Quantifying the Uncertainty of the Heisenberg Uncertainty Relation:

Preparation vs. Measurement Uncertainty

Jacob "J" Collings

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Jean-François S. Van Huele, Advisor

Department of Physics and Astronomy

Brigham Young University

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ABSTRACT

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Jacob "J" Collings
Department of Physics and Astronomy, BYU
Bachelor of Science

Since Heisenberg introduced the relation $p_1 q_1 \sim h$ in 1927, it has been the subject of discussion and further investigation. Recent work has shown that the term 'uncertainty' applies to two different quantum properties. The first pertains to preparation uncertainty, the principle that one cannot prepare a quantum system such that two incompatible observables are arbitrarily well-defined. The second pertains to measurement uncertainty, the principle that the measurement with a certain degree of accuracy of one observable disturbs the subsequent measurement of a second incompatible observable. We review recent experiments showing evidence for a violation of the measurement uncertainty. We illustrate various reformulations of the Heisenberg uncertainty relation with examples using spin measurements.

Keywords: Uncertainty, Error, Disturbance, Heisenberg Uncertainty Relation, Measurement, Preparation

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Chapter 1

How Uncertain is the Heisenberg Uncertainty Relation?

1.1 Origin of the Heisenberg Uncertainty Relation (HUR)

In 1927, Heisenberg introduced a thought experiment concerning a γ-ray microscope [1]. In this thought experiment, Heisenberg has us imagine an electron having its position measured by means of a photon. In order to detect the electron, the photon collides with it, providing an accurate measurement of the electron's position; however, the collision jolts the electron, causing its momentum to be disturbed. The more accurate the measurement, the shorter the photon wavelength required, and thus the larger the momentum transfer. Conversely, if we were to measure the momentum of the electron, its position would be disturbed by the photon used to measure it. This principle - that one cannot measure the position of a quantum system without disturbing its momentum and vice versa - and its corresponding relation, is what we today call the Heisenberg Uncertainty Relation (HUR) .

When Heisenberg first proposed this relation, he supported his thought experiment with a math-

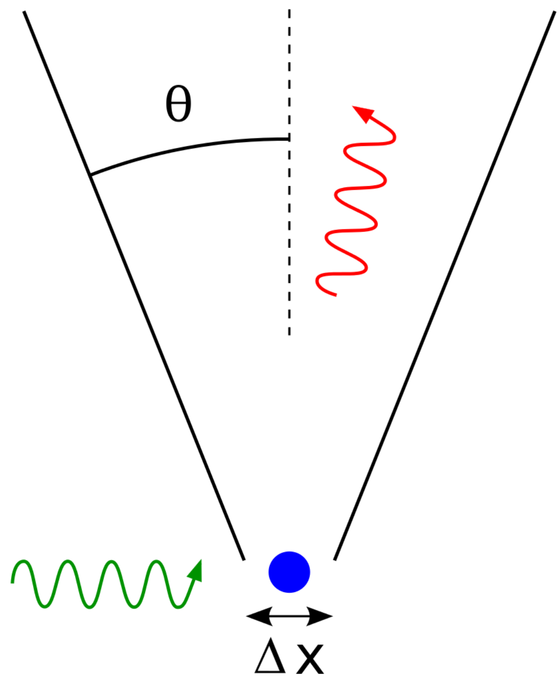

Figure 1.1 Heisenberg's γ-ray microscope thought experiment. The incoming γ-ray (green) scatters against the electron (blue), producing a position measurement with the scattered γ-ray (red). The more precise the position measurement, the stronger the γ-ray disturbs the momentum of the electron.

ematical proof using Gaussian states for the electron. While the formal proof is rather tedious [2], and will not be reproduced here, the results are as we now would expect; namely, that the standard deviation of momentum and position are inversely proportional to one another as their product is bounded by a lower limit. When Heisenberg first produced this proof, his final relation was $p_1 q_1 = h$. Today, this is more commonly written as $\Delta x \Delta p \geq \frac{\hbar}{2}$, where q_1 (or Δx) and p_1 (or Δp) represent the standard deviation of the position and momentum, respectively.

This relation has been a fundamental ingredient of quantum mechanics as we know it today. It is obedience to this law that allows us to set the limits and perform the necessary calculations for quantum systems. For years this relation has set the bar for the maximum precision with which experimentalists can prepare quantum systems, as well as measure any quantum states. Many experiments have demonstrated the validity of this relation beyond any doubt. An example is the

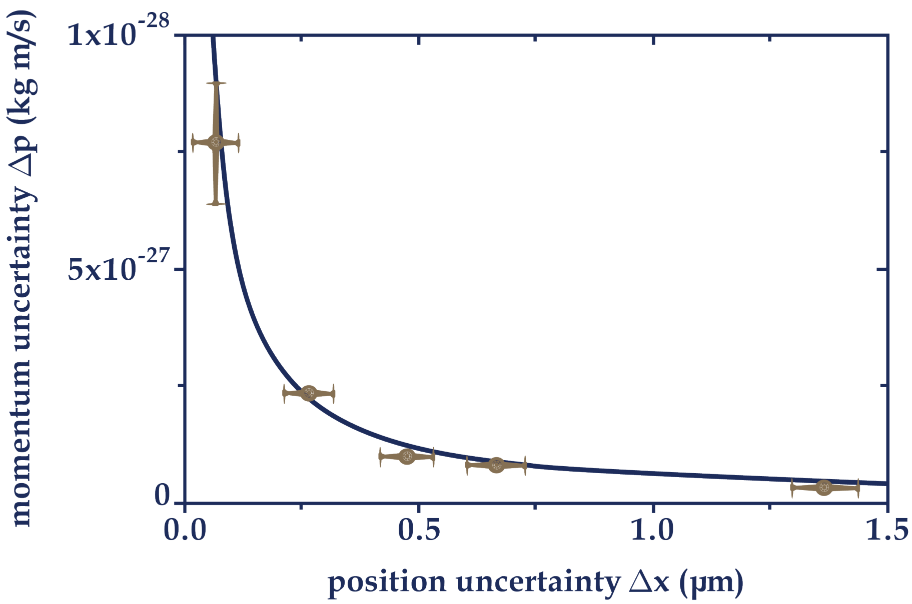

Figure 1.2 Experimental verification of the Heisenberg uncertainty relation for C_{70}, by Nairz et al. [3]

work done by Nairz [3], wherein the lower bound of the HUR was found using a system of C_{70} fullerene (see Fig. 1.2) . Yet, despite all the overwhelming evidence of its validity, the HUR has come into question in the past few years.

1.2 Ozawa's and Branciard's Reformulations of the HUR

Recently, two new types of quantum measurement have brought the validity of the HUR into question; namely, weak measurement [4] and triple-state measurement [5]. By taking advantage of quantum mechanical properties, these two measurement techniques are in principle capable of making a physical measurement without creating any disturbance. Triple-state measurement accomplishes this by making three distinct measurements of a system. Each measurement is precisely arranged such that whatever disturbance is caused by the first measurement, the second and third measurements cancel it. This is done by varying the angle of measurement in the plane of the two incompatible observables. In a completely different fashion, weak measurement is the process of making very weak measurements that do not collapse the system. The price for minimally affecting the system is that the measurement will produce a result with often very large error, but when done numerous times, the expectation value of the measurements are taken and the result is a highly accurate measurement with no disturbance. Though the two techniques utilize completely different quantum mechanical properties, they both succeed in accomplishing a disturbance-free measurement.

It is in light of these two measurement techniques that the HUR quickly begins to unravel. When referencing a physical measurement, the HUR is written in terms of the error ε of the measurement and the disturbance η of the incompatible observable. The relation is written

$$\varepsilon(A)\eta(B) \geq \frac{1}{2}|\langle[A,B]\rangle|, \tag{1.1}$$

where A is the observable being measured and B is the incompatible observable being disturbed. In the case of an ideal weak or a triple-state measurement, η will be zero. In this instance, the left-hand side being a product, we obtain $0 \geq \frac{\hbar}{2}$. This is obviously false, which means that, in the case of an ideal weak or triple-state measurement, the HUR does not hold.

In 2003, Ozawa [6] noticed this flaw in the Heisenberg Uncertainty Relation (HUR). In his text, Ozawa commented on how the HUR is violated if either the error or the disturbance is zero. To remedy this situation, Ozawa proposed a reformulation of the uncertainty relation

$$\varepsilon(A)\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{1}{2}|\langle[A,B]\rangle|. \tag{1.2}$$

The $\langle ... \rangle$ refers to the quantum mechanical expectation value [2]. The enclosed operator $[A,B]$ represents the commutator $AB - BA$ of two operators, and σ represents the standard deviation - the same σ we recognize in the modern-day HUR and which we will define in Eq. (2.2). It is noteworthy that Ozawa would choose to include not only the error and disturbance of the system, but also the standard deviation of each observable. By doing this, we can see that the left-hand side will never go to zero. Ozawa's reformulation of the HUR did not catch the interest of many researchers until mid-2012, when various experimental researchers validated Ozawa's claims [7, 8, 9].

With these findings, other scientists have since produced their own reformulation of the HUR [10, 11]. Branciard produced a relation [12] that draws upon the key concepts of Ozawa's formulation, but extends it to provide the tighter relation

$$\varepsilon^2(A)\sigma^2(B)+\sigma^2(A)\eta^2(B)+2\varepsilon(A)\eta(B)\sqrt{\sigma^2(A)\sigma^2(B)-C_{AB}^2}\geq C_{AB}^2, \tag{1.3}$$

where $C_{AB}=\frac{1}{2}|\langle[A,B]\rangle|$.

The lower bound of Branciard's relation is larger than Ozawa's lower bound, providing a stricter, 'tighter' relation, while still remaining valid. The HUR is too tight and is violated. The tightness of Branciard's relation, as compared to Ozawa's and Heisenberg's, can be seen in Fig. 1.3 in the optimal state preparation case with equal standard deviation (coherent state).

We should point out that none of the reformulations presented here modify the expression in the right-hand side of the equation, as can be seen in Eqs. (1.2)-(1.3) and Eq. (2.1).

1.3 HUR in a State of Flux

Despite the reformulation proposal [6] and its strong support from scientists around the world, not everyone agrees that the Heisenberg uncertainty relation (HUR) is violated. It has been argued [13] that Ozawa's reformulation ignores the original meaning Heisenberg intended when presenting his uncertainty relation. This claim states that Ozawa and other authors of reformulations are not interpreting the properties of the relation properly. With no proper physical interpretation, Ozawa's relation is thus mathematically sound, yet meaningless in practicality.

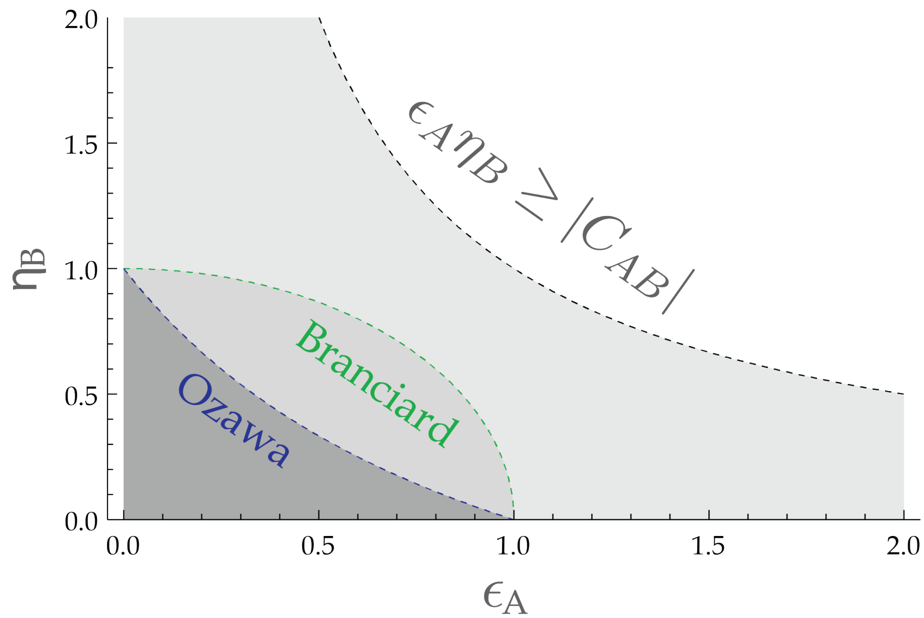

Figure 1.3 Lower bounds for the HUR (black), Ozawa's reformulation (blue), and Branciard's reformulation (green). The shaded regions show the 'forbidden zones,' the quantities that fall below the lower limit of the relations. Original figure.

In response to the defense of the HUR, various experimentalists provided physical results using weak [8, 9] and triple-state measurement [7] techniques. All of these experiments found results within the forbidden zones of the HUR, as can be seen in Fig. 1.4. Yet, as we saw, Nairz et al. provided physical evidence supporting the HUR. This situation presents quite a paradox.

The discussion on the proper interpretation has been ongoing since and is not over yet [14, 15]. The simpler solution is this: both sides are correct. The HUR does hold, as has been claimed, and yet it is violated, as others have claimed. This is possible because, though both the HUR and Ozawa's formulations concern uncertainty of a quantum state, they are addressing two completely different physical properties of the quantum state. This is why we thus argue for the necessity to redefine quantum uncertainty; namely, by separating what was once covered by the HUR into two definitions: preparation uncertainty and measurement uncertainty.

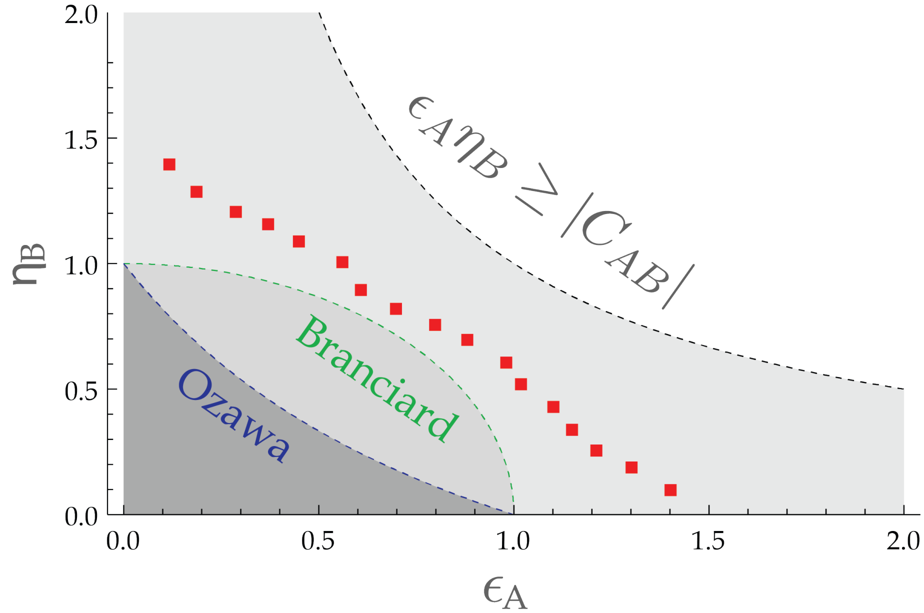

Figure 1.4 Results of a triple-state measurement by Ringbauer, et al. [16], clearly indicating a violation of the Heisenberg Uncertainty Relation. Original figure.

1.4 Overview

In the following chapters, we present a more precise definition for the two types of uncertainty. In Section 2.1, we discuss what is meant when we say 'uncertainty.' In Sections 2.2 and 2.3, we take a detailed look at preparation uncertainty and measurement uncertainty, respectively. In Section 2.4, we examine the flaws of the original measurement uncertainty definitions and relations. In Section 2.5 we examine various reformulations that have been proposed for measurement uncertainty. In Section 2.6 we look at experimental evidence and in Section 2.7 we present a weak measurement simulation, all supporting the need to reformulate the measurement uncertainty relation. Finally, in Section 3.1 we establish the current status of the HUR and then in Section 3.2 discuss where we need to go with it in the future.

Chapter 2

The Uncertainty About Uncertainty

2.1 Understanding Uncertainty

Heisenberg's original mathematical proof of his uncertainty relation involved two variables, the canonical position q and the canonical momentum p, and an expression for the minimum uncertainty containing Planck's fundamental quantum of action h. Today, the more common format for this relation between arbitrary incompatible observables A and B is [17]

$$\sigma(A)\sigma(B) \geq \frac{1}{2}|\langle[A,B]\rangle|, \tag{2.1}$$

where σ is the standard deviation of the related observable. Here, σ is defined as

$$\sigma(A) = \sqrt{\langle A^2\rangle - \langle A\rangle^2}, \tag{2.2}$$

where $[A,B] = AB - BA$ and $\langle A\rangle$ is the expectation value of A. While standard deviation is a well-defined and well-known statistical property, Heisenberg did not use standard deviation to qualitatively define his uncertainty relation. In his thought experiment, Heisenberg referenced an error and a disturbance that occur from a physical measurement. For many years, standard deviation has been the accepted mathematical interpretation for the physical properties of error and disturbance. However, Ozawa argued that a more accurate definition for error and disturbance was needed [6]. In his 2003 paper, Ozawa defined error ε and disturbance η as

$$\varepsilon \equiv \langle (U^\dagger (\mathbb{1} \otimes M) U - A \otimes \mathbb{1})^2 \rangle^{\frac{1}{2}} \tag{2.3}$$

and

$$\eta \equiv \langle (U^\dagger (B \otimes \mathbb{1}) U - B \otimes \mathbb{1})^2 \rangle^{\frac{1}{2}}. \tag{2.4}$$

Here, A and B are the incompatible observables, M is the measurement probe, and U is a unitary transform relating the interaction between the probe and the system. These definitions of error and disturbance are quite different from the standard deviation definition in Eq. (2.1). While standard deviation is a static value that relies entirely upon the nature of the observable and the state it is calculated in, we see that Ozawa's error and disturbance are dynamic properties involving the probe, the system, and how they interact, along with the observable being measured. This change in definition provides two different physical interpretations of the uncertainty relation: namely, a static interpretation, given by the traditional use of standard deviation for error and disturbance, and a dynamic measurement interpretation, given by Ozawa's definition of error and disturbance.

Much of the debate over the Heisenberg Uncertainty Relation (HUR) is due to these two different physical interpretations of error, disturbance, and uncertainty. To resolve this issue, we present two new, separate definitions for uncertainty to accomodate these different physical interpretations; namely, preparation uncertainty and measurement uncertainty.

2.2 Preparation Uncertainty

We define preparation uncertainty as the principle that one cannot prepare a quantum system such that two incompatible observables are arbitrarily well-defined. This is the uncertainty taught in the quantum textbooks [2, 18]. For example, if one wanted to prepare a spin system, the z-component of the spin could be prepared to a particular degree of certainty. However, as the z-component is better prepared, the x-component and y-component will both have degraded certainty in their initial state. Likewise, as the x-component or y-component are prepared to a particular degree of certainty, the z-component will lose certainty in its preparation. This trade-off of certainty in a state's preparation reflects the preparation uncertainty.

Mathematically, preparation uncertainty is defined as in Eq. (2.1). This was shown for the case of position and momentum in Fig. 1.2, referring to the experiment by Nairz et al. [3] This relation remains valid. As we move forward, it is important to keep this in mind: preparation uncertainty is not being called into question. Many experiments illustrate the validity of preparation uncertainty, and the reformulations to the Heisenberg uncertainty relation (HUR) do not apply.

A key physical property of preparation uncertainty that has not been emphasized in debates concerning the HUR is that preparation uncertainty is static. This is to be expected; the preparation of a state does not evolve with time, since the state is prepared at a given time, say $t = 0$, and any evolution after that no longer pertains to the initial preparation of the state. Just as we saw in Section 2.1, we have a static interpretation of error $\sigma(A)$, disturbance $\sigma(B)$, and uncertainty. The evolution presented in Eqs. (2.3)-(2.4) is absent from preparation uncertainty. We will see that this static property distinguishes preparation uncertainty from its counterpart, measurement uncertainty.[1]

When Heisenberg first produced his uncertainty relation, he proved it using Gaussian states. By doing so, Heisenberg moved away from his original thought experiment and presented a mathematical proof for preparation uncertainty. This mix-up is what has caused so much misunderstanding in the scientific community. When we mention Heisenberg's original relation and his thought ex-

[1] Of course, states evolve in time (as seen in the spreading of the wave functions), so it is possible to consider $\Delta x(t)$ and $\Delta p(t)$, but their product can only get larger as our state evolves.

periment, we are referring to two different types of uncertainty. The thought experiment portion of Heisenberg's microscope dealt with measurement uncertainty, which will be discussed in the next section.

2.3 Measurement Uncertainty

We define measurement uncertainty as the principle that the measurement with a certain degree of accuracy of one observable disturbs the subsequent measurement of a second, incompatible observable. This uncertainty pertains to physical measurements only; it has nothing to do with the original state of the system. For example, when measuring the z-component of a spin, one disturbs the x-component and y-component. Not only will the measurement have a degree of error in the z-component, but the x- and y- components will no longer be in the same state as they were before measurement. Their values will be disturbed, and the original values will no longer be recoverable. To reduce this disturbance η, one would have to increase the allowed error in the z-component.

Traditionally, measurement uncertainty takes the same mathematical definition as preparation uncertainty

$$\varepsilon(A)\eta(B) \geq \frac{1}{2}|\langle[A,B]\rangle|, \tag{2.5}$$

where ε is the error of the measurement of observable A and η is the disturbance of the second, incompatible observable B. While the relation has the same basic form as preparation uncertainty, the crucial difference comes in the definition of error and disturbance, as given in Eqs. (2.3)-(2.4). First we have the measurement probe-system interaction, defined by $U^\dagger(\mathbb{1}\otimes M)U$. Here, the probe is represented by M, and the physical interaction between the probe and the system is given by the unitary transform U. Notice that the probe M undergoes a tensor product with the identity operator, $\mathbb{1}$. This brings the probe into the proper space so that it can interact with the rest of the system. The probe is then acted upon by the unitary transform. This interaction provides the physical measurement result obtained by the probe. In other words, if we had a meter providing a measurement result, the interaction between the probe and the system would be the result printed out on the meter. In the definition of error, we find the difference between this probe result and what the system actually is supposed to be without measurement, given by $A \otimes \mathbb{1}$. If the probe had made a perfect measurement, this difference would result in a zero value, and we would have no error. Disturbance follows a similar reasoning, although instead of having the probe interacting with the system, we only consider the incompatible observable B interacting with the probe. The disturbance is then found by finding the difference between this disturbed value for B and the value B would have had if it had not been disturbed.

These definitions for error and disturbance are more in line with what Heisenberg was discussing in his original thought experiment. However, these definitions do not work with the uncertainty of a prepared state. Error and disturbance are calculated in a dynamic state that evolves through interactions. This does not satistfy the static conditions that physically define the uncertainty of a prepared state. As such, these definitions of error and disturbance only apply to the uncertainty involved in taking a physical measurement. This satisfies Heisenberg's thought experiment, but they are not in line with the static uncertainty he proved mathematically using Gaussian states. This justifies the redefining of uncertainty into two distinct types of uncertainty, as we have done here.

2.4 Flaw in the Original Measurement Uncertainty Relation

The limiting flaw in the original measurement uncertainty relation is the requirement that both error and disturbance are assumed to be non-zero. Traditionally, this has been a given assumption. However, newer sophisticated measurement techniques in quantum mechanics have presented means to make a physical measurement with zero disturbance [4, 7]. Looking at the original measurement uncertainty relation, we see that having a zero disturbance causes a breakdown of the relation; the right-hand side, being non-zero, cannot be smaller than the left-hand side. This is also true for any situation wherein the error is zero, although any possible means of performing such a measurement are, to our knowledge, unknown.

As previously stated, this measurement uncertainty is what Heisenberg had in mind with his thought experiment. However, when Heisenberg first conceived this experiment, he did not account for the ability to have zero disturbance. This understandable lack of foresight causes the original relation to break down. It is for this reason that a reformulation of the measurement uncertainty relation is required. Note, however, that the preparation uncertainty relation is completely accurate; we currently have no means of providing a zero-standard-deviation preparation for any quantum mechanical system, so the violation discussed here for measurement uncertainty does not apply to preparation uncertainty. The mathematical proof is solid and the assumptions underlying it have not been contradicted by any experimental fact.

2.5 Measurement Uncertainty Reformulations

In light of the development of zero-disturbance measurements, we need a new relation for measurement uncertainty. Ozawa was the first to propose a new uncertainty relation, although in his original paper, he suggested replacing the HUR as a whole. Noticing that having either error or disturbance as zero would break the HUR, Ozawa proposed revising the relation to

$$\varepsilon(A)\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{1}{2}|\langle[A,B]\rangle|. \tag{2.6}$$

Here, Ozawa introduces the standard deviation σ terms from preparation uncertainty into the measurement uncertainty relation. As a result, when either error or disturbance are zero, the product terms including σ prevent the left-hand side from going to zero, preventing a violation of the relation. Introducing preparation uncertainty to the measurement uncertainty makes sense: the degree of accuracy by which one can measure the system will be impacted by how well the state is prepared before making the measurement.

Ozawa's reformulation gained little attention until 2012, when Erhart et al [7] performed an experiment to validate Ozawa's claims (as will be discussed in Section 2.6). This also brought Ozawa's reformulation to the attention of Branciard, who noticed that Ozawa's relation was not as tight as it could be [12]. Using Ozawa's work as a starting point, Branciard produced the new relation

$$\varepsilon^2(A)\sigma^2(B)+\sigma^2(A)\eta^2(B)+2\varepsilon(A)\eta(B)\sqrt{\sigma^2(A)\sigma^2(B)-C_{AB}^2}\geq C_{AB}^2. \tag{2.7}$$

Like Ozawa, Branciard includes the preparation uncertainty σ terms. However, Branciard also introduces the preparation uncertainty limit into the left-hand side to compensate for a non-minimized preparation uncertainty. This inclusion provides a tighter relation than Ozawa's.

Though initially met with much skepticism, these reformulations have since been verified experimentally. Here, we will study four independently conducted experiments, two of which use triple-state and two of which use weak measurements to validate the necessity of these reformulations.

2.6 Experimental Evidence of the Measurement Uncertainty

We now discuss the results of four different experiments: Erhart [7] and Ringbauer [16], who used triple-state measurement techniques, and those of Rozema [8] and Kaneda [9] who used weak measurement techniques. We compile the data into four error vs. disturbance graphs which we present in Fig. 2.1. As the data shows, when using the original HUR, every one of the four

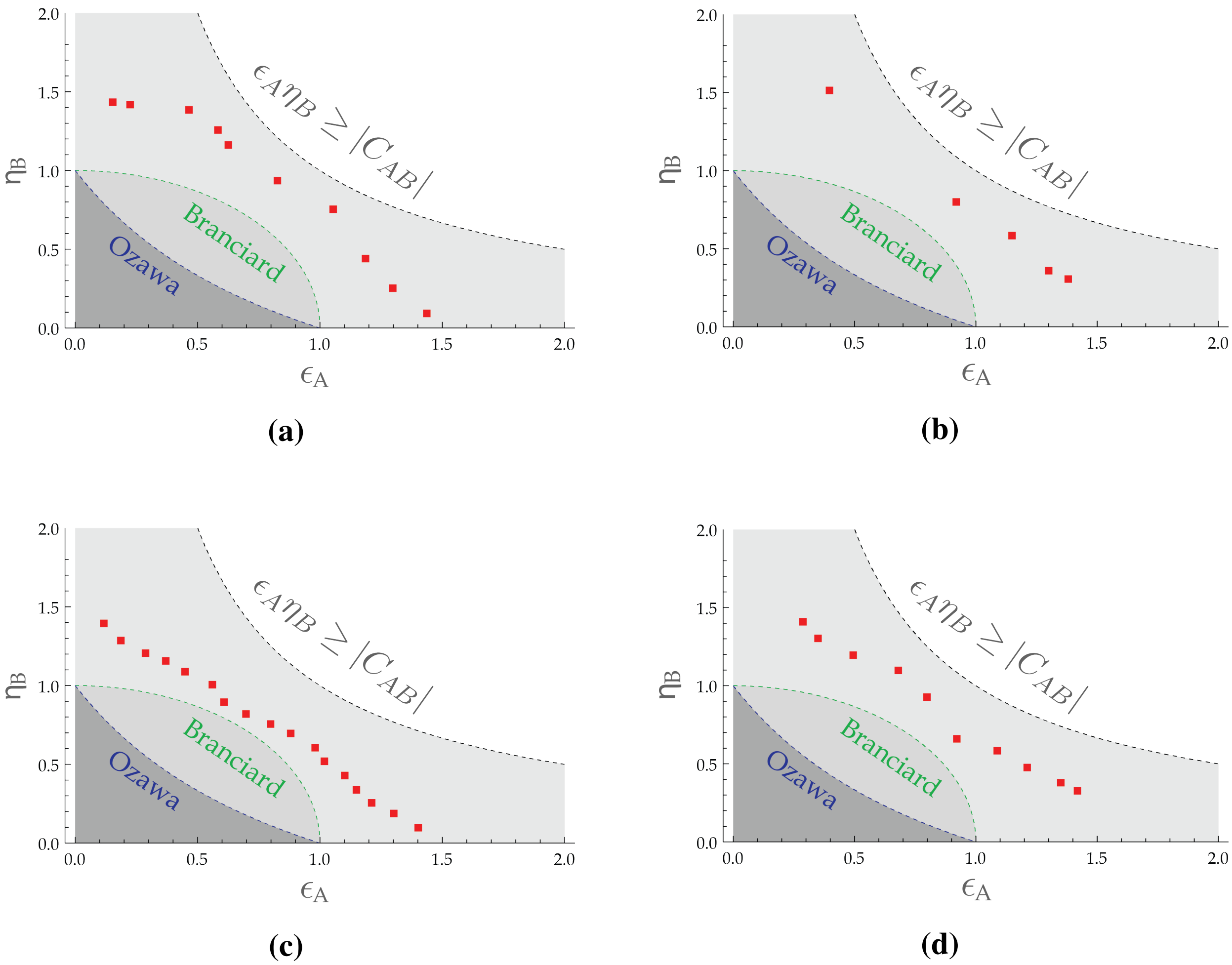

Figure 2.1 Violation of HUR in (a) Erhart, (b) Rozema, (c) Ringbauer, and (d) Kaneda. The Ozawa and Branciard relations still hold in all four figures. Original figures.

experiments results in a violation of the lower limit.[2] However, when we take into account our new definitions for measurement uncertainty and then apply the proposed relations of Ozawa or Branciard, we find that the data does not violate the lower bounds. Once again, this is due to the dynamic nature of measurement uncertainty; we simply cannot use the static properties of preparation uncertainty to calculate the dynamic uncertainty inherent in a physical measurement, especially when those measurements contain a near-zero disturbance.

[2]In these graphs, we use units such that $|C_{AB}| = 1$

2.7 Simulating Weak Measurements

To further test the difference between static and dynamic natures in uncertainty, we use Mathematica to simulate the weak measurement setup proposed by Lund and Wiseman [4] and shown in Fig. 2.2. We use the definition of a physical measurement as discussed in [19]. We measure a spin-$\frac{1}{2}$ particle in z and in x. To do so, we construct a unitary transform for the CNOT (Control-NOT) gates:

$$U_1 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.8}$$

$$U_2 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

Here, U_1 (resp. U_2) flips the z (resp. x) component of the second bit, depending on the value of the first bit. This represents the interaction between the system and the probe.

We choose A to be the observable that we are measuring and B to be the observable that is disturbed. First, we find the error of a z measurement with a disturbance in x. We set our probe M to be in z to get a measurement. To better see the effect of measurement strength, we choose for an input $|\psi\rangle$ prepared in Y that varies with the measurement strength θ. With a spin prepared along the y-component, we are able to achieve the maximum degree of uncertainty for x and z measurements. This is critical; by maximizing the possible uncertainty, we make sure to probe each relation where it is most likely to fail (at the greatest extreme possible). Doing so gives

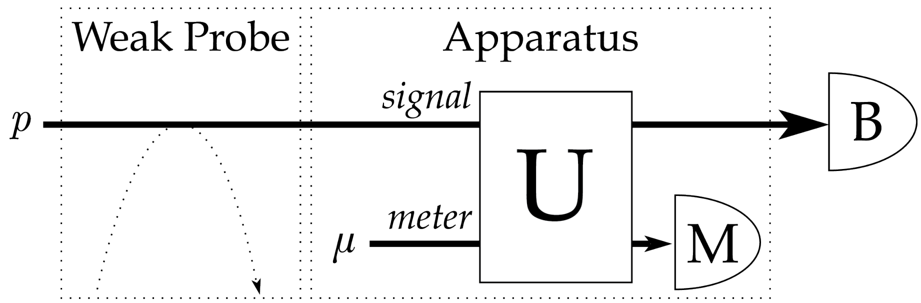

Figure 2.2 Weak measurement setup proposed by Lund and Wiseman [4]. By using a weak probe M, the quantum system (here represented by p) can be measured with probe M without causing any disturbance. The error can be quite large, however, so many repeated measurements must be made in order to produce a valid result. Original figure based on [4]

$$\varepsilon(Z) = \langle\psi|\,(U_1^\dagger(\mathbb{1}\otimes Z)U_1 - Z\otimes\mathbb{1})^2\,|\psi\rangle^{\frac{1}{2}},$$

with

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta\\ \sin\theta\\ i\cos\theta\\ i\sin\theta\end{pmatrix}$$

where $\langle\psi|\,\alpha\,|\psi\rangle$ is the expectation value of α in state $|\psi\rangle$ with $\mathbb{1}$ being the identity matrix and Z being the Pauli operator for a spin-$\frac{1}{2}$ particle in the z direction, we find that

$$\mathbb{1}\otimes Z = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

and

$$Z \otimes \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We simulate the weak measurement using the U_1 CNOT gate as a unitary transform on the probe. We define Q_ε as

$$Q_\varepsilon = U_1^\dagger (\mathbb{1} \otimes Z) U_1, \tag{2.10}$$

represented by

$$Q_\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here, Q_ε is the actual observable measured by the probe. To find the error, we must take the difference between Q_ε and the ideal observable had no measurement occurred, namely $Z \otimes \mathbb{1}$. We define W_ε as

$$W_\varepsilon = (U_1^\dagger (\mathbb{1} \otimes Z) U_1) - Z \otimes \mathbb{1}, \tag{2.11}$$

which is represented by

$$W_\varepsilon = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

This gives

$$W_\varepsilon^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}. \tag{2.12}$$

Using our choice for $|\psi\rangle$, we find the error in the operator z,

$$\varepsilon^2(Z) = \langle\psi| W_\varepsilon^2 |\psi\rangle \tag{2.13}$$

$$= 4\sin^2\theta. \tag{2.14}$$

Since the error is defined as a positive quantity, this gives

$$\varepsilon(Z) = 2|\sin\theta| \tag{2.15}$$

for the error of a z measurement. To find $\eta(X)$, the disturbance in X, we follow a similar procedure, again using the U_1 CNOT gate as our unitary transform U, giving

$$\eta(X) = \langle (U_1^\dagger (X\otimes \mathbb{1}) U_1 - X\otimes\mathbb{1})^2 \rangle^{\frac{1}{2}}.$$

We find that

$$X\otimes\mathbb{1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and thus

$$Q_\eta = U_1^\dagger (X\otimes\mathbb{1}) U_1 \tag{2.16}$$

$$Q_\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$W_\eta = (U_1^\dagger (X \otimes \mathbb{1}) U_1) - X \otimes \mathbb{1} \tag{2.17}$$

$$W_\eta = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

This gives

$$W_\eta^2 = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}. \tag{2.18}$$

Applied to our particular $|\psi\rangle$, we find the disturbance in the operator x,

$$\eta^2(X) = \langle\psi| W_\eta^2 |\psi\rangle$$
$$\eta^2(X) = 2(\cos\theta - \sin\theta)^2,$$

which gives

$$\eta(X) = \sqrt{2}|\cos\theta - \sin\theta| \tag{2.19}$$

for the disturbance in x of a z measurement.

Following this same idea, we generate Table 3.1 by alternating taking a measurement in Z and in X. Note that whenever making a measurement in X, we must use a different CNOT gate, namely U_2 for the unitary transform U. We analyze these results in Chapter 3.

Chapter 3

Discussion of the HUR

3.1 Math and Semantics

The weak measurement simulation presented in Section 2.7 provides insight into how measurement uncertainty differs from preparation uncertainty. The data in Table 3.1 shows that no matter how strong the measurement, the HUR is always violated.

Measurement	ε	η
U=U_1, A=Z, B=X	$2\|\sin\theta\|$	$\sqrt{2}\|\cos\theta-\sin\theta\|$
U=U_2, A=X, B=Z	$\sqrt{2}\|\cos\theta-\sin\theta\|$	$2\|\sin\theta\|$
U=U_1, A=Z, B=Z	$2\|\sin\theta\|$	0
U=U_2, A=X, B=X	$2\|\sin\theta\|$	0

Table 3.1 Error ε and disturbance η for measurements using various unitary transforms. θ varies between 0 and $\frac{\pi}{4}$.

Applying this to the original HUR (Eq. 2.5) and Ozawa's relation, we generate Table 3.2. This shows how the left-hand side of the HUR differs from that of the Ozawa relation. The key factors are the $|\sin\theta|$ and $|\cos\theta-\sin\theta|$ terms in the HUR. When θ is equal or near 0, $|\sin\theta|=0$, thus the left-hand side of the HUR goes to 0. When θ is equal or near $\frac{\pi}{4}$, $|\cos\theta - \sin\theta| = 0$, thus the left-hand side of the HUR again goes to 0. In either case, the HUR inequality is violated. However, the Ozawa relation has both a $|\sin\theta|$ term and a $|\cos\theta - \sin\theta|$ term, preventing the left-hand side from going to 0 at either of these points.

U A B	LHS of Eq. (2.5)	LHS of Eq. (2.6)
U_1, Z, X	$2\sqrt{2}\|\cos\theta - \sin\theta\|\|\sin\theta\|$	$2\sqrt{2}\|\cos\theta - \sin\theta\|\|\sin\theta\| + \sqrt{2}\|\cos\theta - \sin\theta\| + 2\|\sin\theta\|$
U_2, X, Z	$2\sqrt{2}\|\cos\theta - \sin\theta\|\|\sin\theta\|$	$2\sqrt{2}\|\cos\theta - \sin\theta\|\|\sin\theta\| + \sqrt{2}\|\cos\theta - \sin\theta\| + 2\|\sin\theta\|$
U_1, Z, Z	0	$2\|\sin\theta\|$
U_2, X, X	0	$\sqrt{2}\|\cos\theta - \sin\theta\|$

Table 3.2 Left-hand side of various uncertainty relations for the various measurements.

This additional term comes from Ozawa's inclusion of the preparation uncertainty in calculating his measurement uncertainty. Without this inclusion, the relation would be violated, just as the HUR is violated.

To better see the dependence of the measurement uncertainty on the strength of the weak measurement, we graph the results of a measurement in x with disturbance in z using the original HUR, the Ozawa relation, and the Branciard relation in Fig 3.1. When θ is zero, the measurement is entirely in the z direction, resulting in the weakest possible measurement. Since error $\varepsilon(x) \propto |\cos\theta - \sin\theta|$ (see Table 3.1), this measurement produces a large value for error; however, since disturbance $\eta(z) \propto |\sin\theta|$, the measurement produces a zero value for $\eta(z)$. This results in an HUR value of zero. As θ increases, the measurement begins to vary between the z and x components, resulting in an increase of measurement strength and thus a decrease in error and an increase in disturbance. When θ reaches the maximum value at $\frac{\pi}{4}$ the measurement is a mix of up and down in z, resulting in a measurement entirely in x. This causes disturbance $\eta(z) \propto |\sin\theta|$ to be large but error $\varepsilon(x) \propto |\cos\theta - \sin\theta|$ goes to zero. Thus the HUR value once again returns to

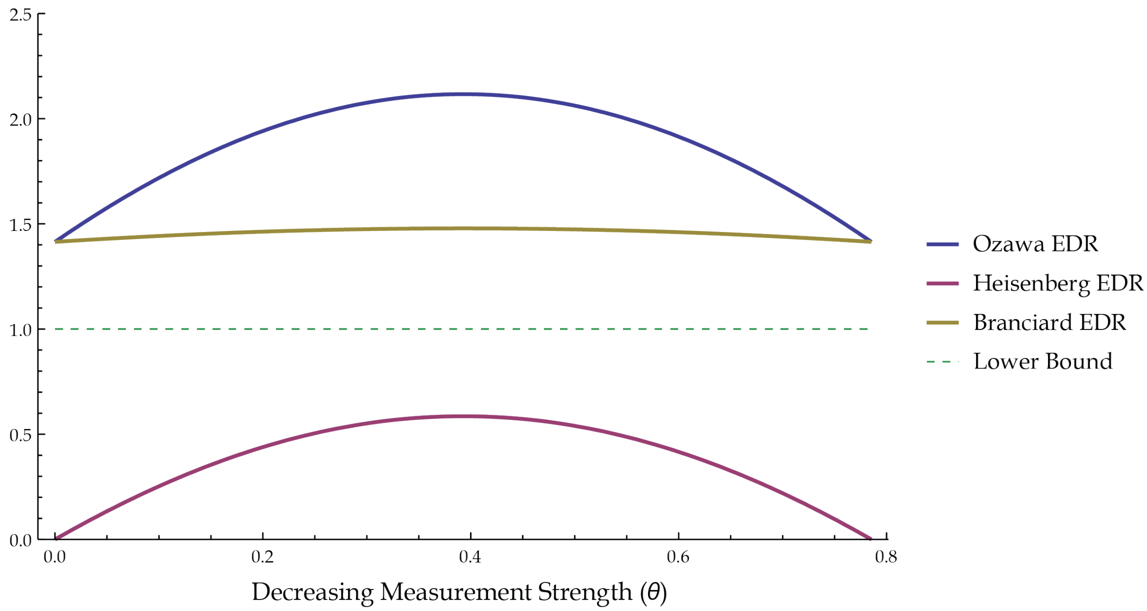

Figure 3.1 Weak Measurement Simulation. By varying the strength of the measurement, we obtain the following error-disturbance relation (EDR) limits. When θ is zero, the measurement is fully in the Z direction. When θ is equal to $\frac{\pi}{4}$, the measurement is a mix of up and down in Z, which results in an measurement entirely in X.

zero. We see that regardless of how strong the weak measurement is, the HUR is violated by measurement uncertainty, although it is more violated around $\theta = 0$ and $\frac{\pi}{4}$. It is important to note that while both Ozawa's and Branciard's relations hold, Branciard's relation provides a tighter bound. Our results clearly indicate a need to redefine uncertainty in regards to quantum measurements. More specifically, we must concur on a valid reformulation for measurement uncertainty, as the original HUR falls short when probed with modern quantum measurement techniques.

A large part of the discussion confronting a reformulation of the HUR relies on semantics. Some scientists [13] refer to Heisenberg's original words to defend the HUR and their calculations, in contrast to the work examined here. They argue that the wording Heisenberg used makes the measurements done by Ozawa, Branciard, Erhart, and others, a moot point, stating that they aren't examining what Heisenberg originally intended. Whether this is the case or not, it is clearly evident from the results of the experiments examined in this paper and those we performed that the HUR falls short when applying to certain particular quantum measurement techniques. An argument of what Heisenberg meant when he said this or when he proved that is mostly of historic interest and irrelevant for our purpose; what is important are the data confronting us. A way has been found to violate the HUR; what we must do now is find a new definition that will universally qualify every quantum measurement technique, regardless of semantics. This is not the first time we have had to redefine a founding principle, and as science progresses, it will certainly not be the last.

3.2 A Future for HUR Reformulations

It will be some time before the scientific community settles on the answer of how the HUR should be presented. We have presented here a redefinition of uncertainty, refining the current definition into two separate concepts: preparation uncertainty and measurement uncertainty. The preparation uncertainty maintains the original HUR in regard to the uncertainty inherent in the preparation of a quantum system. This has been experimentally verified; we have no need to revise the HUR in regards to preparation uncertainty. Indeed, it may very well be said that the HUR is preparation uncertainty, for it is exactly what was mathematically proved in Heisenberg's original paper. The measurement uncertainty requires a new definition to be universally valid for every quantum measurement technique available today. Currently, the Branciard relation is the tightest relation. Using this relation, we are able to provide meaningful and valid error and disturbance relations to weak and triple-state measurement experiments - something we are unable to do with the HUR.

While a redefinition of uncertainty is certainly a step in the right direction, this quandary is far from resolved. True, we have been provided with a relation that compensates for the inherent flaws in the HUR, but that does not mean we have found the best relation. Though the Branciard relation is the best we currently have, we have yet to prove that it is the best we can ever have. Moving forward, we will need to continue to search for an even tighter relation with even lower bounds.

We will also need to formulate a theory to confirm whether any particular relation is the lowest possible, or whether we need to continue looking. These are daunting tasks that must be addressed in the near future. The need to redefine uncertainty is clear; all that is left is to decide on the new definition.

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