each other on the sphere represent orthogonal states. This way of representing qubits is called the Bloch sphere (see Fig. 1.1).

Quantum states are vectors in a Hilbert space. A useful generalization of this formalism is the density matrix formalism. In this formalism, states are written as matrices rather than vectors. As an example an arbitrary qubit $\alpha|0\rangle + \beta|1\rangle$ is represented by the matrix $\big(\alpha|0\rangle + \beta|1\rangle\big)\big(\alpha\langle 0| + \beta\langle 1|\big)$. The purpose of rewriting states this way is that they can be easily generalized to include what are called mixed states. Consider a machine that prepares a qubit when turned on. Specifically, the machine will either prepare the qubit $|0\rangle$ or the qubit $|1\rangle$. Which it prepares is left up to chance: the machine flips a coin and uses the outcome to decide which of $|0\rangle$ or $|1\rangle$ to output. It does this without signifying which choice was made. How does one describe the output of this machine? A naive guess might be that the output is $|\psi\rangle = \sqrt{\frac{1}{2}}\big(|0\rangle + |1\rangle\big)$. After all, $|\psi\rangle$ has probability $\frac{1}{2}$ of being $|0\rangle$ upon measurement or of being $|1\rangle$ upon measurement, just like the output of the machine does. However, the output of the machine is not really a superposition of $|0\rangle$ and $|1\rangle$; though the superposition $|\psi\rangle$ will satisfy *some* of the same statistics as the output of the machine,

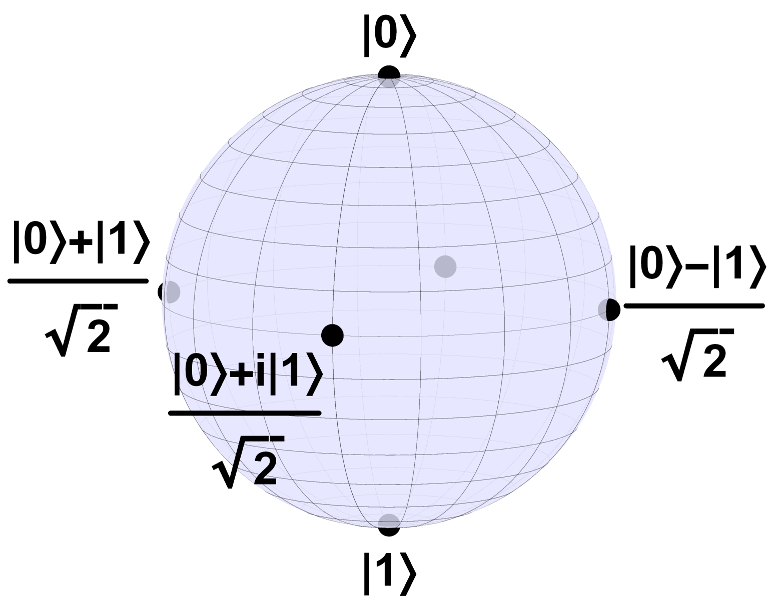


Figure 1.1 The Bloch sphere gives a convenient representation of the space of qubit states. Each state is a point on the surface of the sphere. Shown are a few states and their corresponding points.

it will not satisfy *all* of the same statistics. So it does not accurately describe the output of the machine. The output of the machine is a statistical mixture of quantum states, a *mixed state*. The density matrix formalism gives us the correct way to represent mixed states. As an example, the output of the machine would be correctly represented by the matrix $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ which can be thought of as the weighted average of the states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$.

The space of qubits, including mixed state qubits, is topologically a ball. One can visualize such a generalized qubit as a point in the unit ball. Mixed states are points in the interior, while *pure states*, states that are not mixed, are on the surface. This way of visualizing states is called the Bloch ball. The surface of the Bloch ball is simply the Bloch sphere mentioned above. The *purity* of a state is how mixed it is, i.e. how close it is to the surface of the Bloch ball. For a detailed discussion of qubits, mixed states, and the Bloch ball, see [3].

Chapter 2

Quantum Channels

2.1 Formalism

In quantum mechanics the evolution of a (closed) system over some time interval is described by a unitary operator acting on the state vector of the system. Say the system starts in some initial state represented by the density operator $|\text{initial}\rangle\langle\text{initial}|$. The state at some later time will have evolved to

$$|\text{final}\rangle\langle\text{final}| = U\,|\text{initial}\rangle\langle\text{initial}|\,U^{\dagger},$$

where U is some appropriate unitary transformation. This is the paradigm for evolution of any closed quantum system. What about open systems, i.e. systems that interact with an environment? The answer comes via the trick of viewing the system and its environment together as one larger system. This joint system is itself closed (though neither of its constituent parts is), and therefore its evolution will be described by a unitary operator. For example, consider a system described by the state vector $|\text{system}\rangle$ and its environment, described by $|\text{environment}\rangle$. The system and environment together are initially described as

$$|\text{initial}\rangle\langle\text{initial}| = |\text{system}\rangle\langle\text{system}| \otimes |\text{environment}\rangle\langle\text{environment}|,$$

which evolves unitarily; at a later time the system and environment together will be

$$|\text{final}\rangle\langle\text{final}| = U\big(|\text{system}\rangle\langle\text{system}| \otimes |\text{environment}\rangle\langle\text{environment}|\big)U^{\dagger}.$$

To examine just the system itself, we trace out the environment, i.e. sum over all possible configurations of the environment. Let $\{|1\rangle, |2\rangle, \ldots, |N\rangle\}$ be a basis for the environment Hilbert space. Then the system after evolution is described by the density matrix ρ_{system}, such that

$$\begin{aligned}\rho_{\text{system}} &= \sum_{i=1}^{N} \langle i|\text{final}\rangle\langle\text{final}|i\rangle \\ &= \sum_{i=1}^{N} \langle i|U\big(|\text{system}\rangle\langle\text{system}| \otimes |\text{environment}\rangle\langle\text{environment}|\big)U^{\dagger}|i\rangle.\end{aligned}$$

Let $\mathscr{E}$ be the linear map defined by

$$\mathscr{E}(\rho) = \sum_{i=1}^{N} \langle i|U\big(\rho \otimes |\text{environment}\rangle\langle\text{environment}|\big)U^{\dagger}|i\rangle.$$

The evolution of the open system is given by this map $\mathscr{E}$. In general, the evolution of an open quantum system is described by a map like $\mathscr{E}$. Such maps are called completely positive trace-preserving maps, quantum operations, or quantum channels . More details on quantum channels, including a precise definition, can be found in [3]. For our purposes it suffices to know the following two properties:

- Where $\mathscr{H}_{\text{i}}$ is the Hilbert space describing the initial system and $\mathscr{H}_{\text{f}}$ is the Hilbert space of the final system, a quantum channel is a linear map from density matrices on $\mathscr{H}_{\text{i}}$ to density matrices on $\mathscr{H}_{\text{f}}$.
- Any quantum channel $\mathscr{E}$ can be written in the form $\mathscr{E}(\rho) = \sum_i A_i \rho A_i^{\dagger}$ where the A_i, called Kraus operators, are a finite set of operators satisfying $A_i^{\dagger} A_i = \mathbb{1}$.

2.1.1 Examples

As an example of a quantum channel, imagine a quantum memory that stores a qubit but with some failure, so that over a certain time interval there is a probability p that the qubit switches value, $|1\rangle$

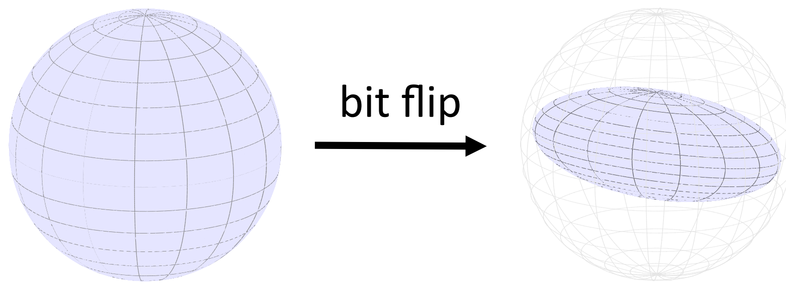


Figure 2.1 The effect of a bit flip on Bloch ball is to deform it into an ellipsoid whose smallest axes have length $2\,|1-2p|$.

becoming $|0\rangle$ and $|0\rangle$ becoming $|1\rangle$. The evolution over this interval is described by the map

$$\mathcal{E}_{\text{bit flip}}(\rho) = (1-p)\rho + p\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\rho\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The first term represents the probability $1-p$ that there is no error so that the initial state ρ remains ρ. The second term represents the probability p of an error switching the bit. This is a quantum channel called the *bit-flip channel*. For more examples of simple single-qubit quantum channels with clear physical interpretations, see [3]. To further illustrate the concept of a quantum channel, some of these channels are described in Table 2.1.

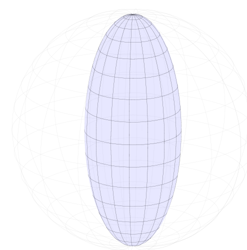
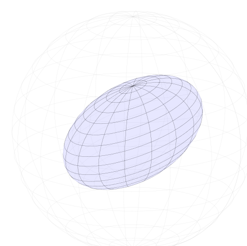
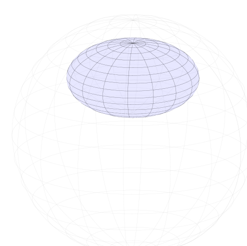
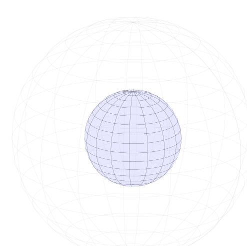
We can visualize single-qubit channels by their effect in the Bloch ball. The effect of a single-qubit channel on the Bloch ball is an affine transformation [3]. (An affine transformation is simply some combination of stretching, rotating, and translating.) For instance, when we input the Bloch ball to the bit flip channel $\mathcal{E}_{\text{bit flip}}$ described above, the output is an ellipsoid as pictured in Fig. 2.1.

The transformations for the simple channels listed in Table 2.1 are given in Table 2.2. Note that the effect of a unitary transformation on the Bloch ball is simply a rotation. As is reflected in the figures in Table 2.2, the bit flip, phase flip and bit-phase flip channels are related to each other

Table 2.1 The action of a quantum channel $\mathscr{E}$ on a state ρ can be given as $\mathscr{E}(\rho) = \sum_j A_j \rho A_j^\dagger$ where A_j are the Kraus operators. Listed are simple single-qubit channels and corresponding Kraus operators, where p and α are constants.

Channel	Kraus operators
phase flip	$\sqrt{1-p}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sqrt{p}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
bit flip	$\sqrt{1-p}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sqrt{p}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
bit-phase flip	$\sqrt{1-p}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sqrt{p}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$
generalized amplitude damping	$\sqrt{1-p}\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-\alpha}\end{pmatrix}, \quad \sqrt{1-p}\begin{pmatrix}0 & \sqrt{\alpha}\\ 0 & 0\end{pmatrix},$ $\sqrt{p}\begin{pmatrix}\sqrt{1-\alpha} & 0\\ 0 & 1\end{pmatrix}, \quad \sqrt{p}\begin{pmatrix}0 & 0\\ \sqrt{\alpha} & 0\end{pmatrix}$
depolarizing channel	$\sqrt{1-p}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sqrt{\frac{p}{3}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sqrt{\frac{p}{3}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},$ $\sqrt{\frac{p}{3}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Table 2.2 The action of a single-qubit channel on the Bloch ball is an affine transformation. This table shows the image of the Bloch ball under some simple channels.

Channel	Image of Bloch ball
phase flip	
bit-phase flip	
generalized amplitude damping	
depolarizing channel	

by unitary transformations. Also, it is easy to see that the depolarizing channel is equivalent to a bit flip, a phase flip, and a bit-phase flip applied in sequence.

As mentioned above, every affine transformation is the composition of rotations, scalings of the axes, and translations. These three roughly correspond to unitary transformations, bit (or phase or bit-phase) flips, and generalized amplitude dampings respectively. It seems reasonable that it should be possible to create *any* affine transformation of the Bloch ball from the transformations available from unitary transformations, bit flips, and generalized amplitude dampings alone. We would thus be able to consider any single-qubit quantum channel as the composition of unitaries, bit flips, and generalized amplitude dampings, three particularly simple channels with clear physical interpretations. The question of whether this is indeed possible is the subject of Section 2.4.

2.1.2 Choi-Jamiołkowski Isomorphism

Though any channel can be written in terms of Kraus operators, i.e. $\mathscr{E}(\rho) = \sum_i A_i \rho A_i^\dagger$, it cannot always be written *uniquely* in this form. There can be many choices of Kraus operators that give the same channel. This redundancy can complicate searches for channels with certain properties. So Kraus operators are in some sense not the best way to parametrize the space of quantum channels. A better way is to make use a standard tool of quantum information called the Choi-Jamiołkowski isomorphism. Where $\mathscr{H}$ and $\mathscr{H}'$ are any Hilbert spaces, this isomorphism is a one-to-one correspondence between quantum channels from $\mathscr{H}$ to $\mathscr{H}'$ and density matrices for the space $\mathscr{H} \otimes \mathscr{H}'$ [5]. This isomorphism associates the channel $\mathscr{E}$ with the density matrix

$$\chi = \frac{1}{N} \sum_{j,k}^{N} |j\rangle \langle k| \otimes \mathscr{E}\big(|j\rangle \langle k|\big),$$

where $\{|1\rangle, |2\rangle, \ldots, |N\rangle\}$ is an orthonormal basis for $\mathscr{H}$. Or, in the reverse direction, it associates the density matrix χ with the channel given by

$$\mathscr{E}(\rho) = N \, \mathrm{Tr}_{\mathscr{H}} \big(\chi(\rho^T \otimes \mathbb{1}_{\mathscr{H}'})\big),$$

where the trace is over the $\mathscr{H}$ system, i.e. $\chi(\rho^T \otimes \mathbb{1}_{\mathscr{H}'})$ is an operator on $\mathscr{H} \otimes \mathscr{H}'$ and $\mathrm{Tr}_{\mathscr{H}}\big(\chi(\rho^T \otimes \mathbb{1}_{\mathscr{H}'})\big)$ is an operator on $\mathscr{H}'$.

With this isomorphism the quantum channels on a space are parametrized by density matrices of a larger space. This can reduce a search for a desired channel to a search for a certain state. We will use this idea in Section 2.3.

2.2 Quantum Cloning Channels

The identification of quantum channels with states allows us to more easily understand the space of quantum channels. In particular, it allows us to more easily optimize over this space. In Section 2.3 we will present a simple example of this. In this section we introduce a prerequisite concept, that of a quantum cloning channel. For classical information the operation of copying the information is important and ubiquitous. For instance, it happens in a literal sense every time you download a file. In the case of quantum information however no such copying operation is possible [1]. Suppose on the contrary there was a quantum channel that could copy a qubit, so $|0\rangle \mapsto |0\rangle|0\rangle$ and $|1\rangle \mapsto |1\rangle|1\rangle$. By linearity we would have $|0\rangle + |1\rangle \mapsto |0\rangle|0\rangle + |1\rangle|1\rangle$. This contradicts the premise that the operation copies its input which should mean

$$|0\rangle + |1\rangle \mapsto \big(|0\rangle + |1\rangle\big)\big(|0\rangle + |1\rangle\big) \neq |0\rangle|0\rangle + |1\rangle|1\rangle .$$

So no such quantum channel exists! The result that quantum information cannot be copied is called the no-cloning theorem. Though quantum information cannot be copied, it can be *approximately* copied. One can devise quantum channels that can copy states to some approximation. Such channels are called quantum cloning channels, or cloners [6]. More precisely, a quantum cloning channel is a channel $\mathscr{E}\colon \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ whose purpose is to (at least approximately) copy the input state. We can quantify the accuracy with which a cloner copies its input with a function called fidelity [7]. The fidelity $F(\rho, \sigma)$ between states ρ and σ is, in a sense, a measure of how close they are to each other. A fidelity of 1 means the two are the same state. A fidelity of 0 means they are orthogonal. If ρ and σ are qubits, as they will be in the case of interest below, the fidelity is given by

$$F(\rho,\sigma) = \mathrm{Tr}(\rho\sigma) + \sqrt{1-\mathrm{Tr}(\rho^2)}\sqrt{1-\mathrm{Tr}(\sigma^2)}.$$

Fidelity has several important properties, among which are convexity

$$F(\rho, \alpha\sigma_1 + \beta\sigma_2) \geq \alpha F(\rho,\sigma_1) + \beta F(\rho,\sigma_2), \tag{2.1}$$

and invariance under unitary transformation

$$F(\rho,\sigma) = F\big(\mathscr{V}(\rho), \mathscr{V}(\sigma)\big) \quad \text{for all unitary transformations } \mathscr{V}. \tag{2.2}$$

We will make use of these two properties later.

One way to quantify the accuracy of a cloner $\mathscr{E}$ is to average the fidelity between the input and first output with the fidelity between the input and second output and then to average this over all possible input states. So the accuracy Acc of $\mathscr{E}$ is

$$\mathrm{Acc}(\mathscr{E}) = \int_{\substack{\text{input}\\\text{states}}} \frac{F\big(\rho, \mathrm{Tr}_2(\mathscr{E}(\rho))\big) + F\big(\rho, \mathrm{Tr}_1(\mathscr{E}(\rho))\big)}{2} \, \mathrm{d}\rho \Bigg/ \int_{\substack{\text{input}\\\text{states}}} \mathrm{d}\rho, \tag{2.3}$$

where Tr_1 is the trace over the first output and Tr_2 the trace over the second.

2.3 Cloner for Known Purity

Another way of phrasing the no-cloning theorem is that an *unknown* quantum state cannot be cloned perfectly. Of course a *known* state can be cloned perfectly. If for example we know beforehand that the state to be cloned is $|1\rangle$ we can simply prepare the state $|1\rangle|1\rangle$. This is an illustration of a more general principle: the more one knows about the state before attempting to clone it the more accurately it can be cloned. In general, we can ask the question of how well we can clone a state, given *partial* information about the state. Given some information about the state to be cloned, the optimal cloning problem asks what quantum cloning channel will copy the input with the highest accuracy, on average. We will use the Choi-Jamiołkowsk isomorphism to solve such a problem.

Consider the problem of optimally cloning a qubit given only the purity of that qubit. That is, we are given a qubit about which we know nothing except its radius r in the Bloch ball, and we wish to copy it as well as we can. Our desired cloner can be described as a quantum channel $\mathscr{E}$ from a two-dimensional Hilbert space $\mathscr{H}_{in}$ to a four dimensional Hilbert space $\mathscr{H}_{\text{out}} = \mathscr{H}_1 \otimes \mathscr{H}_2$ so that $\rho_{\text{out}} = \mathscr{E}(\rho_{\text{in}})$ where ρ_{in} is the state to be cloned and ρ_{out} is the state that results from the cloning.

Via the Choi-Jamiołkowski isomorphism the map $\mathscr{E}$ can be represented by a positive operator χ such that $\rho_{\text{out}} = \mathscr{E}(\rho_{\text{in}}) = \text{Tr}_{\text{in}}\left(\chi(\rho_{\text{in}}^T \otimes \mathbb{1}_{\text{out}})\right)$, where $\mathbb{1}_{\text{out}}$ is the identity operator in $\mathscr{H}_{\text{out}}$. The problem of finding an optimal cloner reduces to the problem of finding the optimal corresponding operator χ.

Using the definition of Eq. (2.3), the accuracy of the cloner $\mathscr{E}$ is then

$$Acc(\mathscr{E}) = \frac{1}{4\pi r}\int_0^{\pi}\int_0^{2\pi} \frac{F\left(\rho_{\text{in}}, \text{Tr}_2(\mathscr{E}(\rho_{\text{in}}))\right) + F\left(\rho_{\text{in}}, \text{Tr}_1(\mathscr{E}(\rho_{\text{in}}))\right)}{2} r \sin\theta \,\mathrm{d}\phi \,\mathrm{d}\theta, \tag{2.4}$$

where the general input state with the specified radius r is given by

$$\rho_{\text{in}}(\theta,\phi) = \begin{pmatrix} r\left(\cos\frac{\theta}{2}\right)^2 + \frac{1-r}{2} & re^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ re^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & r\left(\sin\frac{\theta}{2}\right)^2 + \frac{1-r}{2} \end{pmatrix}.$$

The goal is to find a cloner $\mathscr{E}$ that will maximize the quantity in Eq. (2.4).

Let $\mathscr{U}$ be the quantum operation that switches the first and second qubits of a two-qubit state, so that $\mathrm{Tr}_2\left(\mathscr{U}(\sigma)\right) = \mathrm{Tr}_1\left(\sigma\right)$ and $\mathrm{Tr}_1\left(\mathscr{U}(\sigma)\right) = \mathrm{Tr}_2\left(\sigma\right)$. Then

$$\begin{aligned}
\mathrm{Acc}(\mathscr{E}) =& \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\left(\frac{F\left(\rho_{\mathrm{in}}, \mathrm{Tr}_2\left(\mathscr{E}(\rho_{\mathrm{in}})\right)\right)}{2} + \frac{F\left(\rho_{\mathrm{in}}, \mathrm{Tr}_1\left(\mathscr{E}(\rho_{\mathrm{in}})]\right)\right)}{2}\right)\sin\theta \mathrm{d}\phi \mathrm{d}\theta \\
=& \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\left(\frac{\frac{1}{2}F\left(\rho_{\mathrm{in}}, \mathrm{Tr}_1\left(\mathscr{E}(\rho_{\mathrm{in}})\right)\right) + \frac{1}{2}F\left(\rho_{\mathrm{in}}, \mathrm{Tr}_1\left(\mathscr{U}\mathscr{E}(\rho_{\mathrm{in}})\right)\right)}{2}\right. \\
&\left. + \frac{\frac{1}{2}F\left(\rho_{\mathrm{in}}, \mathrm{Tr}_2\left(\mathscr{E}(\rho_{\mathrm{in}})\right)\right) + \frac{1}{2}F\left(\rho_{\mathrm{in}}, \mathrm{Tr}_2\left(\mathscr{U}\mathscr{E}(\rho_{\mathrm{in}})\right)\right)}{2}\right)\sin\theta \mathrm{d}\phi \mathrm{d}\theta,
\end{aligned}$$

and by convexity we have

$$\begin{aligned}
\mathrm{Acc}(\mathscr{E}) \leq& \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\left(\frac{F\left(\rho_{\mathrm{in}}, \frac{1}{2}\mathrm{Tr}_2\left(\mathscr{E}(\rho_{\mathrm{in}})\right) + \frac{1}{2}\mathrm{Tr}_2\left(\mathscr{U}\mathscr{E}(\rho_{\mathrm{in}})\right)\right)}{2}\right. \\
&\left. + \frac{F\left(\rho_{\mathrm{in}}, \frac{1}{2}\mathrm{Tr}_1\left(\mathscr{E}(\rho_{\mathrm{in}})\right) + \frac{1}{2}\mathrm{Tr}_1\left(\mathscr{U}\mathscr{E}(\rho_{\mathrm{in}})\right)\right)}{2}\right)\sin\theta \mathrm{d}\phi \mathrm{d}\theta \\
=& \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\left(\frac{F\left(\rho_{\mathrm{in}}, \mathrm{Tr}_2\left(\mathscr{E}'(\rho_{\mathrm{in}})\right)\right)}{2} + \frac{F\left(\rho_{\mathrm{in}}, \mathrm{Tr}_1\left(\mathscr{E}'(\rho_{\mathrm{in}})\right)\right)}{2}\right)\sin\theta \mathrm{d}\phi \mathrm{d}\theta \\
=& \mathrm{Acc}(\mathscr{E}'),
\end{aligned}$$

where

$$\mathscr{E}' = \frac{1}{2}(\mathscr{E} + \mathscr{U}\mathscr{E}).$$

Notice that $\mathscr{E}'$ is symmetric in its outputs, i.e. $\mathscr{E}' = \mathscr{U}\mathscr{E}'$. The accuracy of the cloner $\mathscr{E}'$ is greater than or equal to that of $\mathscr{E}$. Thus if $\mathscr{E}$ is a cloner of maximum accuracy then $\mathscr{E}'$ is as well. So in our search for a cloner of maximum accuracy we only need consider those cloners that are symmetric in their outputs.

Because every input state contributes equally to the accuracy, the quantity of Eq. (2.4), we can simply substitute $\mathscr{V}(\rho_{\mathrm{in}})$ for ρ_{in}, where $\mathscr{V}$ is a unitary transformation, without changing the integral. So

$$\mathrm{Acc}(\mathscr{E}) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\left(\frac{F\left(\mathscr{V}(\rho_{\text{in}}),\mathrm{Tr}_2\left(\mathscr{E}\mathscr{V}(\rho_{\text{in}})\right)\right)}{2}+\frac{F\left(\mathscr{V}(\rho_{\text{in}}),\mathrm{Tr}_1\left(\mathscr{E}\mathscr{V}(\rho_{\text{in}})\right)\right)}{2}\right)\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta. \tag{2.5}$$

Now applying to Eq. (2.5) the fact that fidelity is invariant under the unitary transformation $\mathscr{V}^\dagger$ gives

$$\begin{aligned}\mathrm{Acc}(\mathscr{E}) =& \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\left(\frac{F\left(\mathscr{V}^\dagger\mathscr{V}(\rho_{\text{in}}),\mathscr{V}^\dagger\,\mathrm{Tr}_2\left(\mathscr{E}\mathscr{V}(\rho_{\text{in}})\right)\right)}{2}\right.\\ &\left.+\frac{F\left(\mathscr{V}^\dagger\mathscr{V}(\rho_{\text{in}}),\mathscr{V}^\dagger\,\mathrm{Tr}_1\left(\mathscr{E}\mathscr{V}(\rho_{\text{in}})\right)\right)}{2}\right)\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\\ =& \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\left(\frac{F\left(\rho_{\text{in}},\mathrm{Tr}_2\left(\left(\mathscr{V}^\dagger\otimes\mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}(\rho_{\text{in}})\right)\right)}{2}\right.\\ &\left.+\frac{F\left(\rho_{\text{in}},\mathrm{Tr}_1\left(\left(\mathscr{V}^\dagger\otimes\mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}(\rho_{\text{in}})\right)\right)}{2}\right)\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\\ =& \mathrm{Acc}\left(\left(\mathscr{V}^\dagger\otimes\mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}\right),\end{aligned}$$

where the second equality is from the unitarity of $\mathscr{V}$ and the properties of the partial trace. So the accuracy is invariant under the conjugation of its argument by any unitary transformation, i.e. $\mathscr{E}$ and $\left(\mathscr{V}^\dagger\otimes\mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}$ have the same accuracy. Let

$$\mathscr{E}' = \int_{\text{unitaries}}\left(\mathscr{V}^\dagger\otimes\mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}\;\mathrm{d}\mathscr{V}\Bigg/\int_{\text{unitaries}}\mathrm{d}\mathscr{V}.$$

Note that the form of $\mathscr{E}'$ gives it the special property $\mathscr{E}' = \left(\mathscr{V}^\dagger\otimes\mathscr{V}^\dagger\right)\mathscr{E}'\mathscr{V}$ for any unitary $\mathscr{V}$. By the convexity of fidelity we can obtain

$$\begin{aligned}\mathrm{Acc}(\mathscr{E}') &= \mathrm{Acc}\left(\int \left(\mathscr{V}^\dagger \otimes \mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}\ \mathrm{d}\mathscr{V} \middle/ \int \mathrm{d}\mathscr{V}\right)\\ &\geq \int \mathrm{Acc}\left(\left(\mathscr{V}^\dagger \otimes \mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}\right)\ \mathrm{d}\mathscr{V} \middle/ \int \mathrm{d}\mathscr{V}\\ &= \int \mathrm{Acc}\left(\mathscr{E}\right)\ \mathrm{d}\mathscr{V} \middle/ \int \mathrm{d}\mathscr{V}\\ &= \mathrm{Acc}(\mathscr{E}).\end{aligned}$$

Thus if $\mathscr{E}$ is a cloner of maximum accuracy then $\mathscr{E}'$ is as well. This implies that in our search for an optimal cloner we need only consider cloners $\mathscr{E}$ with the special property that $\mathscr{E} = \left(\mathscr{V}^\dagger \otimes \mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}$ for any unitary $\mathscr{V}$.

There is a density operator χ corresponding to an optimal cloner map $\mathscr{E}$ according to the relation $\chi = \frac{1}{N}\sum_{j,k} |j\rangle\langle k| \otimes \mathscr{E}\left(|j\rangle\langle k|\right)$. The assumptions $\mathscr{E} = \mathscr{U}\mathscr{E}$ and $\mathscr{E} = \left(\mathscr{V}^\dagger \otimes \mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}$ dictate the form that χ can take. First, because $\mathscr{E}$ is symmetric

$$(\mathbb{1} \otimes \mathscr{U})(\chi) = \frac{1}{N}\sum_{j,k} |j\rangle\langle k| \otimes \mathscr{U}\mathscr{E}\left(|j\rangle\langle k|\right) = \frac{1}{N}\sum_{j,k} |j\rangle\langle k| \otimes \mathscr{E}\left(|j\rangle\langle k|\right) = \chi, \tag{2.6}$$

where $\mathscr{U}$ is the unitary channel that swaps the first and second qubits of a two-qubit state. Because $\mathscr{E} = \left(\mathscr{V}^\dagger \otimes \mathscr{V}^\dagger\right)\mathscr{E}\mathscr{V}$, and hence $(\mathscr{V} \otimes \mathscr{V})\mathscr{E} = \mathscr{E}\mathscr{V}$, for any unitary $\mathscr{V}$, we get

$$\begin{aligned}(\overline{\mathscr{V}} \otimes \mathscr{V} \otimes \mathscr{V})(\chi) &= \frac{1}{N}\sum_{j,k} \overline{\mathscr{V}}\left(|j\rangle\langle k|\right) \otimes (\mathscr{V} \otimes \mathscr{V})\mathscr{E}\left(|j\rangle\langle k|\right)\\ &= \frac{1}{N}\sum_{j,k} \overline{\mathscr{V}}\left(|j\rangle\langle k|\right) \otimes \mathscr{E}\mathscr{V}\left(|j\rangle\langle k|\right)\\ &= (\mathbb{1} \otimes \mathscr{E})(\overline{\mathscr{V}} \otimes \mathscr{V})\left(\frac{1}{N}\sum_{j,k} |j\rangle\langle k| \otimes |j\rangle\langle k|\right)\\ &= (\mathbb{1} \otimes \mathscr{E})\left(\frac{1}{N}\sum_{j,k} |j\rangle\langle k| \otimes |j\rangle\langle k|\right)\\ &= \chi,\end{aligned} \tag{2.7}$$

where $\overline{\mathscr{V}}$ is the complex conjugate of $\mathscr{V}$. The fourth equality can be seen by expanding $\mathscr{V}$ and $\overline{\mathscr{V}}$ in the standard basis, after which it becomes straightforward that $\sum_{j,k} |j\rangle\langle k| \otimes |j\rangle\langle k|$ does not change when acted upon by $\overline{\mathscr{V}} \otimes \mathscr{V}$. Equations (2.6) and (2.7) constitute constraints on the form that χ can take. Two more constraints on χ come from the fact that it is a density matrix: χ must be have trace 1 and χ must be positive, i.e. its eigenvalues must be nonnegative. Taking into account all four of these constraints gives the form of χ as

$$\chi = \frac{1}{2}\begin{pmatrix} A & 0 & 0 & 0 & 0 & A+B-\frac{1}{2} & A+B-\frac{1}{2} & 0 \\ 0 & B & \frac{1}{2}-2B & 0 & 0 & 0 & 0 & A+B-\frac{1}{2} \\ 0 & \frac{1}{2}-2B & B & 0 & 0 & 0 & 0 & A+B-\frac{1}{2} \\ 0 & 0 & 0 & 1-A-2B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-A-2B & 0 & 0 & 0 \\ A+B-\frac{1}{2} & 0 & 0 & 0 & 0 & B & \frac{1}{2}-2B & 0 \\ A+B-\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2}-2B & B & 0 \\ 0 & A+B-\frac{1}{2} & A+B-\frac{1}{2} & 0 & 0 & 0 & 0 & A \end{pmatrix}$$

with the eigenvalues of χ, namely $1-A-2B$, $\frac{1}{2}(4A+2B-1)$, and $\frac{1}{2}(6B-1)$, nonnegative. Using this form for χ and our definition of accuracy, it is a straightforward calculation to evaluate the accuracy in terms of A, B and r. One gets

$$\mathrm{Acc}(\mathscr{E}) = r^2(A+B) + \frac{1-r^2}{2} + \sqrt{(1-r^2)\left(-r^2(A+B)^2 + r^2(A+B) + \frac{1-r^2}{4}\right)}. \tag{2.8}$$

Maximizing Eq. (2.8) with respect to A and B while taking into consideration the positivity constraints $1-A-2B \geq 0$, $\frac{1}{2}(4A+2B-1) \geq 0$, and $\frac{1}{2}(6B-1) \geq 0$ gives a maximum accuracy of

$$\mathrm{Acc}(\mathscr{E}) = \frac{1}{3}r^2 + \frac{1}{2} + \sqrt{(1-r^2)\left(\frac{1}{4} - \frac{1}{9}r^2\right)} \tag{2.9}$$

when $A = 2/3$ and $B = 1/6$. These values of A and B correspond to a well-known cloner called the the Hillery-Bužek cloner [8]. Thus, when only the radius of a state in the Bloch ball is known *a priori*, the optimal cloner is the Hillery-Bužek cloner, with accuracy given by Eq. (2.9). When r is 1 this evaluates to $\frac{5}{6}$ as expected [8]. When r is 0 it evaluates to 1. This signifies that the maximally mixed state can be cloned perfectly.

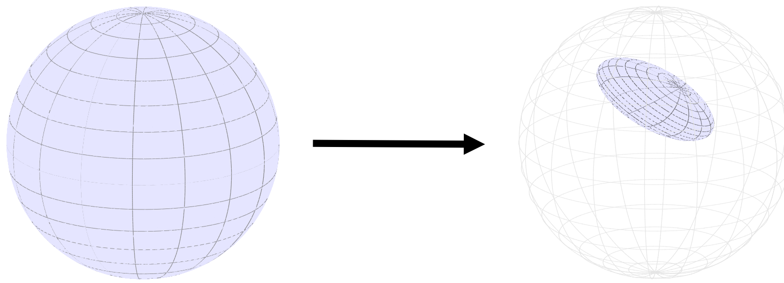


Figure 2.2 The effect of $\mathscr{E}$ on the Bloch ball is a more complicated affine transformation.

2.4 Decomposition of Single-Qubit Channels

In Section 2.1.1 we mentioned in particular three types of channels: unitary transformations, bit flips, and generalized amplitude damplings. These are simple and have easily understood physical meaning [3]. On the other hand consider the quantum channel $\mathscr{E}$ with the Kraus matrices

$$\begin{pmatrix} 0.78901 & -0.32682 \\ 0.16276 & 0.51813 \end{pmatrix}, \begin{pmatrix} 0.17911 & 0.69911 \\ 0.23097 & 0.095671 \end{pmatrix}, \begin{pmatrix} -0.23097 & -0.095671 \\ 0.40005 & 0.16571 \end{pmatrix}, \begin{pmatrix} -0.13279 & 0.055003 \\ -0.18582 & 0.29528 \end{pmatrix}.$$

The effect of $\mathscr{E}$ on the Bloch ball is pictured in Fig. 2.2. This affine transformation is visibly more complicated than those pictured in Section 2.1.1.

Unitary transformations are simply rotations of the Bloch ball, bit flips involve only scalings of the axes, and generalized amplitude dampings involve axis scalings along with a translation. Rotations, scalings and translations are the building blocks of any affine transformation. This suggests that the complicated affine transformation induced by $\mathscr{E}$ might by the composition of an appropriate generalized amplitude damping (to provide the correct translation) and a combination of unitary transformations and bit flips (to provide the correct rotation and scalings). In fact, by judiciously choosing a generalized amplitude damping whose translation is the same as that of $\mathscr{E}$ we can see that $\mathscr{E}$ can be written as the composition of a bit flip, a unitary transformation and a generalized amplitude damping. The effects of each factor in the composition are illustrated in Fig. 2.3.

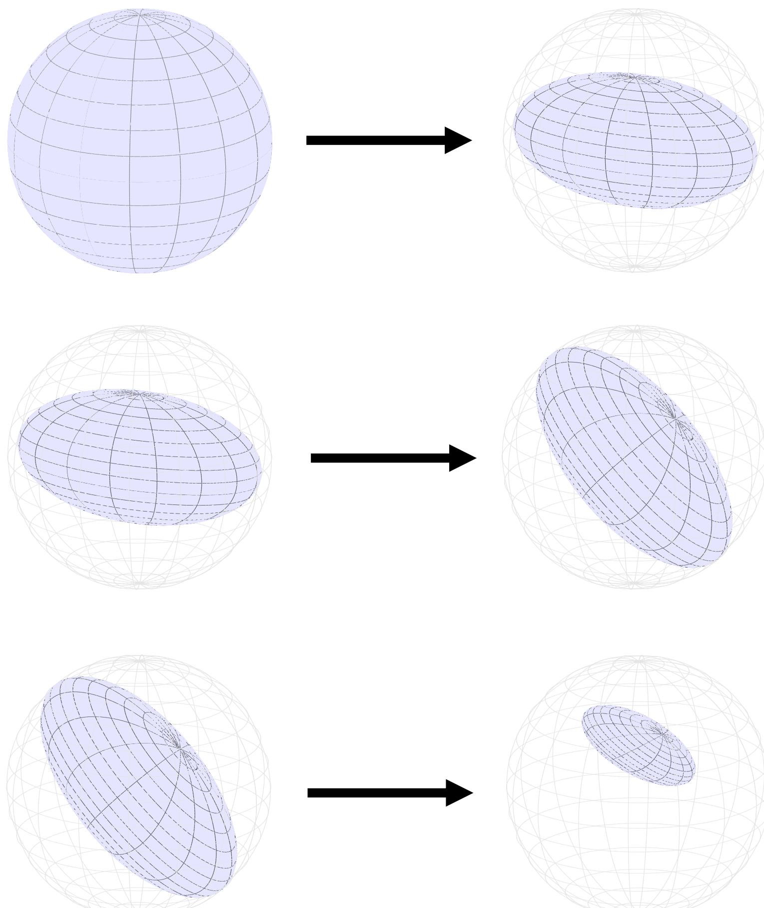


Figure 2.3 Decomposition into simpler channels: an example. The channel $\mathscr{E}$ is the composition of a bit flip followed by a unitary transformation followed by a generalized amplitude damping. The effects these three composition factors have on the Bloch ball are shown.

The question arises whether we can similarly write any single-qubit quantum channel as the composition of these simpler channels. Such a decomposition would allow us to understand any evolution of a qubit in terms of simple channels whose physical interpretations are clear. Unfortunately, no such decomposition exists. The image of a single-qubit quantum channel is necessarily contained by the Bloch ball, and its intersection with the Bloch sphere, the surface of the Bloch ball, is often much less than the full surface. All of the simple quantum channels described above have images that intersect the Bloch sphere at two antipodal points, one point, or nowhere. Compositions of these channels will have images that intersect the Bloch sphere in a similar way. Not all single-qubit quantum channels have images whose intersection with the Bloch sphere has this property, showing that they are not all compositions of the simple channels. This is made precise in the following theorem:

Theorem. *Single-qubit quantum channels cannot in general be written as the composition of bit flips, generalized amplitude dampings, and unitary transformations.*

Proof. Given a single-qubit quantum channel $\mathscr{E}$, define $I(\mathscr{E})$ to be the intersection of the image of $\mathscr{E}$ with the set of pure states, i.e. those states on the surface of the Bloch ball.

Let $\mathscr{E}$ be a nonunitary channel that is the composition of bit flip, generalized amplitude damping, and unitary channels $\{\mathscr{E}_i\}$, so $\mathscr{E} = \mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_n$. Let $\mathscr{E}_k$ be the left-most channel in the composition that is not a unitary transformation. (If they are all unitaries then $\mathscr{E}$ itself would be unitary, which is not the case). First note that

$$\operatorname{image}\mathscr{E} \subseteq \mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_{k-1}(\operatorname{image}\mathscr{E}_k).$$

And note that for any unitary channel $\mathscr{U}$ and set of states S we have

$$\{\text{pure states}\}\cap\mathscr{U}(S) = \mathscr{U}\left(\{\text{pure states}\}\cap S\right),$$

as unitary transformations do not alter the purity of states. As $\mathscr{E}_1$ through $\mathscr{E}_{k-1}$ are all unitary this means

$$
\begin{aligned}
I(\mathscr{E}) &= \{\text{pure states}\} \cap \text{image}\,\mathscr{E} \\
&\subseteq \{\text{pure states}\} \cap \mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_{k-1}(\text{image}\,\mathscr{E}_k) \\
&= \mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_{k-1}\big(\{\text{pure states}\} \cap \text{image}\,\mathscr{E}_k\big) \\
&= \mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_{k-1}\big(I(\mathscr{E}_k)\big).
\end{aligned}
$$

There are two cases:

1. If $\mathscr{E}_k$ is a generalized amplitude damping then $I(\mathscr{E}_k)$ is either one point or empty, in which case $\mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_{k-1}\big(I(\mathscr{E}_k)\big)$ is one point or empty.

2. If $\mathscr{E}_k$ is a bit flip then $I(\mathscr{E}_k) = \big\{\, |0\rangle\langle 0|\,, |1\rangle\langle 1|\,\big\}$. So $\mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_{k-1}\big(I(\mathscr{E}_k)\big)$ is a set of two orthogonal pure states.

In either of these two cases $\mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_{k-1}\big(I(\mathscr{E}_k)\big)$, and hence $I(\mathscr{E})$, is contained in a set that consists of two orthogonal pure states. We have shown that the image of any nonunitary single-qubit channel that is the composition of the simple channels will intersect the Bloch sphere in at most two points and that these two points must be orthogonal pure states.

We will now construct a nonunitary single-qubit quantum channel $\mathscr{F}$ for which $I(\mathscr{F})$ is not contained in a set of two orthogonal points, implying that $\mathscr{F}$ is not the composition of bit flips, generalized amplitude dampings, and unitary transformations. Let $|\psi_1\rangle = \big(\sqrt{1+\alpha}\,|0\rangle + \sqrt{1-\alpha}\,|1\rangle\big)/\sqrt{2}$ and $|\psi_2\rangle = \big(\sqrt{1-\alpha}\,|0\rangle + \sqrt{1+\alpha}\,|1\rangle\big)/\sqrt{2}$ where $0 < \alpha < 1$. Let $A_1 = |\psi_1\rangle\langle\psi_1|0\rangle\langle 0|$, $A_2 = |\psi_1\rangle\langle\psi_1|1\rangle\langle 0|$, $A_3 = |\psi_2\rangle\langle\psi_2|1\rangle\langle 1|$, and $A_4 = |\psi_2\rangle\langle\psi_2|0\rangle\langle 1|$. One can check that the A_i satisfy the condition to be Kraus operators for a quantum channel which we will call $\mathscr{F}$.

A straightforward calculation shows that $\mathscr{F}(|0\rangle\langle 0|) = |\psi_1\rangle\langle\psi_1|$ and $\mathscr{F}(|1\rangle\langle 1|) = |\psi_2\rangle\langle\psi_2|$. So $I(\mathscr{F})$ includes $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$. The overlap $\big|\langle\psi_1|\psi_2\rangle\big|^2 = \sqrt{1-\alpha^2}$ is between 0 and 1 (as α is between 0 and 1), so $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$ are distinct but not orthogonal. Finally, note that that $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ are orthogonal while their images aren't, implying that $\mathscr{F}$ is not a unitary transformation. So by the above arguments $\mathscr{F}$ cannot be written as the composition of bit flips, generalized amplitude dampings, and unitary transformations. $\square$

Chapter 3

Conclusions

When the only information known about a qubit is its purity the accuracy of cloning is bounded by Eq. (2.9), where r is the radius of the qubit in the Bloch ball. This bound is achieved by the Hillery-Bužek cloner described in [8]. The accuracy with which a state can be cloned increases as the purity, and hence the amount of information contained in the state, decreases. This is to the point where a completely mixed state, which contains no information, can be reproduced with perfect accuracy. This hints at a principle of conservation of information. Mixed states are "easier" to reproduce while pure states, containing a full bit of information, can only be copied approximately. A message encoded in mixed states, as opposed to pure states, could be copied more accurately per qubit, i.e. less information would be lost per qubit. But it would also take more qubits to encode, as mixed states hold less information. This trade-off enforces the general principle of quantum mechanics that (quantum) information cannot be copied.

Another way of viewing the above result is seen in the general principle that the more specified the qubit is prior to cloning, the more accurately it can be cloned. Knowing that the radius in the Bloch ball of the state to be cloned is 0 entirely specifies it as the maximally mixed state, the unique state at the center of the Bloch ball. It is thus no surprise that states with radius 0 can be cloned perfectly; we know perfectly the state to be cloned and can thus simply prepare another copy. Similarly states of radius nearly 0 are all similar to each other, so knowing that the radius of the unknown state to be cloned is close to 0 will nearly specify the state. Thus they are easier to clone. For an unknown state closer to the surface of the Bloch ball, however, there is intuitively more distance between the different possibilities of what the unknown state could be. This leaves the state less specified and thus harder to clone.

A qubit is the simplest quantum system and can be thought of as the building blocks of the universe. The evolution of a qubit, like any quantum system, is described by a quantum channel. In Section 2.4 we found that single-qubit quantum channels cannot in general be written as the composition of certain simple channels, namely unitaries, bit flips, and generalized amplitude dampings. This suggests that even the evolution of the simplest quantum systems can be complex.

Bibliography

[1] W. K. Wootters and W. H. Zurek, “A single quantum cannot be cloned,” Nature **299,** 802–803 (1982).

[2] V. Scarani, C. Lynn, and L. Shi Yang, *Six quantum pieces: a first course in quantum physics* (World Scientific, Singapore, 2010).

[3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*, 10th ed. (Cambridge University Press, New York, NY, USA, 2011).

[4] P. Hayden and J. Preskill, “Black holes as mirrors: quantum information in random subsystems,” Journal of High Energy Physics **2007,** 120 (2007).

[5] A. Jamiołkowski, “Linear transformations which preserve trace and positive semidefiniteness of operators,” Reports on Mathematical Physics **3,** 275–278 (1972).

[6] V. Scarani, S. Iblisdir, N. Gisin, and A. Acín, “Quantum cloning,” Rev. Mod. Phys. **77,** 1225–1256 (2005).

[7] R. Jozsa, “Fidelity for Mixed Quantum States,” Journal of Modern Optics **41,** 2315–2323 (1994).

[8] V. Bužek and M. Hillery, “Quantum copying: Beyond the no-cloning theorem,” Phys. Rev. A **54,** 1844–1852 (1996).

Index