Instructional Lab for Undergraduates Utilizing

the Hanbury Brown and Twiss Effect

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ABSTRACT

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Using the classical Hanbury Brown and Twiss effect, students will measure correlation at two detectors far from an aperture in order to discover the diameter of the aperture. The light source will consist of a laser, spatially modulated so that the effect can still be observed and measurements can easily be made. The students will gain an increased understanding of angular size, spatial coherence, interference, and Rayleigh's criterion.

Contents

Chapter 1

Introduction

1.1 The Hanbury Brown and Twiss Effect

The classical Hanbury Brown and Twiss (HBT) effect is a method for determining the angular size of stars. It was an improvement made in the 1950's off of the established Michelson interference method. The method created by Robert Hanbury Brown, an astronomer, and Richard Q. Twiss, a mathematician, was to measure the intensity of the light sources with a normal detectors then by analyzing the currents from the detectors, determine the spatial coherence and thus the angular size of stars. By transforming the setup from predominantly optical to predominantly electronic the setup became more stable and precision increased. The classical HBT effect then is a way of determining angular size of light sources by measuring just the intensity at two different points.

1.2 History of the Hanbury Brown and Twiss Effect

In 1956, Hanbury Brown and Twiss (HBT) published a paper (Brown & Twiss 1956) on a method of determining the angular size of a star by comparing the intensities gathered from two detectors. This effect was seen by using two photomultiplier tubes and by increasing the distance between

them, seeing a drop in the correlation. Because the correlation is made while only measuring the intensity of the light and with the similarities to previous interferometry techniques, the effect is sometimes referred to as intensity interferometry. The size of the source can be calculated from measuring how the correlation drops off as a function of the distance between the detectors.

At the time this method was a serious improvement on the previous method of traditional interferometry. Interferometers required stable and delicate optics in order to create any interference. The HBT technique instead requires large light gathering ability and precision electronics. Through this method, hundreds of stars were measured with unprecedented accuracy. However, as optical engineering improved, the primary method of measuring the size of stars eventually reverted back to the interferometer. Although the HBT effect is still used in other areas, it is no longer being utilized to determine the size of stars.

The idea arose to use the HBT effect to teach undergraduates various principles of light in a lab setting. The application of this effect would allow students to experiment easily with temporal and spatial coherence, correlation, angular size and the frequency spectrum of light received at the two detectors.

1.3 Learning Outcomes

At BYU there exists an active interest in developing the instructional labs used to teach undergraduates experimental methods. This challenge is not as easy one. There are many different and well known experiments used to teach undergraduates but they each suffer from some of the same problems that prevent the students from actively learning during the class. A common problem is too much "dirt." This is to mean that students spend far too much working with imperfect equip-

ment instead of spending the time to learn the principles or methods of the experiment. A second problem is the other end of the spectrum. Students may instead be presented with a "cookie cutter" experiment that simply involves turning on and copying numbers from a digital display. The result of this type of instruction is similar as students fail to learn but only confirm what they already knew. This simpler experiment could just as easily be reproduced by a computer simulation with the same outcome.

These problems can be avoided by creating an experiment that is robust enough to consistently work while also being complex enough to stretch the students and help them learn to the principles behind the success of the experiment. To accomplish these goals the physics department regularly reviews it current arsenal of experiments, tweaking and redesigning the classes to enable the students to learn most with the time available.

The purpose of my research has been to design an instructional lab that utilizes the classical Hanbury Brown and Twiss(R. Hanbury Brown 1957) effect as it applies to chaotic light sources. Beginning in the 1950's, this effect was used to measure the angular size of distant stars. It was praised because although it had electronic complexity to analyze the source, the nature of the effect allows for some instability in the optics. I expect that this flexibility will help the students so they do not have too much "dirt" while the principles used by this effect are sufficiently complex so that the data gathering and analysis will still require the students to learn the principles that govern the experiment. The principles with which the students will become familiar include spatial coherence and how it is related to the Rayleigh criterion. They also learn the techniques for taking data and calculating the correlation function.

Chapter 2

Explanation of the HBT Effect

2.1 Description of the Hanbury Brown and Twiss Effect

The original application of the Hanbury Brown and Twiss (HBT) effect was used to determine the spatial coherence of light by analyzing the fluctuations in the intensity of the light. The idea is that chaotic light sources consist of many frequencies coming from many source locations that create many fluctuations of the intensity at a detector. However, at a different point that fluctuation pattern is different because each of the frequency components coming from different points on the source shifts relative to each other producing a different set of constructive and destructive interferences at the detector. Clearly the fluctuation patterns are very rapid and comparing one fluctuation pattern to another is fruitless unless they were taken at the same time. The HBT effect was a technique for measuring the spatial coherence of a chaotic light source by measuring a light source while varying the distance between the detectors.

The only information gathered by the detectors was this intensity fluctuation pattern. When the two detectors are close together, the intensity patterns are nearly identical but as the detectors are

separated the fluctuations eventually look completely different. The way to measure the similarity of the intensity fluctuations is by using the second-order correlation function. The second-order correlation can be related to the original electric fields as long as the light is chaotic. That means that making measurements of the intensity and its fluctuations can be used to measure the spatial coherence of the light source. The spatial coherence of a chaotic light source is directly related to its angular size. This all means that by measuring how the correlation of the the intensity fluctuations of light changes with distance, the angular size can be determined.

2.1.1 Analogy to Young's Two-Slit Experiment

There are similarities between the HBT effect and Young's two-slit experiment. The two-slit experiment sends light through two different slits and an interference pattern is seen on a screen behind the slits. The two-slit experiment can also be used to measure the ability to create an interference pattern on the screen as the two slits are separated. The HBT effect can be thought of as a measurement made at the slits to see if an interference pattern would form on the wall behind. In this way it is understood that there is not interference occurring but rather by measuring the intensity fluctuations at the two separate positions the coherence of the light can be measured.

2.2 Mathematical Description of the Correlation Function

Here I present an introduction to the second-order correlation function and how it is utilized by Hanbury Brown and Twiss. First we describe the first-order correlation function in order to draw the parallels with the second-order correlation function. The first-order correlation function is defined as

$$g^{(1)}(\tau) = \frac{\langle E_1(t)E_2(t+\tau)\rangle}{\langle E_1(t)\rangle\langle E_2(t+\tau)\rangle},$$

where E_1 and E_2 represent the electric field at two separate detectors separated by a time τ. The $\langle$ and $\rangle$ denote a time average. This function tells of the coherence between two different electric fields at two separate detectors. The magnitude of the first-order correlation function can be determined from the second-order correlation function by the relation

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2.$$

This relationship is only true for any ensemble of uncorrelated emitting atoms.(Dayan 2012b)

The second-order correlation appears very similar to the first,

$$g^{(2)}(\tau) = \frac{\langle I_1(t)I_2(t+\tau)\rangle}{\langle I_1(t)\rangle\langle I_2(t+\tau)\rangle},$$

where I_1 and I_2 represent the intensity at two detectors separated by a time τ. The power of the equation lies in multiplying the intensities before time-averaging. I will explore this equation by considering an idealized situation where the source consists of two points separated by a distance R from each other and a distance L from two detectors. The detectors are separated by a distance d. See Figure 2.1. There exists four possible paths for the light to travel.

The light from source A that is seen by detector 1 can be expressed as

$$\alpha e^{i(k*L_{A1}+\phi_A)}$$

where L_{A1} is the distance between source A and detector 1 and ϕ_A is random phase fluctuations of source A that characterize an incoherent source. Similar expressions are assumed for the other three paths shown in Figure 2.1. The total electric field measured at detector 1 is

$$E_1 = \alpha e^{i(k*L_{A1}+\phi_A)} + \beta e^{i(k*L_{B1}+\phi_B)},$$

with a similar expression for the electric field at detector 2. The intensity measured at detector 1 would then be

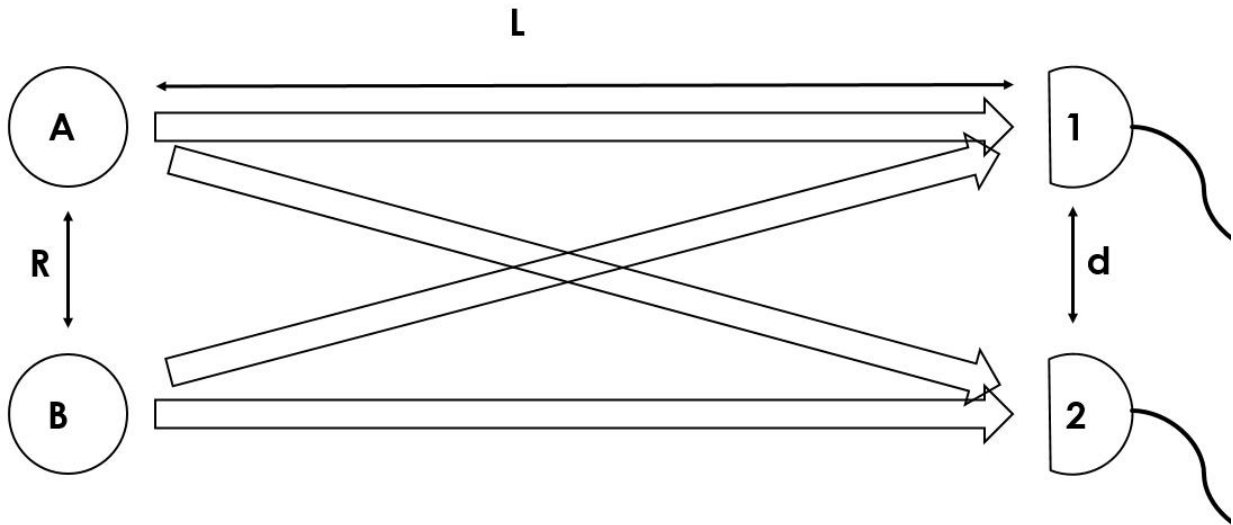$$I_1 = \alpha^2 + \beta^2 + 2\alpha\beta \cos(k*(L_{A1}+L_{B1}) + \phi_A - \phi_B).$$

Figure 2.1 Two uncorrelated light sources separated by a distance R are seen by two detectors separated by a distance d. The sources and detectors are separated by a distance L. The four possible paths are shown.

Time averaging the intensity over many coherence times gives

$$\langle I_1 \rangle = \alpha^2 + \beta^2,$$

as the cosine term averages to zero due to the random phase fluctuations.

Multiplying the separate time-averaged intensities gives simply

$$\langle I_1 \rangle \langle I_2 \rangle = (\alpha^2 + \beta^2)^2.$$

However, if the intensities from detector 1 and detector 2 are first multiplied and then time-averaged the result is

$$\langle I_1 * I_2 \rangle = \alpha^4 + \beta^4 + 2\alpha^2\beta^2[1 + \cos(k * (L_{A1} - L_{A2} - L_{B1} + L_{B2}))].$$

As mentioned before, this is the more powerful equation. Because it has no time dependance but rather consists of a constant term along with a term that varies only based on the relative distance between sources and detectors. Although it may appear that this example may only work for one

wavelength, it actually does not work if there is only one wavelength. The random phase accounts for the incoherence of the light and without this term the second-order correlation would always yield the same number regardless of how far apart the detectors were.

2.3   Determining Angular Size from Spatial Coherence

The Rayleigh criterion is an acceptable standard for minimum resolvable detail. It tells of the diameter of aperture necessary to resolve an angular size. For a slit aperture the Rayleigh criterion is defined as

$$\sin\theta_R = \frac{\lambda}{d}.$$

The angle (θ_R) is the smallest angle resolved by an aperture with length d of a light with peak wavelength λ. Using the HBT effect the angular size is determined by increasing the distance between detectors simulating a larger aperture. When the d is determined then the angular size of the star is determined. If the distance to the source is known, then the diameter of the source can be determined from the angular size.

Chapter 3

Numerical Exploration of a Possible Lab Experiment

3.1 Numerical Calculations of the Second-Order Correlation Function

In order to better understand the second-order correlation function as well as predict the curves that would be obtained after the experiment was assembled, I conducted numerical calculations of the second-order correlation function in Mathematica. Here I will outline the method I used to numerically simulate the source.

To generate the source I assumed that the light was really a phase jitter on top of a single frequency carrier wave. I chose 650 nm light to simulate available light sources and added on top a random phase component at each point in time that would approximate an incoherent source with a bandwidth of 200 kHz. I then split up the source into equal portions by generating separate waves coming from the separate source points. I then had one detector move along a path reading a sum

of the signals at each of the separate positions. By shifting the phase of each frequency of the light because of the distance it travelled between source and detector, I calculated a signal at each of the detector positions. For this initial calculation I assumed a detector fas enough to measure changes as fast as the period of the light oscillations (discussion of the effects of limited detector speed are discussed later). Unfortunately the calculation time proved to be much longer than expected. To cut down on the time the number of sources and detector positions was reduced but this no longer created the expected correlation curve.

After analyzing the problem with my advisor (Durfee 2015), it was determined that instead of a phase shift on top of the carrier wave, we could simulate the source by only generating and analyzing the phase jitter that existed on top of the carrier. For the full analysis see Appendix B. The numerical analysis was considerably simplified and it became possible to generate curves in a few minutes. A curve generated from a source split into 100 parts and detectors moved through 30 positions is shown in Figure 3.1.

The figure shows the expected behavior of the second-order correlation function for an incoherent light source. The correlation is a maximum of 2 when the detector is compared to itself and drops to 1 as the distance is increased. The relationship between this correlation function and correlation of the electric fields is

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2.$$

Connecting to the numerical results means that the first-order correlation goes from 1 to 0, again as expected.
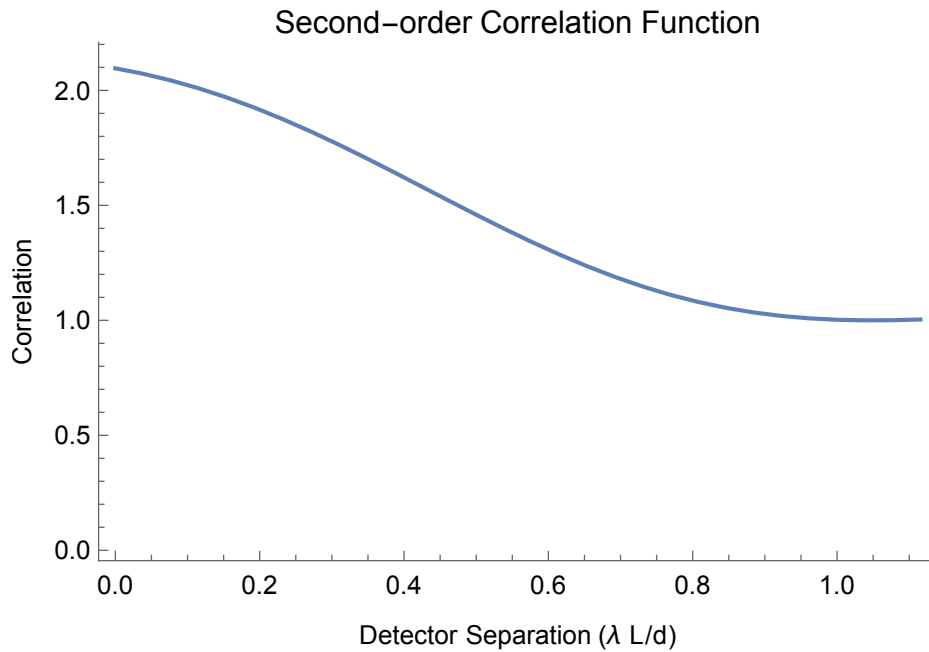
Second−order Correlation Function

Figure 3.1 Graph of the second-order correlation function generated from a source consisting of 100 parts and detectors moved through 30 positions.

3.2   Generating a Spatially Incoherent Light Source

Here I address the signal-to-noise ratio. The SNR for this experiment is a function of both the coherence time of the light t_c as well as the response time of the photodiode, T_r. (Dayan 2012a) The relationship (Gamo 1966) exists as

$$SNR \propto \frac{t_c}{T_r}.$$

Thermal light has a coherence time on the order of a femtosecond and the faster detectors in our lab have a time response of 20 ns making $\frac{t_c}{T_r}$ approximately equal to 5×10^{-8}. To see the signal would require very long integration times which is not optimal for an instructional lab. The solution is to instead lengthen the coherence time. The method that we decided upon to lengthen the coherence time would be to use a high-powered laser as the light source and send the light through a spinning piece of glass. The piece of glass can be attached to a variable motor and by

changing the speed of the motor we can change the spatial incoherence that is introduced onto the original laser beam. In a similar method to the numerical introduced in section 3.1, we simply created a random phase component onto an existing carrier wave generated by the laser.

Chapter 4

Conclusions

4.1   Possibility of Doing the Experiment

From the numerical calculations and research we have concluded that it will be possible to conduct the experiment as planned. By creating noise on a coherent source we can measure the correlation much easier than originally expected. This experiment also holds the potential for teaching students about several complicated principles in an accessible way.

4.2   Progress Made

We have acquired analog multiplier chips that have a bandwidth much greater than the time scale of fluctuations we plan to introduce onto the coherent source. We attempted to procure a portable laser that had sufficient power output so that the small amount that came out the aperture would still be sufficient, but the laser we purchased did not perform to specifications.

4.3   Next Steps

We have considered the idea of using the line-in port on a computer to gather the data so that the correlation can be digitally calculated. A program would also have to be created to calculate the correlation. This would make the experiment much easier to assemble since the students' equipment would consist of just the photodiode and the computer. A high-powered laser pointer still needs to be acquired and the light source assembled. The project seems to be fulfilling the expectations outlined at the beginning of this thesis. The apparatus may be complex in assembly but if successful it will simplify the data acquisition so that the student does not have to deal with the "dirt" of the experiment. The project must now be continued by another student.

Appendix A

Mathematica Code for Numerical Analysis

A.1 Random Phase Shift on Carrier Wave

This is the code from Mathematica used to generate the predicted second-order correlation curve from a discrete number of sources. The code creates a carrier wave and adds phase jitter to that wave.

```
Clear["*"]

Tpeak = 650*^-9/3.*^8;

dt = Tpeak/1000;

Δt = dt * 2^20;

Nc = 1*^-15/dt;

dϕ = 2 * π * Sqrt[Nc];

ωt = 3*^8/650*^-9 * dt;

nωt = Range[2^20] * ωt;

jitter:=Module[{},

rand = Accumulate[RandomInteger[{−1, 1}, 2^20] * dϕ];

nωtϕ = nωt + rand;

wave = Sin/@nωtϕ

]

pieces = 10;

sources = Array[jitter&, pieces];

fourier = Array[Fourier[sources[[#]]]&, pieces];

L:=10;

width = 1*^-4;

d = width/(pieces − 1);

posnum = 10; posstart = 0; posend = .07;

posres = (posend − posstart)/(posnum);

positions = Range[posstart, posend, posres];

While[Length[positions] ≠ posnum,

If[Length[positions] < posnum, AppendTo[positions, Last[positions] + posres],

positions = Drop[positions, −1]]];
```

```
sections = Range[0, width, d];

distance = Table[Sqrt[L^2 + (pos + sig)^2], {sig, sections}, {pos, positions}];

freq = 2π ∗ Join[Range[0, 1/(2 ∗ dt), 1/Δt], Range[−1/(2 ∗ dt) + 1/Δt, −1/Δt, 1/Δt]];

ϕshift = Table[f ∗ dis, {dis, distance}, {f, freq}];

phase = E^(I ∗ ϕshift/3∗^8);

If[Length[freq] ≠ Dimensions[fourier][[2]],

MessageDialog["Fourier generated more fequencies than expected."];

Quit[]]

Dimensions[fourier]

{10, 1048576}

Dimensions[phase]

{10, 1048576, 10}

pattern = Total[Array[phase[[#, All, #2]] ∗ fourier[[#]]&, {pieces, posnum}]];

signal = Array[InverseFourier[pattern[[#]]]&, posnum];

inten = Chop[signal ∗ signal∗];

mean = Array[Mean[inten[[#]]]&, posnum];

corrolations = Array[Mean[inten[[1]] ∗ inten[[#]]]/mean[[1]]/mean[[#]]&, posnum]

corr = Transpose[{positions, corrolations}];

ListPlot[corr]
```

A.2 Random Phase Shift Only

This is the code from Mathematica used to generate the predicted second-order correlation curve from a discrete number of sources. This code analyzes only the phase jitter.

```
Clear["*"]

Δω = 20.*^3;

Δt = 1/Δω;

Tpeak = 1/(20*^3);

dt = Tpeak/10;

T = dt * 2.^20;

Nc = Δt/dt;

dφ = 2 * π/Sqrt[Nc];

jitter:=Module[{},

rand = Accumulate[RandomInteger[{−1, 1}, 2^20] * dφ];

Fourier[Sin/@rand]

]

pieces = 100;

sources = Array[jitter&, pieces];

flaser = 3*^8/650*^-9;

freq = 2π * Join[Range[0, 1/(2 * dt), 1/T], Range[−1/(2 * dt) + 1/T, −1/T, 1/T]]+

flaser;

L:=2;

width = 1*^-3;

d = width/(pieces − 1);

posnum = 30; posstart = 0; posend = .0015;

posres = (posend − posstart)/(posnum);

positions = Range[posstart, posend, posres];

While[Length[positions] ≠ posnum,

If[Length[positions] < posnum, AppendTo[positions, Last[positions] + posres],
```

```
positions = Drop[positions, −1]]];

sections = N[Range[0, width, d]];

distance = N[Table[Sqrt[L^2 + (pos + sig)^2], {pos, positions}, {sig, sections}]];

inten = ConstantArray[0, {posnum, 2^20}];

corrmaker[detector_]:=Module[{ϕshift, phase, signal, pattern},

ϕshift = Array[Exp[−I ∗ 2π ∗ (distance[[detector, #]] ∗ freq)/3∗^8]&, pieces];

pattern = Total[Array[ϕshift[[#]] ∗ sources[[#]]&, pieces]];

signal = InverseFourier[pattern];

inten[[detector]] = Abs[signal]^2;

]

Reap[Do[

Array[corrmaker[#]&, posnum];

mean = Array[Mean[inten[[#]]]&, posnum];

corrolations = Array[Mean[inten[[1]] ∗ inten[[#]]]/mean[[1]]/mean[[#]]&,

posnum];

corr = Transpose[{positions, corrolations}];

Sow[ListPlot[corr, PlotRange → All, Joined → True]],

{1}]]

650.∗^-9 ∗ L/width

positions/%

corr = Transpose[{positions/.0013, corrolations}]

ListPlot[corr, Joined->True]

Show[%, PlotLabel → HoldForm["Second-order Correlation Function"],

LabelStyle → {GrayLevel[0]}, Frame → {True, True, False, False},

FrameLabel → {"Detector Separation (λ L/d)", "Correlation"}, Axes → False]
```

Appendix B

Theoretical Analysis to Use Only the Phase Jitter

This chapter contains the private communication between Dr. Durfee and myself. This analysis is the work of Dr. Durfee. See (Durfee 2015).

Each point on the source has an electric field waveform $E_n(t)$. This can be written in the form

$$E_n(t) = A_n(t)e^{i\omega_0 t}.$$

This is completely general if we let A_n be an arbitrary complex function of time. When this wave reaches detector m, it has traveled a distance s_{nm}, causing a time delay $\tau_{nm} = s_{nm}/c$, becoming

$$E_{nm}(t) = A_n(t - \tau_{nm})e^{i\omega_0(t-\tau_{nm})} = A_n(t - \tau_{nm})e^{i\omega_0 t}e^{-i\omega_0 \tau_{nm}}.$$

We can do the time shift by taking the Fourier transform, multiplying by $e^{-i2\pi(f/c)s}$, and then taking the inverse transform:

$$E_{nm}(t) = \mathscr{F}^{-1}\left[\mathscr{F}\left[A_n(t)e^{i\omega_0 t}\right]e^{-i2\pi\frac{f}{c}s}\right].$$

We can use the convolution theorem to write this as

$$= \mathscr{F}^{-1}\left[\left[\mathscr{F}\left[A_n(t)\right] \otimes \mathscr{F}\left[e^{i\omega_0 t}\right]\right]e^{-i2\pi\frac{f}{c}s}\right] = \mathscr{F}^{-1}\left[\left[\mathscr{F}\left[A_n(t)\right] \otimes \delta(\omega - \omega_0)\right]e^{-i2\pi\frac{f}{c}s}\right].$$

Convolving a spectrum with a delta function will just move all of the frequencies up by an amount ω_0. So if you calculate this numerically, you can do an FFT of A, and then just change your interpretation of the frequencies to be ω_0 higher. Then apply the exponent to give phase shifts, remembering the f applies to the shifted frequencies. This means to get to this point, you don't need to sample the signal fast enough to catch the oscillations at frequencies near ω_0, just fast enough to catch changes in $A_n(t)$.

When we try to do the inverse transform, we have a problem because our array of data doesn't have the low and negative frequency terms. But this won't be a problem. To see why, let's take the Fourier transform of both sides:

$$\mathscr{F}[E_{nm}(t)] = [\mathscr{F}[A_n(t)] \otimes \delta(\omega - \omega_0)]e^{-i2\pi\frac{f}{c}s}.$$

Now, just for kicks, let's convolve both sides with the Fourier transform of $e^{-i\omega_0 t}$:

$$\mathscr{F}[E_{nm}(t)] \otimes \mathscr{F}\left[e^{-i\omega_0 t}\right] = \left[[\mathscr{F}[A_n(t)] \otimes \delta(\omega - \omega_0)]e^{-i2\pi\frac{f}{c}s}\right] \otimes \mathscr{F}\left[e^{-i\omega_0 t}\right].$$

Which becomes:

$$\mathscr{F}\left[E_{nm}(t)e^{-i\omega_0 t}\right] = \left[[\mathscr{F}[A_n(t)] \otimes \delta(\omega - \omega_0)]e^{-i2\pi\frac{f}{c}s}\right] \otimes \delta(\omega - \omega_0).$$

But, again, convolving with a delta function just offsets all of the frequencies. So the right side is shifted back to the original frequency bins that we had when we took the Fourier transform of $A_n(t)$. So I can use my standard inverse FFT to take the inverse transform of the right side, giving me

$$E_{nm}(t)e^{-i\omega_0 t} = \mathscr{F}^{-1}\left[[[\mathscr{F}[A_n(t)] \otimes \delta(\omega - \omega_0)]e^{-i2\pi\frac{f}{c}s}] \otimes \delta(\omega - \omega_0)\right].$$

Which becomes:

$$E_{nm}(t) = e^{i\omega_0 t}\mathscr{F}^{-1}\left[\mathscr{F}[A_n(t)]e^{-i(2\pi f + \omega_0)\frac{s}{c}}\right].$$

And since we only care about intensity (that's what the photodiode measures), we can just drop the $e^{i\omega_0 t}$:

$$\left[\Sigma_{n=0}^{N}\Sigma_{m=0}^{M}E_{nm}(t)e^{-i\omega_0 t}\right]^{*}\left[\Sigma_{n=0}^{N}\Sigma_{m=0}^{M}E_{nm}(t)e^{-i\omega_0 t}\right]$$

$$= \left[\Sigma_{n=0}^{N}\Sigma_{m=0}^{M}E_{nm}^{*}(t)e^{i\omega_0 t}\right]\left[\Sigma_{n=0}^{N}\Sigma_{m=0}^{M}E_{nm}(t)e^{-i\omega_0 t}\right]$$

$$= \left[\Sigma_{n=0}^{N}\Sigma_{m=0}^{M}E_{nm}^{*}(t)\right]\left[\Sigma_{n=0}^{N}\Sigma_{m=0}^{M}E_{nm}(t)\right]$$

The size of the arrays we work with in a simulation depends not on the carrier frequency ω_0, but on the rate at which $A(t)$ changes. If $A(t)$ changes slowly, then it will only broaden the carrier a little bit. So simulating a narrow linewidth source is much easier than a broadband source.

When doing an actual experiment, the speed that we need our detectors to work at only depends on $A(t)$, and not ω_0. So a narrow band source doesn't require a really fast detector. A broadband source requires either a detector fast enough to catch changes on the scale of a single oscillation of the carrier, or very good signal to noise ratio so that the tiny bit of correlation in the bandwidth you can detect can be measured.

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