Analyzing the Dynamics of Coupled

Quantum Harmonic Oscillators

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ABSTRACT

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The dynamics of a coupled ground and coherent state are explored. The approach is focused on solving for the time evolution operator and then applying it to a tensor product of a ground and coherent state representing a physical system and environment respectively. The coherent state is then partially traced to extract the dynamics of the ground state. The time evolution operator is found by solving a series of eleven coupled differential equations. The results demonstrate that a change in coupling results in a change in the evolution of the ground state.

Keywords: quantum physics, coupled harmonic oscillators, squeezing, decoherence

ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. Jean Francois Van Huele for all his guidance and help. I would also like to thank my husband Ryan for all the hours he watched Everett so I could work on this thesis. Thank you to my sister Analisa for her help in editing. Finally, I would like to thank my parents for all their help and support throughout my college experience and especially after Everett was born.

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Chapter 1

Introduction

1.1 Overview

Quantum mechanics helps us understand nature at the most fundamental level. Everything is composed of atoms and quantum particles and, as we delve deeper into the quantum world, we uncover more of the mysteries of the universe. My project approaches a small part of the workings of quantum mechanics by attempting to understand more about the dynamics of a quantum system. There are several approaches to understand the dynamics of a system within classical mechanics. The Newtonian, Lagrangian, and Hamiltonian approaches to classical mechanics predict where a ball will fall and how a spaceship will fly in space. However, quantum mechanics does not currently have a simple method for solving for the time-dependence of any system. Many time-independent approaches are available for simple systems, but our research group has aimed at using an approach to solve for the time-dependence of more intricate systems, such as systems depending on time-dependent parameters, for example, time-dependent harmonic oscillators. The simple harmonic oscillator is commonly known in a classical sense as a mass m on a spring with a specific spring constant k or a swinging pendulum as represented in Fig. (1.1). The importance of the simple harmonic oscillator (SHO) follows from the fact that any system with a local minimum can be approximated by it. The time-dependence of the SHO with constant m and k has been worked out in introductory quantum physics courses [1] and will not be addressed here.

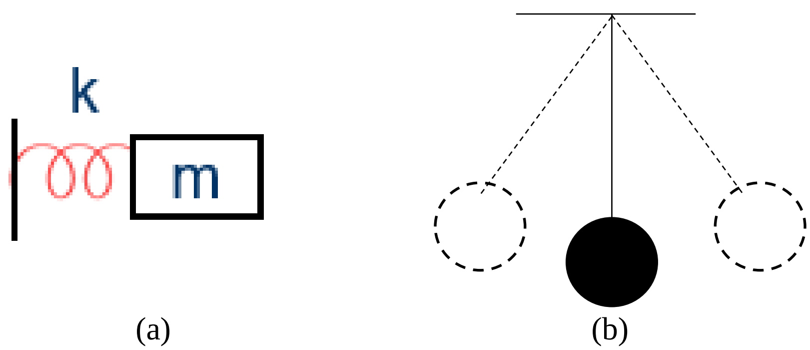


Figure 1.1 Examples of Simple Harmonic Oscillators. (a) A mass on a spring and (b) a swinging pendulum.

Within the research group students have succeeded in approaching topics such as simple harmonic oscillators with a time-dependent mass, a time-dependent parametric oscillator, or a time-dependent driving force [2][3]. My specific project builds upon this previous work by coupling two simple harmonic oscillators and attempting to solve for their time-dependence. In my research, I create a time-dependent model of a quantum system interacting with its environment and by so doing, gain new insight into the effect of an environment on a quantum system. This effect of the environment on a quantum system is termed decoherence. Decoherence occurs as multiple quantum systems interact. This phenomenon can thus greatly hinder the ability to control a quantum system and, as needed, in the creation of a quantum computer.

1.2 Background

As mentioned, other students within the research group have approached topics related to the SHO. The results of this work have shown that quantum control is possible. Squeezing of a coherent state has been shown to be possible through varying both the mass and the oscillation of the system [4]. Research conducted by another group has also shown that analysis of the coupled quantum oscillator can lead to squeezing [5]. This previous research, conducted on analyzing the coupled quantum oscillator, similarly approached coupling a ground and coherent state but solved for the evolution by using the normal mode coordinates. This technique demonstrates that when the oscillators are in resonance squeezing is observed. They found that the position uncertainty decreases while the momentum uncertainty increases in the equal-mass and equal-spring constant case [5].

I use coupled harmonic oscillators to model the coupling of the environment to a physical system. As the coupling increases, the connection between the state and the environment around it strengthens and, as the coupling disappears, we expect to see a decrease in the influence of the environment. Interestingly, it may be possible to observe the disappearance of the environment, without a complete disappearance of the effects of the coupling. This concept of removing coupling while yet still observing a connection can help us learn more about how the environment affects a system. We hope to be able to establish control over the state through the strength, length, or time-dependence of the coupling. This control can give us power to create new states, but also teach us more about how an environment can influence a quantum state.

A major difficulty when working with quantum states is maintaining coherence. It is difficult both mathematically and technically to observe a state for very long. Mathematically, due to the uncertainty principle and, technically due to complications creating instruments that work at that level. Decoherence is the term used to describe the breakdown of information that occurs as you get further from the original quantum state. A typical example is Schrödinger's cat. The experiment is set up where you have a cat in a box with a Geiger counter and a box of poison. The question comes when you ask what part of the experiment is no longer "quantum." Can you describe a cat as being in a superposition state? Where does the ability to describe a state in a quantum sense disappear? This is a question I hope to explore through my model. By expanding my model to include the ability to adjust the mechanics of the system and environment we can adjust the "mass" of one of the two oscillators and continuously increase it until we reach a point where the dynamics no longer make sense. This point becomes our breaking point for "coherence" and the beginning of decoherence.

The wide applicability of the harmonic oscillator makes it an easy choice for this model. Almost every system can be approximated by the harmonic oscillator at a minimum. The potential energy well of a SHO is a simple curve and can fit into any local minimum. Also, the dynamics of the SHO are well understood and covered in the undergraduate curriculum [1]. These characteristics make it an approachable problem. My rough model uses the idea of two masses connected to springs coupled together as described in Fig. (2.1) in the next chapter. This model allows us to utilize the known and researched coupled oscillator and extrapolate by varying the parameters [1].

1.3 Lie algebra approach

The Lie algebra approach to solving complex differential equations has been used since 1963 [6]. This approach utilizes the idea that if a set of operators and their commutators generates a Lie algebra then the time evolution operator can be factorized into exponentials of the Lie algebra basis operators multiplied by time-dependent functions [3]. Using this approach, we can take a very complicated partial differential operator equation and turn it into a coupled set of nonlinear ordinary differential equations. We can turn a time-dependent operator problem into a time-independent series of operator equations and a time-dependent algebraic formulation.

Chapter 2

Methods

2.1 System Overview

The system used throughout this thesis is a set of two coupled harmonic oscillators. I am using the most straightforward formulation where the two masses are constant and equal, the two outer spring constants are equal whereas the coupling spring constant is time-dependent and not equal to the outer spring constants. Throughout, I denote m for the mass, k(t) for the outer spring constants and $k_3(t)$ for the coupling spring constant. A graphical representation of the system can be seen in Fig. (2.1).

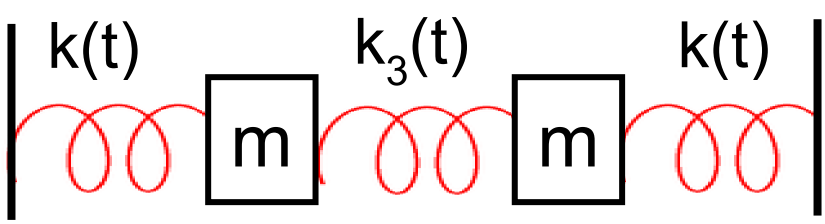


Figure 2.1 This is a representation of the model of two coupled harmonic oscillators. The simple approach has m1=m2=m and k1=k2=k(t)

To describe the configuration mathematically we write down the Hamiltonian H(t) for the system. Eq. (2.1) describes the Hamiltonian in terms of the position operators, x_1 and x_2, and momentum operators, p_1 and p_2,

$$\hat{H}(t) = \frac{(\hat{p}_1)^2}{2m_1} + \frac{(\hat{p}_2)^2}{2m_2} + \frac{1}{2}k_1(\hat{x}_1)^2 + \frac{1}{2}k_2(\hat{x}_2)^2 + \frac{1}{2}k_3(\hat{x}_2 - \hat{x}_1)^2, \tag{2.1}$$

while Eq. (2.2) describes the Hamiltonian in terms of the raising and lowering operators, a and $a^\dagger$,

$$\hat{H}(t) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) + \frac{k_3\hbar}{4m\omega}\left(\hat{b}^2 + \hat{b}^{\dagger 2} + \hat{a}^2 + \hat{a}^{\dagger 2} + 2\left(\hat{b}^\dagger\hat{b} - \hat{b}\hat{a} - \hat{b}\hat{a}^\dagger - \hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + 1\right)\right) \tag{2.2}$$

if $m_1 = m_2$ and $k_1 = k_2$ where $\omega = \sqrt{\frac{k}{m}}$. In Eq. (2.2) a and $a^\dagger$ are related to x_1 through $x_1 = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a})$ similarly $x_2 = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b}^\dagger + \hat{b})$ [1]. We will impose later that a and $a^\dagger$ describe the system and b and $b^\dagger$ describe the environment degrees of freedom. We are then able to insert this Hamiltonian into Schrödinger's equation for the time evolution operator U(t),

$$i\hbar\frac{\partial\hat{U}(t)}{\partial t} = \hat{H}(t)\hat{U}(t). \tag{2.3}$$

Here i is the imaginary number, ħ is Plank's constant, and U(t) is the time evolution operator which evolves states such that $\hat{U}(t_1, t_2)|\psi(t_1)\rangle = |\psi(t_2)\rangle$. Following [6], the time evolution operator U(t) can further be written as a product of exponentials,

$$\hat{U}(t) = \prod_{n=1}^{11} e^{s_n(t)\hat{A}_n}. \tag{2.4}$$

The eleven operators A_n that comprise the time evolution operator can be found in the Hamiltonian and are: $\hat{a}\hat{a}, \hat{b}\hat{b}, \hat{a}\hat{b}, \hat{a}^\dagger\hat{a}, \hat{b}^\dagger\hat{b}, \hat{a}\hat{b}^\dagger, \hat{b}^\dagger\hat{a}^\dagger, \hat{a}^\dagger\hat{b}, \hat{a}^\dagger\hat{a}^\dagger, \hat{b}^\dagger\hat{b}^\dagger, \hat{1}$, while the eleven functions $s_n(t)$ are functions associated with each operator and will be plotted in Chapter 3. To ensure that we can use the Lie algebra approach explained in Chapter 1 we evaluated the commutation between all operators. A table comprising all the commutations between each operator is found in the Appendix. The table demonstrates that the Lie algebra is closed since there are no new operators contained in the table. Since the system forms a Lie algebra basis we know we can move forward in solving for the dynamics.

2.2 Mathematical Approach

The ultimate goal is to solve Schrödinger's equation, Eq. (2.3) above. By solving this equation we will be able to observe the time evolution of our system. We can then apply the evolution operator to specific states.

We begin by re-writing Eq. (2.3) as

$$i\hbar \frac{\partial \hat{U}(t)}{\partial t} \hat{U}(t)^{-1} = \hat{H}(t). \tag{2.5}$$

From here we can substitute the time evolution operator from Eq. (2.4) and the Hamiltonian from Eq. (2.2). Using a Mathematica program written by Ty Beus, a member of the research group, we can then solve Eq. (2.5). The program solves for the unknown $s_n(t)$ equations within the time evolution operator. Once these equations are known, we can input them back into Eq. (2.4), the time evolution operator, and subsequently apply this operator to different states.

We chose to apply the time evolution operator to a tensor product of two coherent states, specifically the ground state (system) and a general coherent state (environment) to compare our method to the method described above found in Ref. [5]. The application to this tensor product of states can be written as

$$\langle x_1| \otimes \langle x_2| (\hat{1} \otimes \hat{1})\, \hat{U}\, |\alpha\rangle \otimes |\beta\rangle, \tag{2.6}$$

where $|\alpha\rangle$ is the system, $|\beta\rangle$ is the coherent state (environment), and $\langle x_1|$ and $\langle x_2|$ are the coordinate eigenstates so as to obtain a position representation of the evolved state. We insert the completeness relation for coherent states found in Ref. [7] as,

$$\hat{1}\otimes\hat{1} = 1/\pi^2 \int d\gamma\, d\delta\, |\gamma\rangle \otimes|\delta\rangle\langle\gamma|\otimes\langle\delta|, \quad (2.7)$$

to allow us to solve for the position representation of the evolved state. Here γ and δ are both coherent states. The application of the time evolution operator on a tensor product of a ground and coherent state written in coordinate representation is

$$1/\pi^2 \iint d\gamma\, d\delta\, \langle x_1|\gamma\rangle \cdot \langle x_2|\delta\rangle \left\langle\gamma|\otimes\langle\delta\middle|\hat{U}\middle|\alpha\right\rangle \otimes|\beta\rangle, \quad (2.8)$$

again found in Ref. [7]. Once we complete this double integral we have a position representation of the time evolved ground and coherent state.

2.3 Partial Tracing

In order to visualize the time evolution of the states we apply partial tracing. Partial tracing is a mathematical technique used with matrices. This technique allows you to integrate over the portion of the function you wish to ignore or trace over. In our case we will integrate over the position of the second oscillator x_2 (the environment in a coherent state) to be able to observe the dynamics of the first oscillator x_1 (the ground state). This will allow us to observe the system in the ground state that has interacted with the environment, while choosing to ignore the environment itself. Upon completing this integration, we can plot the time-evolved ground state and observe its dynamics. These plots are included in Chapter 3.

Chapter 3

Results

3.1 Time Evolution Operator

The time evolution operator is composed of a product of exponentials raised to a time-dependent function multiplied by an operator, see Eq. (2.4). For the coupled harmonic oscillator, the time evolution operator is composed of 11 operators and 11 exponentials. Each time-dependent function is a solution to a coupled differential equation. Mathematica was unable to solve for these functions analytically so after inputting specific values for the system such as the mass, and spring constants k(t) and k_3(t), and ħ, we found solutions to the differential equations numerically. Plots of these functions are included below. The real part of each function is 0, so the imaginary part is plotted. The input values for the plotted functions are: $\omega = 3.29x10^{15}$, $m = 9.1x10^{-31}$, $\hbar = 1.0545718x10^{-34}$, and multiple values were used for the coupling constant k_3(t).

When using a coupling constant $k_3(t) = 0$, or no coupling as seen in Fig. (3.1), we observe what would be expected for a non-coupled simple harmonic oscillator. The Hamiltonian for this situation simplifies to $\hat{H}(t) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)$ therefore, the functions associated with any of the coupling operators are zero while the functions associated with 1, $\hat{a}^\dagger\hat{a}$, and $\hat{b}^\dagger\hat{b}$ have a constant slope as seen in Figs. (3.2) – (3.3), representing the energy. This matches our expectations for two uncoupled SHO's with a lack of dissipation and unitary evolution. The evolution will be explained in the next section.

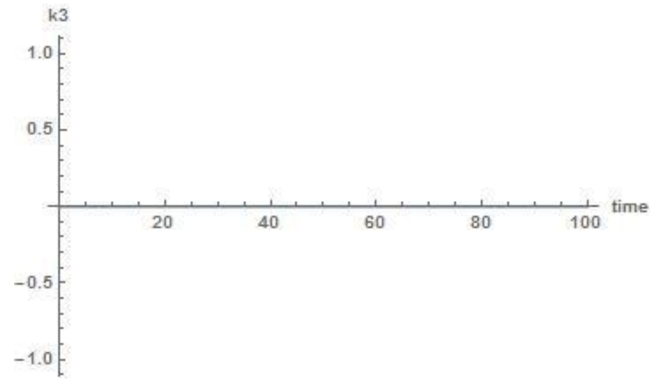


Figure 3.1 Graph of $k_3(t)=0$ associated with no coupling

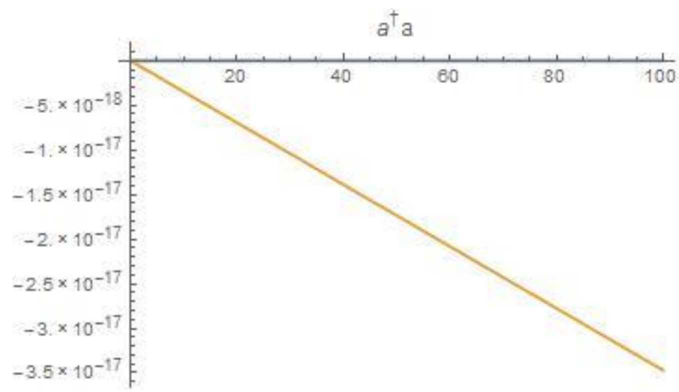


Figure 3.2 Graph of the function associated with the operator $a^\dagger a$. The graphs associated with operators 1 and $b^\dagger b$ are identical

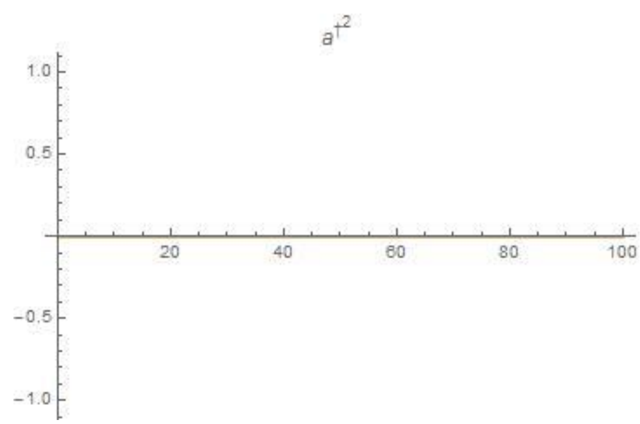


Figure 3.3 Graph associated with the operator $(a^\dagger)^2$. The graphs associated with operators ab, a^2, b^2, $(b^\dagger)^2$, $a^\dagger b^\dagger$, $ab^\dagger$, and $a^\dagger b$ are identical

When using a permanent coupling, represented as a constant $k_3(t) = 5$, Fig. (3.4), we observe two different functions associated with the 11 operators. For the 4 operators associated with a combination of the two oscillators, ab, $a^\dagger b$, $ab^\dagger$, and $a^\dagger b^\dagger$ there is a positively sloped function as seen in Fig. (3.5). While, for the 7 operators associated with the individual oscillators, $(b^\dagger)^2$, $(a^\dagger)^2$, $a^\dagger a$, $b^\dagger b$, b^2, a^2, and 1, there is a negatively sloped function as seen in Fig. (3.6). Again, these functions and the resulting evolution will be described in the next chapter.

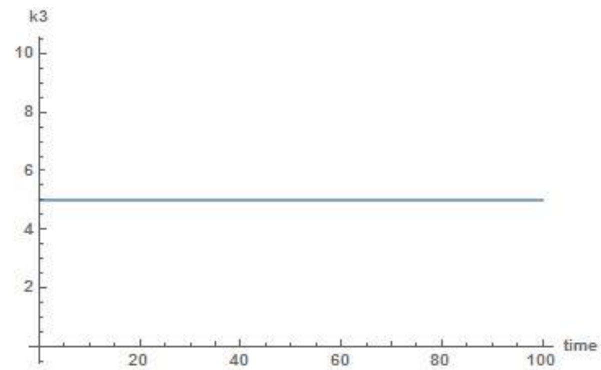


Figure 3.4 The graph of the coupling function $k_3(t)$ with a permanent coupling.

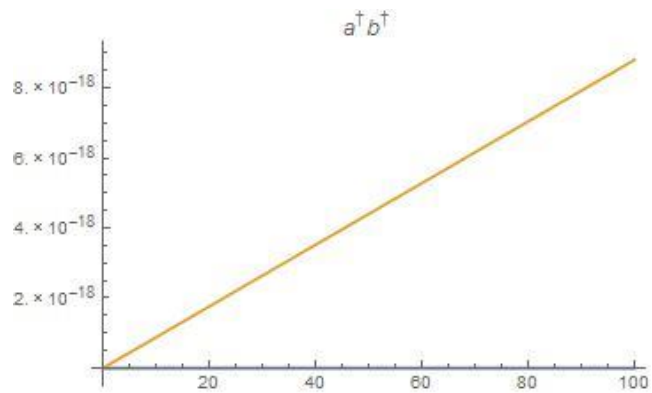


Figure 3.5 Graph of the function associated with the operator $a^\dagger b^\dagger$. The functions associated with the operators ab, $a^\dagger b$, and $ab^\dagger$ are identical.

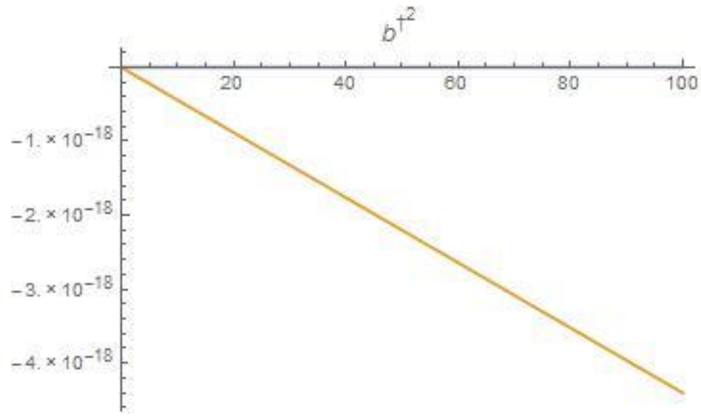


Figure 3.6 Graph of the function associated with the operator $(b^\dagger)^2$. The functions associated with the operators $(a^\dagger)^2$, $a^\dagger a$, $b^\dagger b$, b^2, a^2, and 1 are identical.

Next, we chose a coupling that represents a disappearance of the coupling over time. We used a Gaussian distribution centered at $t = 0$ described by the function, $k_3(t) = \frac{2.5}{\sqrt{2\pi}} e^{-\frac{(t)^2}{20}}$ and seen in Fig. (3.7). When using this disappearing coupling, we observe a peak in the functions associated with the coupling when there is an inflection point in the coupling function itself, as seen in Figs. (3.8) – (3.9). Similar to the previous two situations a negatively sloped function is associated with the non-coupled operators $a^\dagger a$, $b^\dagger b$ and 1, as seen in Fig. (3.10).

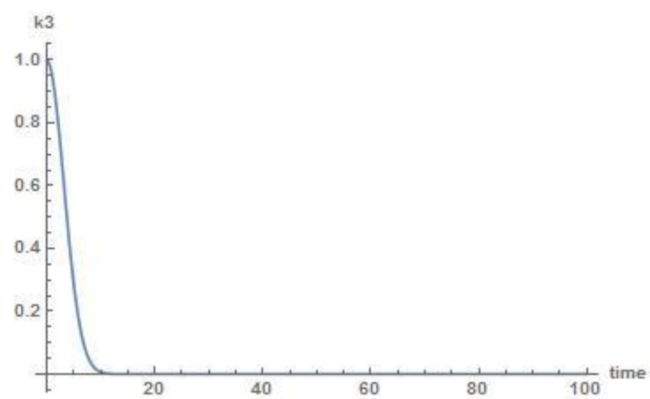


Figure 3.7 Coupling function $k_3(t)$ for a situation where the coupling begins strongest then gradually decreases to zero.

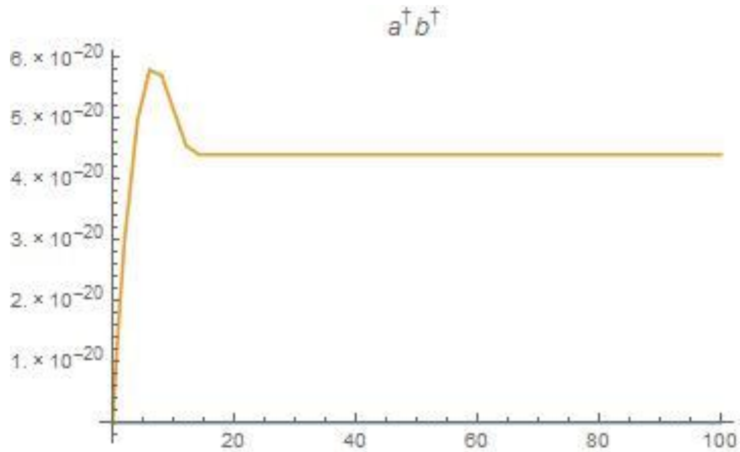


Figure 3.8 Graph of the function associated with the operator $a^\dagger b^\dagger$. Graphs associated with the operators $ab^\dagger$, $a^\dagger b$, and ab are identical.

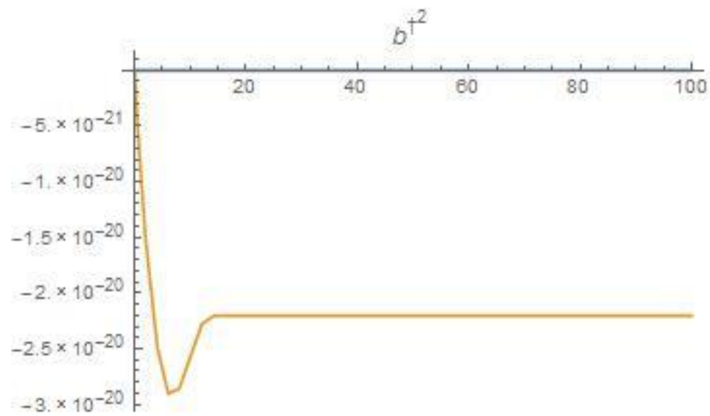


Figure 3.9 Graph of the function associated with the operator $(b^\dagger)^2$. Functions associated with the operators $(a^\dagger)^2$, b^2, and a^2 are identical.

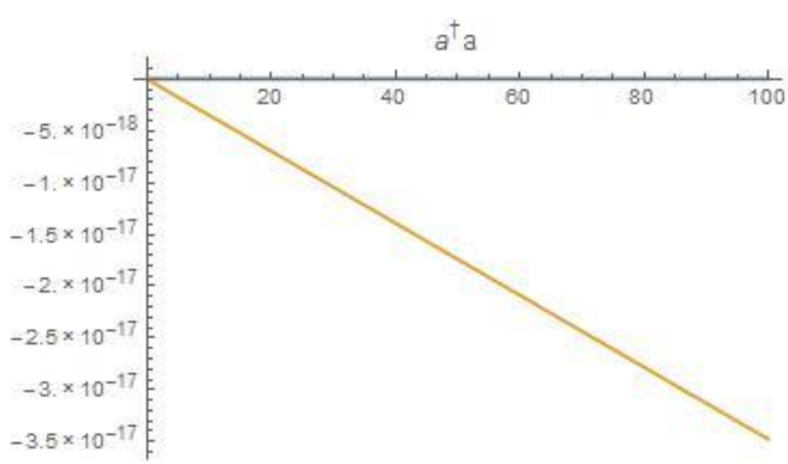


Figure 3.10 Graph of the function associated with the operator $a^\dagger a$. Graphs of the functions associated with the operators 1 and $b^\dagger b$ are identical.

Finally, we chose a coupling that represents beginning with no coupling, introducing coupling, then removing it to see if we observe similar evolution compared to the previous results from uncoupled oscillators and a disappearing coupling. We represented this coupling with a Gaussian distribution centered at $t = 50$ using the function $k_3(t) = \frac{2.5}{\sqrt{2\pi}} e^{-\frac{(t-50)^2}{20}}$, and seen in Fig. (3.11). When comparing with previous results we do see many similarities as with the peaks associated with a decrease in the coupling, Figs. (3.12) and (3.13), and an identical function associated with the non-coupled operators, $a^{\dagger}a$, $b^{\dagger}b$ and 1 as seen in Fig. (3.14) and observed in every coupling previous Figs. (3.10), (3.6), and (3.2).

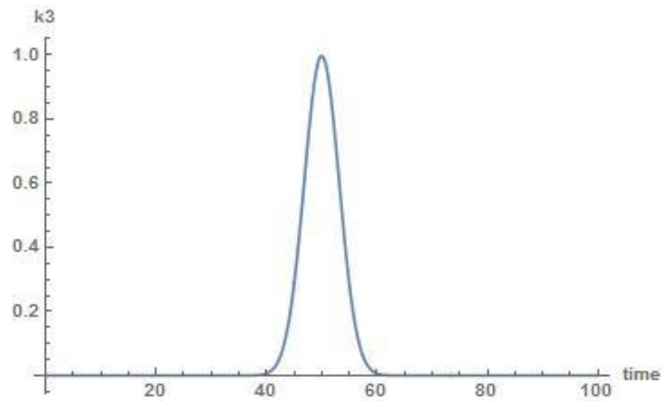


Figure 3.11 Graph of the coupling function $k_3(t)$ where the coupling begins zero, increases to a maximum, then decreases back to zero according to a Gaussian distribution.

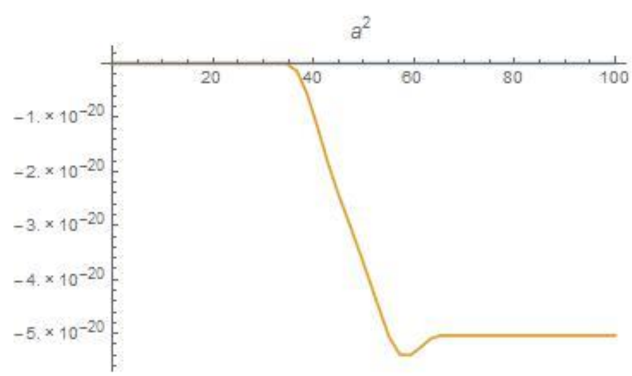


Figure 3.12 This figure represents the imaginary part of the function associated with the operator a^2 within the time evolution operator when the coupling is a Gaussian distribution centered at 50. The functions associated with $(a^{\dagger})^2$, b^2, and $(b^{\dagger})^2$ are identical.

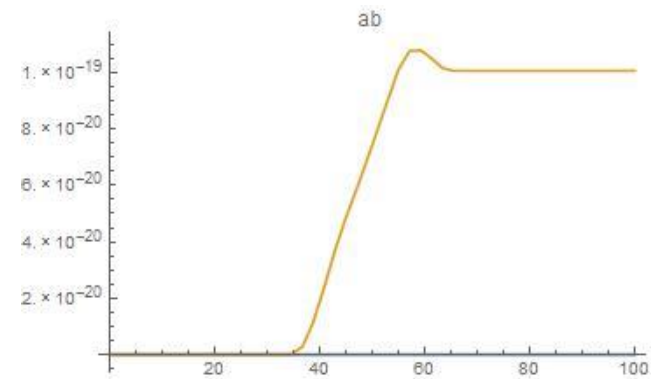


Figure 3.13 This figure represents the imaginary part of the function associated with the operator ab within the time evolution operator when the coupling is a Gaussian distribution centered at 50. The functions associated with $a^{\dagger}b^{\dagger}$, $a^{\dagger}b$, and $ab^{\dagger}$ are identical.

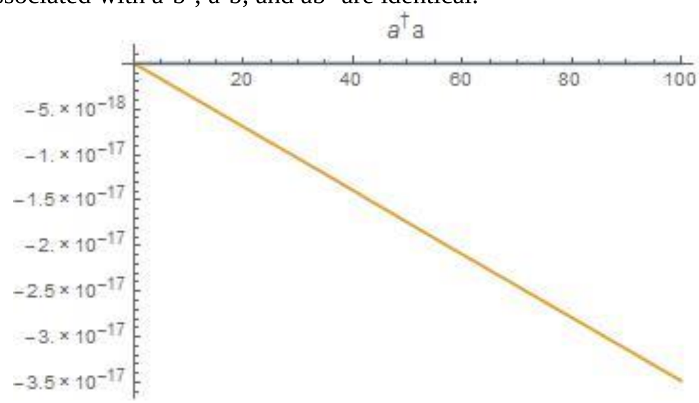


Figure 3.14 This figure represents the imaginary part of the function associated with the operator $a^{\dagger}a$ within the time evolution operator when the coupling is a Gaussian distribution centered at 50. The functions associated with $b^{\dagger}b$ and 1 are identical.

Again, the complete time evolution operator is composed of a product of exponentials raised to the power of the plotted function multiplied by its associated operator, as seen in Eq. (2.4).

3.2 Dynamics of single oscillator

To observe the dynamics of the coupled oscillator, we partially trace over the coherent state (the environment in which we are not interested) to observe the changes to the ground state (the system). To see what happens under different coupling situations I chose various values of $k_3(t)$ as explained above and plotted in Figs. (3.1), (3.4), (3.7), and (3.11). The following are images of the wave function of a ground state coupled with an arbitrary coherent state with the coherent state traced over. The results show any changes that occur in the ground state as a result of being coupled with a coherent state. I could not obtain valid results when leaving the equations in analytic form so I input values for the time and took multiple times to better understand the results.

All the different iterations of coupling, at all times, results in the same plot for the real part of the expression as seen in Fig. (3.15).

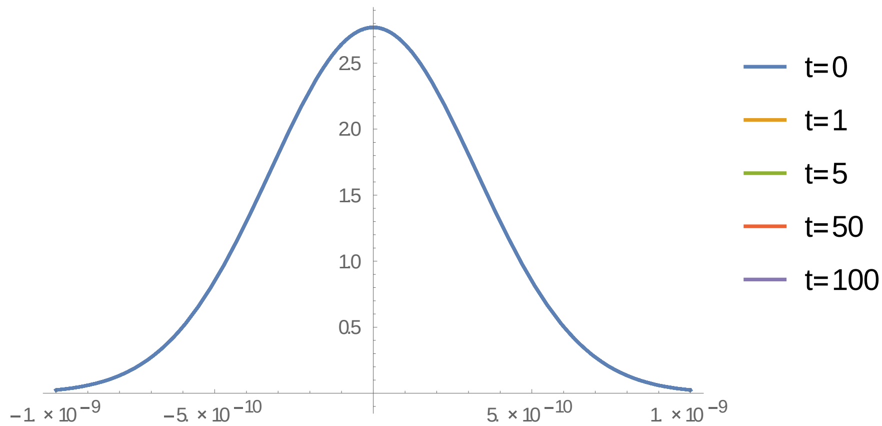


Figure 3.15 This figure is a plot of the real part of the wave function of the ground state after being coupled with an arbitrary coherent state, then having the coherent state traced over. The figure did not change when the coupling was changed. Every time produced the exact same plot so every time is represented in the same line.

Different coupling does result in slight changes to the imaginary part of the wave function. Figs. (3.16) – (3.19) represent the resulting wave function due to the four different values of $k_3(t)$ described in the previous section. Fig. (3.16) represents the wave function when no coupling is present. The different curves represent different times $t = 0, t = 1, t = 5, t = 50, t = 100$.

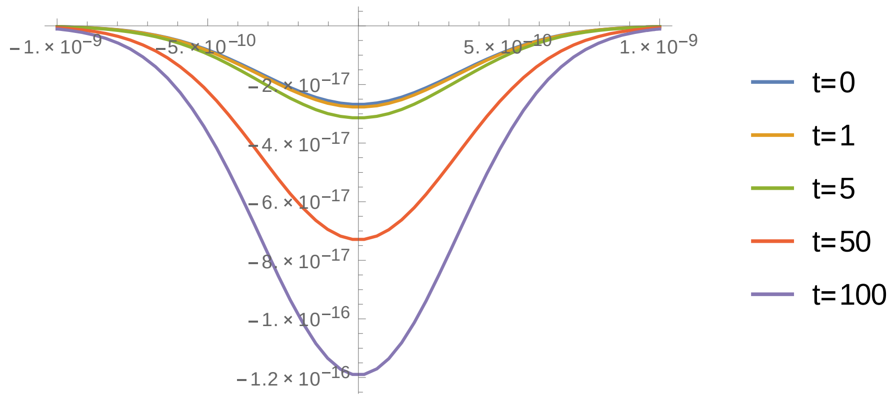


Figure 3.16 The evolution of the ground state with no coupling to the coherent state. The imaginary part of the wave function is plotted against x_1, the real part can be seen in Fig. (3.15).

Fig. (3.17) represents the imaginary part of the wave function, at times $t = 0, t = 1, t = 5, t = 50, t = 100$, of a permanent coupling between the ground and arbitrary coherent state.

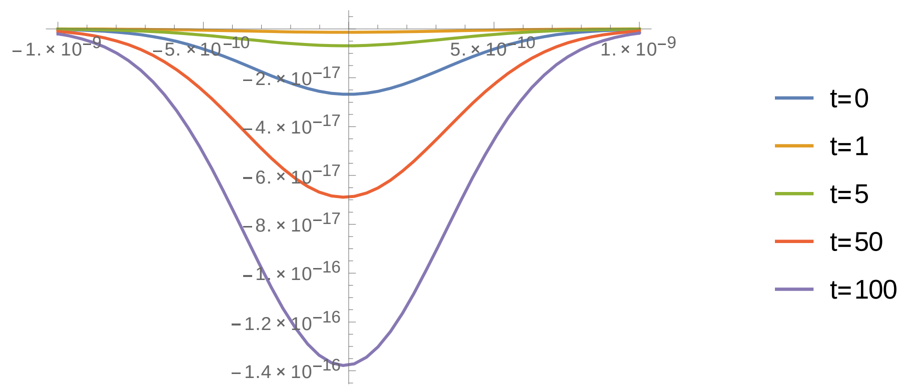


Figure 3.17 The imaginary part of the wave function of a ground state permanently coupled to an arbitrary coherent state plotted against x_1 at five different times. The real part of the wave function can be seen in Fig. (3.15).

Fig. (3.18) represents the wave function at times $t = 0, t = 1, t = 5, t = 50, t = 100$ of a ground state and coherent state with a Gaussian distribution as the coupling. The distribution is centered at zero (meaning the system begins with maximum coupling) then the coupling disappears over time.

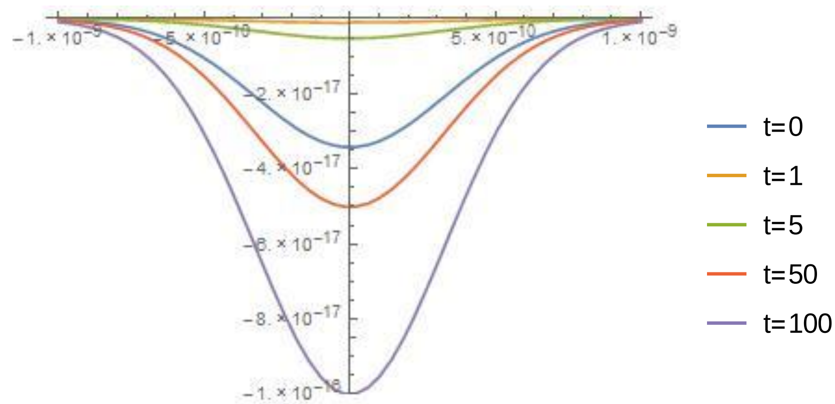


Figure 3.18 The imaginary part of the wave function of a ground state coupled to an arbitrary coherent state plotted versus position x_1. The coupling gradually decreases after t=0 to disappear by t=15. The real part of the wave function can be seen in Fig. (3.15).

Finally, Fig. (3.19) demonstrates the evolution of the ground state as it begins uncoupled, slowly becomes coupled to a maximum coupling at $t = 50$, then slowly becomes uncoupled according to a Gaussian distribution. Again, the different lines represent different times $t = 0, t = 1, t = 5, t = 50, t = 100$.

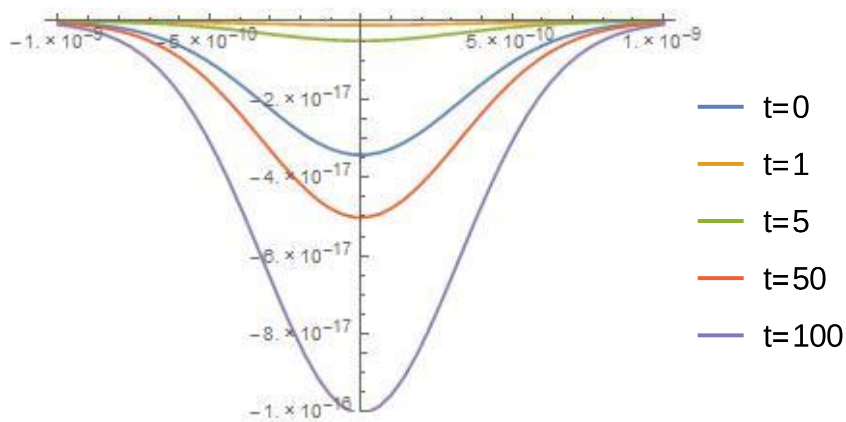


Figure 3.19 The imaginary part of the wave function of a ground state as it begins uncoupled, becomes coupled to an arbitrary coherent state with maximum coupling at t=50, then the coupling disappears to be gone by t=100. The wave function is plotted against position x_1 on the horizontal axis. The real part of the function can be seen in Fig. (3.15).

This figures will be analyzed in the following chapter.

Chapter 4

Conclusion and Outlook

4.1 Conclusion

From the results described in Chapter 3 it is shown that we have been able to solve for the dynamics of the coupled quantum harmonic oscillator with a time-dependent coupling term, when the two masses are constant and equal and the outer spring constants are time-independent, constant and equal. To compare our results against previous results recorded by Ref. [5] we applied our time evolution operator to a tensor product of a ground state and coherent state. By observing the evolution of the ground state, which is the most "classical" quantum state, we can observe if it remains coherent after being coupled with the coherent state. By analyzing both the functions within the time evolution operator and the final dynamics seen in Figs. (3.16) – (3.19) we can come to the following conclusions.

First, by analyzing the time evolution operator we see that different coupling does yield different time evolution operators. This result was expected and matches our hypothesis and previous research. It is interesting to note that identical functions within the time evolution operator can be grouped depending on the coupling. With no coupling, we have two separate groups of operators and associated functions: coupling operators, those found in the Hamiltonian that multiply the coupling function $k_3(t)$, and non-coupling operators, those found in the Hamiltonian that do not multiply the coupling function $k_3(t)$ and can be found in the equation for a simple harmonic oscillator without any coupling. This split in the 11 operators into two separate groups demonstrates the symmetry of the equation.

When looking at the time evolution operator with coupling we see a similar situation of identical functions associated with different groups of operators. With constant coupling, the 11 operators are split into two separate groups. The first group includes operators associated with one oscillator, a^2, b^2, $a^\dagger a$, $b^\dagger b$, $(a^\dagger)^2$, $(b^\dagger)^2$, and 1. While the second group includes operators associated with a combination of the two oscillators, ab, $a^\dagger b$, $ab^\dagger$ and $a^\dagger b^\dagger$. In both situations of Gaussian coupling identical functions are seen associated with the non-coupling operators ($a^\dagger a$, $b^\dagger b$, and 1), the coupling operators associated with one oscillator (a^2, b^2 $(a^\dagger)^2$, $(b^\dagger)^2$), and the coupling operators associated with a combination of two oscillators (ab, $a^\dagger b$, $ab^\dagger$ and $a^\dagger b^\dagger$). This result is expected. The associated functions are found by solving a series of differential equations and it is expected that the functions associated with each separate oscillator, "a" and "b," should be identical. Thus, identical functions should be seen associated with the non-coupling operators 1, $a^\dagger a$ and $b^\dagger b$. Further, each equation associated with the coupling operators, should result in similar results without consideration to which oscillator is being operated on. That is to say, the functions associated with an "a" operator should be identical to those associated with the same "b" operator as seen in the fact that each function associated with the operator a^2 is identical to each function associated with the operator b^2. Again, this result is expected and shows that the coupling acts as would be expected in a classical sense.

We also see what we can term a memory effect. In situations where the coupling does not exist we consistently see a negative sloped function associated with operators $a^\dagger a$, $b^\dagger b$, and 1. We also observe in functions associated with all the other operators that when coupling does not exist the function is constant, when coupling is zero the function is zero, but when coupling returns to zero after being transient we see that the function remains constant but at a new value. This ability for the function to “remember” that there was once coupling even though it now does not exist is what we mean by having a memory effect.

Second, we see through analyzing the final time-dependent wave functions that changes in coupling results in minimal changes in the imaginary part. When there is no coupling, the wave function grows from a minimum at $t = 0$ to a maximum at $t = 100$. With a constant coupling the wave function begins at a value equal to that of no coupling, drops to near zero then grows to a maximum at $t = 100$. The maximum with constant coupling is greater than the maximum in the no coupling situation. Fig. (3.18) associated with the Gaussian coupling centered at zero, shows a wave function that begins at the same value as when no coupling is present, drops near zero, then grows to a value less than both no coupling and constant coupling. The wave function associated with a Gaussian coupling centered at $t = 50$ produces a very similar evolution compared to the function associated with Gaussian coupling centered at $t = 0$. The differences are so small compared to the real part of the wave function plotted in Fig. (3.15), a ratio of $1{:}10^{-16}$, that they seem insignificant. We conclude that changes in the coupling do not produce squeezing and our results do not confirm the results found in Ref. [5].

4.2 Outlook

Looking forward, we hope to be able to compare more closely our results with those found in Ref. [5] and find where the differences lie. We assume there are no coding errors, but further research can re-check my work, and compare more closely with Ref. [5]. To avoid numerical error, we also can work toward dimensionless quantities by redefining $\hbar=1$. Further research can build from these results and investigate the dynamics of coupled oscillators with unequal mass or unequal spring constants. The approach used here of utilizing the Mathematica integrate function does not seem strong enough to tackle such complex integration. Further research can also attempt application of the time evolution operator on different states to see if further information can be gleaned.

Appendix A

	1	$aa^\dagger$	$bb^\dagger$	ab	$ab^\dagger$	$ba^\dagger$	$a^\dagger b^\dagger$	a^2	b^2	$(a^\dagger)^2$	$(b^\dagger)^2$
1	0	0	0	0	0	0	0	0	0	0	0
$aa^\dagger$	0	0	0	$-ab$	$-ab^\dagger$	$ba^\dagger$	$a^\dagger b^\dagger$	$-2a^2$	0	$2(a^\dagger)^2$	0
$bb^\dagger$	0	0	0	$-ab$	$ab^\dagger$	$-ba^\dagger$	$a^\dagger b^\dagger$	0	$-2b^2$	0	$2(b^\dagger)^2$
ab	0	ab	ab	0	a^2	b^2	$1 + aa^\dagger + bb^\dagger$	0	0	$2ba^\dagger$	$2ab^\dagger$
$ab^\dagger$	0	$ab^\dagger$	$-ab^\dagger$	$-a^2$	0	$-aa^\dagger + bb^\dagger$	$(b^\dagger)^2$	0	$-2ab$	$2a^\dagger b^\dagger$	0
$ba^\dagger$	0	$-ba^\dagger$	$ba^\dagger$	$-b^2$	$aa^\dagger - bb^\dagger$	0	$(a^\dagger)^2$	$-2ab$	0	0	$2a^\dagger b^\dagger$
$a^\dagger b^\dagger$	0	$-a^\dagger b^\dagger$	$-a^\dagger b^\dagger$	$-1 - aa^\dagger - bb^\dagger$	$-(b^\dagger)^2$	$-(a^\dagger)^2$	0	$-2ab^\dagger$	$-2ba^\dagger$	0	0
a^2	0	$2a^2$	0	0	0	$2ab$	$2ab^\dagger$	0	0	$2(1 + 2aa^\dagger)$	0
b^2	0	0	$2b^2$	0	$2ab$	0	$2ba^\dagger$	0	0	0	$2(1 + 2bb^\dagger)$
$(a^\dagger)^2$	0	$-2(a^\dagger)^2$	0	$-2ba^\dagger$	$-2a^\dagger b^\dagger$	0	0	$-2(1 + 2aa^\dagger)$	0	0	0
$(b^\dagger)^2$	0	0	$-2(b^\dagger)^2$	$-2ab^\dagger$	0	$-2a^\dagger b^\dagger$	0	0	$-2(1 + 2bb^\dagger)$	0	0

Figure A.1 This table shows the commutation relations between all 11 operators and demonstrates that there are no new operators formed when performing the commutation, thus making the operators in this Hamiltonian form a Lie algebra basis. The operators are listed along the top and left of each row and column.

Appendix B

Included below are my Mathematica documents.

```
info = "starting, Attempting to open Definition NoteBook";
Dynamic["status : " <> info]
SetDirectory["E:"];
EvaluateNotebook[nbname_] :=
 Module[{nb}, nb = NotebookOpen[ToFileName[Directory[], nbname]];
  NotebookEvaluate[nb];
  NotebookClose[nb];];
OpenNotebook[nbname_] := NotebookOpen[ToFileName[Directory[], nbname]];
EvaluateNotebook["\Physics\Research\Coupled Oscillators Project -
   Copy\Mathematica Notebooks/SolveForAlphaEqAndAlgAndKetNoDef7_3universal.nb"];


Alg = M;

InitializeAlgebraMajor[{sho[1], sho[2]}];
id = idsho[1] ** idsho[2];
(*xpħ[i_]["BasisNames"]:={x[i],p[i],idxp[i]}; commutation of x and p is I*)
(*xp[i_]["BasisNames"]:={x[i],p[i],idxp[i]};
commutation of x and p is I ħ*)
(*sho[i_]["BasisNames"]:=
  {a[i]†,a[i],num[i],idsho[i]} commutation of a[i]† and a[i] is idsho;*)
(*aadagar[i_]["BasisNames"]:={a[i]†,a[i],idsho[i]};*)
(*pauli[i_]["BasisNames"]:={ sM[i],sP[i], sZ[i],idpauli[i]}*)

Alg["Alg"] = {a[2]† ** a[2]† ** id, a[1]† ** a[1]† ** id, a[1]† ** a[2]† ** id,
   a[1] ** a[2]† ** id, a[1]† ** a[2] ** id, a[1]† ** a[1] ** id, a[2]† ** a[2] ** id,
   a[2] ** a[2] ** id, a[1] ** a[1] ** id, a[1] ** a[2] ** id, idsho[1] ** idsho[2]};
MakeSubAlg[Alg, Alg["Alg"]];
Hamiltonian = id ** (ħ w (a[1]† ** a[1] + idsho[1] / 2) + ħ w (a[2]† ** a[2] + idsho[2] / 2) +
     (ħ k3) / (4 m w) (a[2] ** a[2] + a[2]† ** a[2]† + a[1] ** a[1] + a[1]† ** a[1]† +
        2 (a[2]† ** a[2] - a[2] ** a[1] - a[2] ** a[1]† - a[2]† ** a[1] -
           a[2]† ** a[1]† + a[1]† ** a[1] + idsho[1] ** idsho[2])));
 (*FullSimplify[id** (w (a[1]†**a[1]+idsho[1]/2)+w(a[2]†**a[2]+idsho[2]/2)+
    k3/(4 m w) ((a[2]+a[2]†)**(a[2]+a[2]†)-
       2(a[1]+a[1]†)**(a[2]+a[2]†)+(a[1]+a[1]†)**(a[1]+a[1]†)))]*)
MAlphaEquations[Alg, Hamiltonian];

SubAlgEqInfoEct[Alg, Hamiltonian]

InitializeCoherentStates[]
```

status : <> info

(I*DUInvUsol-Hamiltonian) in terms of SubAlg :

Took too long..

The mapping between AlgMajor and SubAlg is 1-to-1: True

Commutation table of SubAlg M :

0	0	0	0	$-2 M_3$	0	$-2 M_1$	$-2 (2 M_7 + M_{11})$	0	
0	0	0	$-2 M_3$	0	$-2 M_2$	0	0	$-2 (2 M_6 + M_{11})$	
0	0	0	$-M_1$	$-M_2$	$-M_3$	$-M_3$	$-2 M_5$	$-2 M_4$	-1
0	$2 M_3$	M_1	0	$-M_6 + M_7$	M_4	$-M_4$	$-2 M_{10}$	0	
$2 M_3$	0	M_2	$M_6 - M_7$	0	$-M_5$	M_5	0	$-2 M_{10}$	
0	$2 M_2$	M_3	$-M_4$	M_5	0	0	0	$-2 M_9$	
$2 M_1$	0	M_3	M_4	$-M_5$	0	0	$-2 M_8$	0	
$2 (2 M_7 + M_{11})$	0	$2 M_5$	$2 M_{10}$	0	0	$2 M_8$	0	0	
0	$2 (2 M_6 + M_{11})$	$2 M_4$	0	$2 M_{10}$	$2 M_9$	0	0	0	
$2 M_4$	$2 M_5$	$M_6 + M_7 + M_{11}$	M_9	M_8	M_{10}	M_{10}	0	0	
0	0	0	0	0	0	0	0	0	

α equations :

$$\Big\{\alpha_{M_1}'[t] == -\frac{i \hbar \left(4\, k3\, \alpha_{M_1}[t]^2 + k3\, (-1 + \alpha_{M_3}[t])^2 + \alpha_{M_1}[t]\, (4\, k3 + 8\, m\, w^2 - 4\, k3\, \alpha_{M_3}[t])\right)}{4\, m\, w},$$

$$\alpha_{M_2}'[t] == -\frac{i \hbar \left(4\, k3\, \alpha_{M_2}[t]^2 + k3\, (-1 + \alpha_{M_3}[t])^2 + \alpha_{M_2}[t]\, (4\, k3 + 8\, m\, w^2 - 4\, k3\, \alpha_{M_3}[t])\right)}{4\, m\, w},$$

$$\alpha_{M_3}'[t] == \frac{1}{2\, m\, w}\, i \hbar \Big(k3 + 2\, k3\, \alpha_{M_1}[t]\, (1 + 2\, \alpha_{M_2}[t] - \alpha_{M_3}[t]) - 2\, k3\, \alpha_{M_2}[t]\, (-1 + \alpha_{M_3}[t]) - 2\, k3\, \alpha_{M_3}[t] - 4\, m\, w^2\, \alpha_{M_3}[t] + k3\, \alpha_{M_3}[t]^2\Big),$$

$$\alpha_{M_4}'[t] == -\frac{i\, k3\, \hbar\, (-1 + \alpha_{M_4}[t])\, (1 + 2\, \alpha_{M_1}[t] + \alpha_{M_6}[t] + 2\, \alpha_{M_2}[t]\, \alpha_{M_5}[t] - \alpha_{M_3}[t]\, (1 + \alpha_{M_4}[t]))}{2\, m\, w},$$

$$\alpha_{M_5}'[t] == \frac{1}{2\, m\, w}\, i\, k3\, \hbar\, (1 + 2\, \alpha_{M_1}[t]\, \alpha_{M_5}[t] + 2\, \alpha_{M_4}[t]\, \alpha_{M_5}[t] - \alpha_{M_3}[t]\, (1 + 2\, \alpha_{M_4}[t]\, \alpha_{M_5}[t]) + \alpha_{M_2}[t]\, (2 + (-2 + 4\, \alpha_{M_4}[t])\, \alpha_{M_5}[t])),$$

$$\alpha_{M_6}'[t] == -\frac{i \hbar\, (k3 + 2\, m\, w^2 - 2\, k3\, \alpha_{M_5}[t]\, (-1 + \alpha_{M_4}[t]) + k3\, \alpha_{M_3}[t]\, (-1 + \alpha_{M_4}[t]) - k3\, \alpha_{M_4}[t])}{2\, m\, w},$$

$$\alpha_{M_7}'[t] == -\frac{i \hbar\, (k3 + 2\, m\, w^2 + 2\, k3\, \alpha_{M_1}[t] + k3\, \alpha_{M_4}[t] + 2\, k3\, \alpha_{M_5}[t]\, \alpha_{M_4}[t] - k3\, \alpha_{M_3}[t]\, (1 + \alpha_{M_4}[t]))}{2\, m\, w},$$

$$\alpha_{M_8}'[t] == -\frac{i\, e^{2 \alpha_{M_7}[t]}\, k3\, \hbar\, (1 + (-1 + \alpha_{M_4}[t])\, \alpha_{M_5}[t])^2}{4\, m\, w},$$

$$\alpha_{M_9}'[t] == -\frac{i\, e^{2 \alpha_{M_6}[t]}\, k3\, \hbar\, (-1 + \alpha_{M_4}[t])^2}{4\, m\, w},$$

$$\alpha_{M_{10}}'[t] == -\frac{i\, e^{\alpha_{M_6}[t] + \alpha_{M_7}[t]}\, k3\, \hbar\, (-1 + \alpha_{M_4}[t])\, (1 + (-1 + \alpha_{M_4}[t])\, \alpha_{M_5}[t])}{2\, m\, w},$$

$$\alpha_{M_{11}}'[t] == -\frac{i \hbar\, (k3 + 2\, m\, w^2 + k3\, \alpha_{M_1}[t] + k3\, \alpha_{M_2}[t] - k3\, \alpha_{M_3}[t])}{2\, m\, w},$$

$$\alpha_{M_1}[0] == 0,\ \alpha_{M_2}[0] == 0,\ \alpha_{M_3}[0] == 0,\ \alpha_{M_4}[0] == 0,\ \alpha_{M_5}[0] == 0,\ \alpha_{M_6}[0] == 0,$$

$$\alpha_{M_7}[0] == 0,\ \alpha_{M_8}[0] == 0,\ \alpha_{M_9}[0] == 0,\ \alpha_{M_{10}}[0] == 0,\ \alpha_{M_{11}}[0] == 0\Big\}$$

```
Alg["equationsbaseSimple"] =
  Flatten[{FullSimplify[(Solve[Drop[Simplify[Alg["equationsbase"]], -Length[Alg]],
        D[Alg["α"], t]] /. Rule → Equal)[[1]]],
    Drop[Simplify[Alg["equationsbase"]], Length[Alg]]}];
Alg["equationsbaseSimple s"] = Alg["equationsbaseSimple"] /.
  Flatten[{Alg["αtos"], (Alg["αtos"] /. t → 0)}]
```

$$\Big\{\alpha_{M_1}'[t] == -\frac{1}{4\,m\,w}\,i\,\hbar\,\left(4\,k3\,\alpha_{M_1}[t]^2 + k3\,(-1+\alpha_{M_3}[t])^2 + 4\,\alpha_{M_1}[t]\,\left(k3 + 2\,m\,w^2 - k3\,\alpha_{M_3}[t]\right)\right),$$

$$\alpha_{M_2}'[t] == -\frac{1}{4\,m\,w}\,i\,\hbar\,\left(4\,k3\,\alpha_{M_2}[t]^2 + k3\,(-1+\alpha_{M_3}[t])^2 + 4\,\alpha_{M_2}[t]\,\left(k3 + 2\,m\,w^2 - k3\,\alpha_{M_3}[t]\right)\right),$$

$$\alpha_{M_3}'[t] == \frac{1}{2\,m\,w}\,i\,\hbar\,\left(k3\,(1+2\,\alpha_{M_1}[t])\,(1+2\,\alpha_{M_2}[t]) - 2\,\left(k3 + 2\,m\,w^2 + k3\,(\alpha_{M_1}[t]+\alpha_{M_2}[t])\right)\,\alpha_{M_3}[t] + k3\,\alpha_{M_3}[t]^2\right),$$

$$\frac{1}{2\,m\,w}\,i\,k3\,\hbar\,(-1+\alpha_{M_4}[t])\,(1+2\,\alpha_{M_1}[t] + (1+2\,\alpha_{M_2}[t])\,\alpha_{M_4}[t] - \alpha_{M_3}[t]\,(1+\alpha_{M_4}[t])) + \alpha_{M_4}'[t] == 0,$$

$$\alpha_{M_5}'[t] == \frac{1}{2\,m\,w}\,i\,k3\,\hbar\,(1+2\,\alpha_{M_2}[t] - \alpha_{M_3}[t] + 2\,(\alpha_{M_1}[t] - \alpha_{M_2}[t] + (1+2\,\alpha_{M_2}[t] - \alpha_{M_3}[t])\,\alpha_{M_4}[t])\,\alpha_{M_5}[t]),$$

$$\alpha_{M_6}'[t] == -\frac{1}{2\,m\,w}\,i\,\hbar\,\left(k3 + 2\,m\,w^2 + 2\,k3\,\alpha_{M_2}[t] - k3\,\alpha_{M_3}[t] + k3\,(-1-2\,\alpha_{M_2}[t] + \alpha_{M_3}[t])\,\alpha_{M_4}[t]\right),$$

$$\alpha_{M_7}'[t] == -\frac{1}{2\,m\,w}\,i\,\hbar\,\left(k3 + 2\,m\,w^2 + 2\,k3\,\alpha_{M_1}[t] + k3\,(1+2\,\alpha_{M_2}[t])\,\alpha_{M_4}[t] - k3\,\alpha_{M_3}[t]\,(1+\alpha_{M_4}[t])\right),$$

$$\alpha_{M_8}'[t] == -\frac{i\,e^{2\,\alpha_{M_7}[t]}\,k3\,\hbar\,(1+(-1+\alpha_{M_4}[t])\,\alpha_{M_5}[t])^2}{4\,m\,w},$$

$$\alpha_{M_9}'[t] == -\frac{i\,e^{2\,\alpha_{M_6}[t]}\,k3\,\hbar\,(-1+\alpha_{M_4}[t])^2}{4\,m\,w},$$

$$\alpha_{M_{10}}'[t] == -\frac{1}{2\,m\,w}\,i\,e^{\alpha_{M_6}[t]+\alpha_{M_7}[t]}\,k3\,\hbar\,(-1+\alpha_{M_4}[t])\,(1+(-1+\alpha_{M_4}[t])\,\alpha_{M_5}[t]),$$

$$\alpha_{M_{11}}'[t] == -\frac{i\,\hbar\,\left(k3 + 2\,m\,w^2 + k3\,(\alpha_{M_1}[t] + \alpha_{M_2}[t] - \alpha_{M_3}[t])\right)}{2\,m\,w},\ \alpha_{M_1}[0] == 0,$$

$$\alpha_{M_2}[0] == 0,\ \alpha_{M_3}[0] == 0,\ \alpha_{M_4}[0] == 0,\ \alpha_{M_5}[0] == 0,\ \alpha_{M_6}[0] == 0,$$

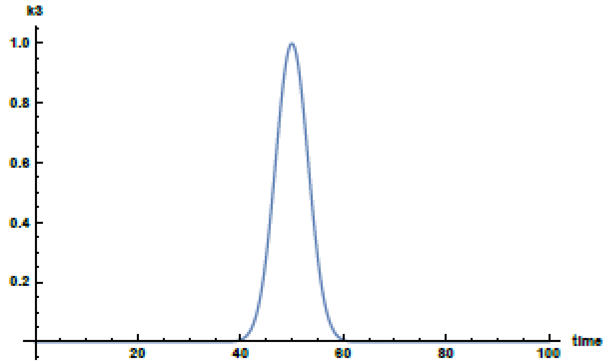
$$\alpha_{M_7}[0] == 0,\ \alpha_{M_8}[0] == 0,\ \alpha_{M_9}[0] == 0,\ \alpha_{M_{10}}[0] == 0,\ \alpha_{M_{11}}[0] == 0\Big\}$$

Attempt to solve analytically:

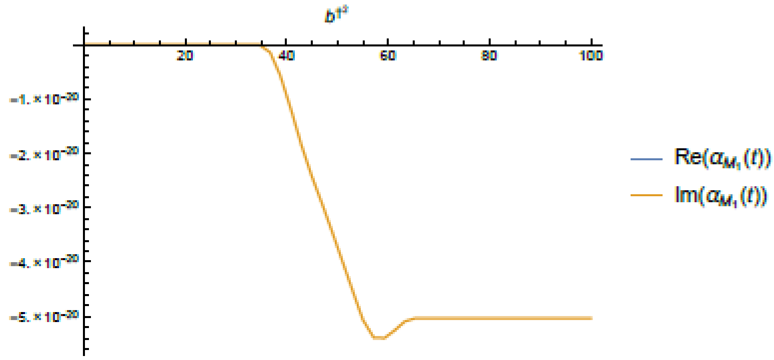
```
DSolve[{D[α_M1[t], t] == -1/(4 m w)
    i ħ (4 k3 α_M1[t]^2 + k3 (-1 + α_M3[t])^2 + 4 α_M1[t] (k3 + 2 m w^2 - k3 α_M3[t])), D[α_M2[t], t] ==
   -1/(4 m w) i ħ (4 k3 α_M2[t]^2 + k3 (-1 + α_M3[t])^2 + 4 α_M2[t] (k3 + 2 m w^2 - k3 α_M3[t])),
  D[α_M3[t], t] == 1/(2 m w) i ħ (k3 (1 + 2 α_M1[t]) (1 + 2 α_M2[t]) -
      2 (k3 + 2 m w^2 + k3 (α_M1[t] + α_M2[t])) α_M3[t] + k3 α_M3[t]^2), D[α_M4[t], t] ==
   -1/(2 m w) i k3 ħ (-1 + α_M4[t]) (1 + 2 α_M1[t] + (1 + 2 α_M2[t]) α_M4[t] - α_M3[t] (1 + α_M4[t])),
  D[α_M5[t], t] == 1/(2 m w) i k3 ħ (1 + 2 α_M2[t] - α_M3[t] +
      2 (α_M1[t] - α_M2[t] + (1 + 2 α_M2[t] - α_M3[t]) α_M4[t]) α_M5[t]), D[α_M6[t], t] ==
   -1/(2 m w) i ħ (k3 + 2 m w^2 + 2 k3 α_M2[t] - k3 α_M3[t] + k3 (-1 - 2 α_M2[t] + α_M3[t]) α_M4[t]),
  D[α_M7[t], t] == -1/(2 m w) i ħ
    (k3 + 2 m w^2 + 2 k3 α_M1[t] + k3 (1 + 2 α_M2[t]) α_M4[t] - k3 α_M3[t] (1 + α_M4[t])),
  D[α_M8[t], t] == -(i e^(2 α_M7[t]) k3 ħ (1 + (-1 + α_M4[t]) α_M5[t])^2)/(4 m w),
  D[α_M9[t], t] == -(i e^(2 α_M7[t]) k3 ħ (-1 + α_M4[t])^2)/(4 m w),
  D[α_M10[t], t] == -1/(2 m w) i e^(α_M4[t] + α_M7[t]) k3 ħ (-1 + α_M4[t]) (1 + (-1 + α_M4[t]) α_M5[t]),
  D[α_M11[t], t] == -(i ħ (k3 + 2 m w^2 + k3 (α_M1[t] + α_M2[t] - α_M3[t])))/(2 m w),
  α_M1[0] == 0, α_M2[0] == 0, α_M3[0] == 0, α_M4[0] == 0, α_M5[0] == 0, α_M6[0] == 0,
  α_M7[0] == 0, α_M8[0] == 0, α_M9[0] == 0, α_M10[0] == 0, α_M11[0] == 0},
 {α_M1[t], α_M2[t], α_M3[t], α_M4[t], α_M5[t], α_M6[t], α_M7[t],
  α_M8[t], α_M9[t], α_M10[t], α_M11[t]}, t];
```

Solve numerically with values chosen for k3, w, m and hbar defined

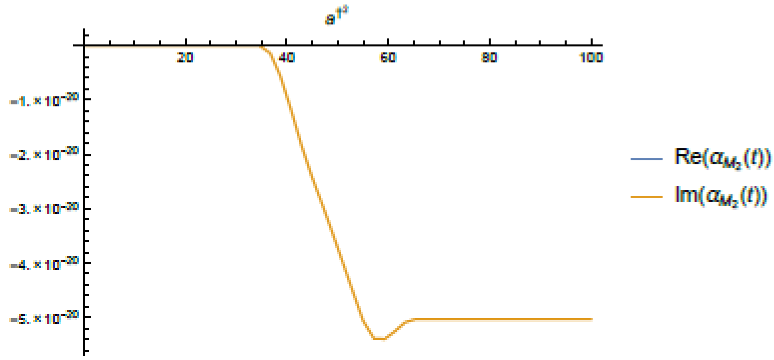
```
k3 = 2.5 / (Sqrt[2 Pi]) Exp[- (t - 50) ^2 / (2 * 10)]
(*Gaussian distribution with peak at 50*); w = 3.29 × 10^15; m = 9.1 × 10^-31;
ħ = 1.0545718 × 10^-34; (*2.5/(Sqrt[2 Pi])Exp[-( t-50)^2/(2 *10)]*)
Plot[k3, {t, 0, 100}, PlotRange → All,
 AxesLabel → {"time", "k3"}, LabelStyle → {Bold}]
sol1 = NDSolve[{D[αM1[t], t] ==
    -1/(4 m w) i ħ (4 k3 αM2[t]^2 + k3 (-1 + αM3[t])^2 + 4 αM1[t] (k3 + 2 m w^2 - k3 αM3[t])), D[αM2[t],
     t] == -1/(4 m w) i ħ (4 k3 αM1[t]^2 + k3 (-1 + αM3[t])^2 + 4 αM2[t] (k3 + 2 m w^2 - k3 αM3[t])),
   D[αM3[t], t] == 1/(2 m w) i ħ (k3 (1 + 2 αM1[t]) (1 + 2 αM2[t]) -
       2 (k3 + 2 m w^2 + k3 (αM1[t] + αM2[t])) αM3[t] + k3 αM3[t]^2), D[αM4[t], t] ==
    -1/(2 m w) i k3 ħ (-1 + αM4[t]) (1 + 2 αM1[t] + (1 + 2 αM2[t]) αM4[t] - αM3[t] (1 + αM4[t])),
   D[αM5[t], t] == 1/(2 m w) i k3 ħ (1 + 2 αM2[t] - αM3[t] +
       2 (αM1[t] - αM2[t] + (1 + 2 αM2[t] - αM3[t]) αM4[t]) αM5[t]), D[αM6[t], t] == -1/(2 m w) i ħ
     (k3 + 2 m w^2 + 2 k3 αM2[t] - k3 αM3[t] + k3 (-1 - 2 αM2[t] + αM3[t]) αM4[t]), D[αM7[t], t] ==
    -1/(2 m w) i ħ (k3 + 2 m w^2 + 2 k3 αM1[t] + k3 (1 + 2 αM2[t]) αM4[t] - k3 αM3[t] (1 + αM4[t])),
   D[αM8[t], t] == -(i e^(2 αM7[t]) k3 ħ (1 + (-1 + αM4[t]) αM5[t])^2)/(4 m w),
   D[αM9[t], t] == -(i e^(2 αM6[t]) k3 ħ (-1 + αM4[t])^2)/(4 m w),
   D[αM10[t], t] == -1/(2 m w) i e^(αM6[t] + αM7[t]) k3 ħ (-1 + αM4[t]) (1 + (-1 + αM4[t]) αM5[t]),
   D[αM11[t], t] == -(i ħ (k3 + 2 m w^2 + k3 (αM1[t] + αM2[t] - αM3[t])))/(2 m w),
   αM1[0] == 0, αM2[0] == 0, αM3[0] == 0, αM4[0] == 0, αM5[0] == 0, αM6[0] == 0,
   αM7[0] == 0, αM8[0] == 0, αM9[0] == 0, αM10[0] == 0, αM11[0] == 0},
  {αM1[t], αM2[t], αM3[t], αM4[t], αM5[t], αM6[t], αM7[t], αM8[t],
   αM9[t], αM10[t], αM11[t]}, {t, 0, 100}, PrecisionGoal → 2000];
```



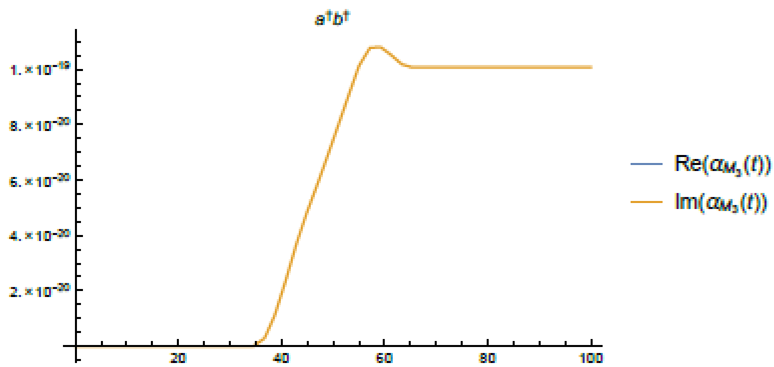

```
αH1[t_] = αH1[t] /. sol1[[1]];
αH2[t_] = αH2[t] /. sol1[[1]]; αH3[t_] = αH3[t] /. sol1[[1]];
αH4[t_] = αH4[t] /. sol1[[1]]; αH5[t_] = αH5[t] /. sol1[[1]];
αH6[t_] = αH6[t] /. sol1[[1]]; αH7[t_] = αH7[t] /. sol1[[1]];
αH8[t_] = αH8[t] /. sol1[[1]]; αH9[t_] = αH9[t] /. sol1[[1]];
αH10[t_] = αH10[t] /. sol1[[1]]; αH11[t_] = αH11[t] /. sol1[[1]];
Plot[{Re[αH1[t]], Im[αH1[t]]}, {t, 0, 100},
 PlotLabel → "b†^2", PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αH2[t]], Im[αH2[t]]}, {t, 0, 100}, PlotLabel → "a†^2",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αH3[t]], Im[αH3[t]]}, {t, 0, 100}, PlotLabel → "a†b†",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αH4[t]], Im[αH4[t]]}, {t, 0, 100}, PlotLabel → "ab†",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αH5[t]], Im[αH5[t]]}, {t, 0, 100},
 PlotLabel → "a†b", PlotLegends → "Expressions"]
Plot[{Re[αH6[t]], Im[αH6[t]]}, {t, 0, 100}, PlotLabel → "a†a",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αH7[t]], Im[αH7[t]]}, {t, 0, 100},
 PlotLabel → "b†b", PlotLegends → "Expressions"]
Plot[{Re[αH8[t]], Im[αH8[t]]}, {t, 0, 100}, PlotLabel → "b^2",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αH9[t]], Im[αH9[t]]}, {t, 0, 100}, PlotLabel → "a^2",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αH10[t]], Im[αH10[t]]}, {t, 0, 100}, PlotLabel → "ab",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αH11[t]], Im[αH11[t]]}, {t, 0, 100},
 PlotLabel → "1", PlotLegends → "Expressions"]
```



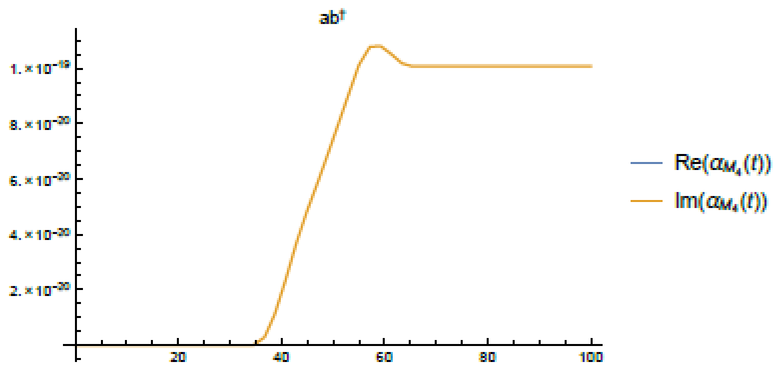
b†²
20
40
60
80
100
-1. × 10⁻²⁰
-2. × 10⁻²⁰
-3. × 10⁻²⁰
-4. × 10⁻²⁰
-5. × 10⁻²⁰
Re(α_{M₁}(t))
Im(α_{M₁}(t))

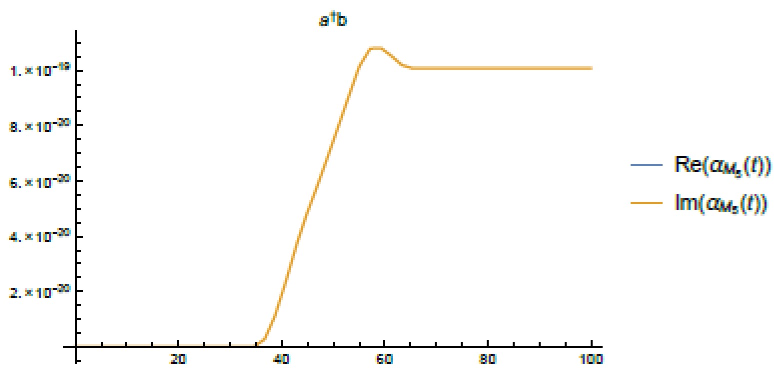


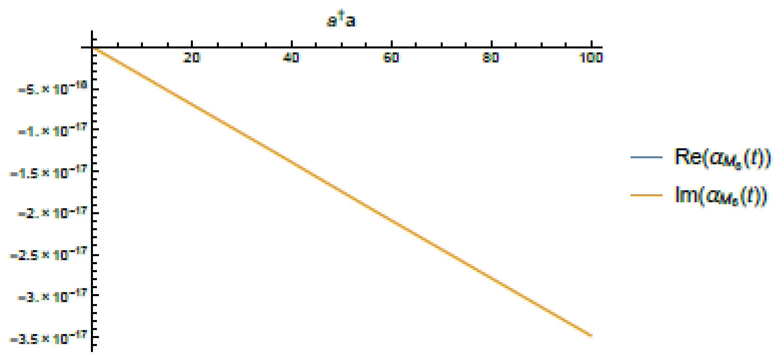
a†²
20
40
60
80
100
-1. × 10⁻²⁰
-2. × 10⁻²⁰
-3. × 10⁻²⁰
-4. × 10⁻²⁰
-5. × 10⁻²⁰
Re(α_{M₂}(t))
Im(α_{M₂}(t))

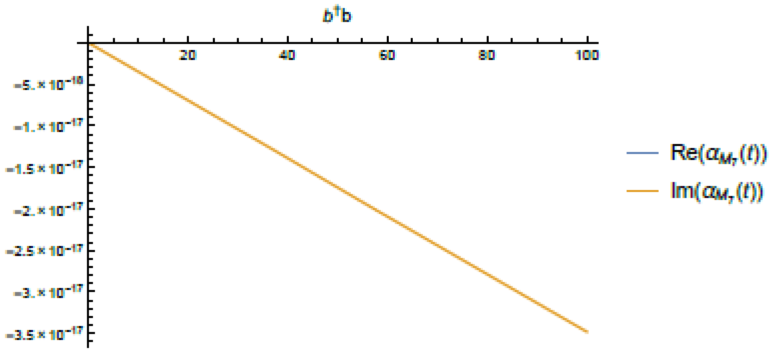


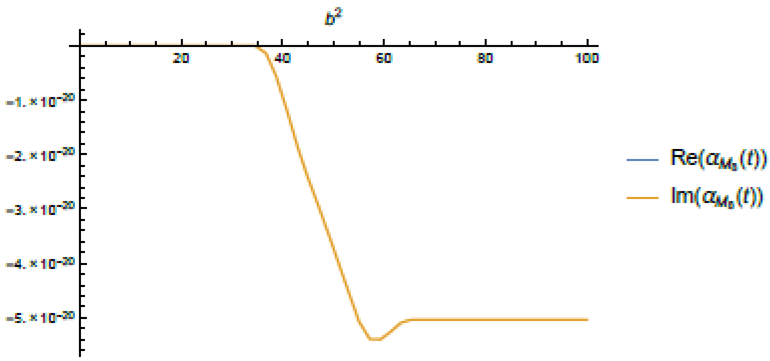
a†b†
1. × 10⁻¹⁹
8. × 10⁻²⁰
6. × 10⁻²⁰
4. × 10⁻²⁰
2. × 10⁻²⁰
20
40
60
80
100
Re(α_{M₃}(t))
Im(α_{M₃}(t))

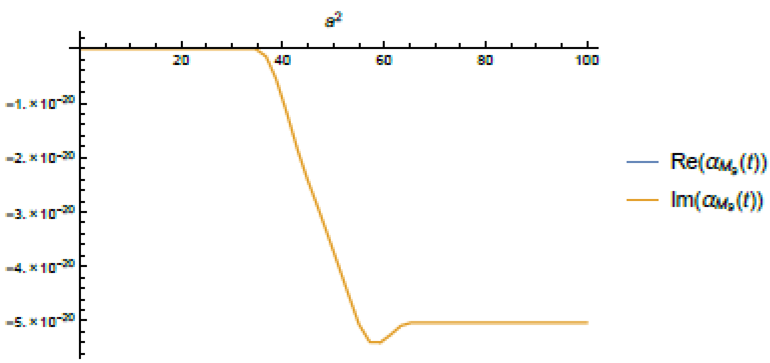

ab†
1. × 10^-19
8. × 10^-20
6. × 10^-20
4. × 10^-20
2. × 10^-20
20
40
60
80
100
Re(α_{M_4}(t))
Im(α_{M_4}(t))

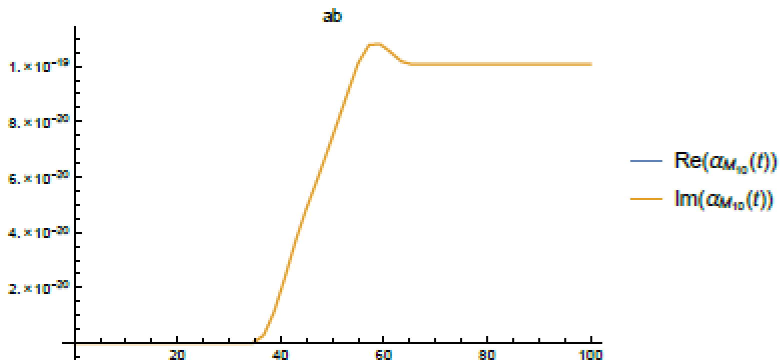

a†b
1. × 10^-19
8. × 10^-20
6. × 10^-20
4. × 10^-20
2. × 10^-20
20
40
60
80
100
Re(α_{M_5}(t))
Im(α_{M_5}(t))

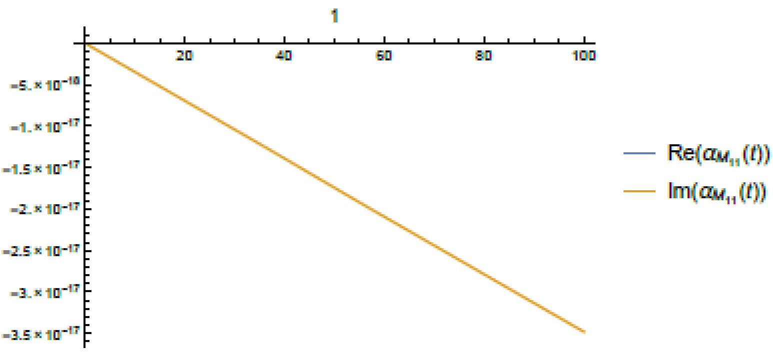

a†a
20
40
60
80
100
-5. × 10^-18
-1. × 10^-17
-1.5 × 10^-17
-2. × 10^-17
-2.5 × 10^-17
-3. × 10^-17
-3.5 × 10^-17
Re(α_{M_6}(t))
Im(α_{M_6}(t))


$b^{\dagger}b$
20
40
60
80
100
−5. × 10^−18
−1. × 10^−17
−1.5 × 10^−17
−2. × 10^−17
−2.5 × 10^−17
−3. × 10^−17
−3.5 × 10^−17
$\mathrm{Re}(\alpha_{M_7}(t))$
$\mathrm{Im}(\alpha_{M_7}(t))$


b^2
20
40
60
80
100
−1. × 10^−20
−2. × 10^−20
−3. × 10^−20
−4. × 10^−20
−5. × 10^−20
$\mathrm{Re}(\alpha_{M_8}(t))$
$\mathrm{Im}(\alpha_{M_8}(t))$


a^2
20
40
60
80
100
−1. × 10^−20
−2. × 10^−20
−3. × 10^−20
−4. × 10^−20
−5. × 10^−20
$\mathrm{Re}(\alpha_{M_9}(t))$
$\mathrm{Im}(\alpha_{M_9}(t))$






```
(* From here it is easy to form our time evolution operator U and by hand
 I have worked out how the operator applies to coherent states which
 become a constant through all the whole process and is given by: *)

uapp[t_] = Exp[αM11[t]] Exp[αM10[t] α β] Exp[αM9[t] α^2]
   Exp[αM8[t] β^2] Exp[-(1/2) β Conjugate[β]] Exp[-(1/2) α Conjugate[α]]
   Exp[(1/2) Exp[αM7[t] β] Conjugate[Exp[αM7[t] β] ]]
   Exp[(1/2) Exp[αM6[t] α] Conjugate [Exp[αM6[t] α]] ]
   (Exp[αM4[t] Exp[αM6[t]] α Conjugate[δ]] Exp[αM5[t] Conjugate[γ] Exp[αM7[t]] β] +
     Exp[αM4[t] Conjugate[δ] ] Exp[αM5[t] Exp[αM7[t]] β])
   Exp[αM3[t] Conjugate[γ] Conjugate[δ]] Exp[αM2[t] Conjugate[γ]^2]
   Exp[αM1[t] Conjugate[δ]^2];

(*We will also need the coherent state which can be written as:
   α=0;
β=X0 Sqrt[m w/(2 ℏ)]; *)

(* So our complete U with the ground state and an
 arbitrary coherent state coupled together can be written as: *)

u[t_] = uapp[t] /. {α → 0, β → X0 Sqrt[m w/(2 ℏ)]};
```

```
(* we now know how to multiply this state by the identity 1 and after that
 we can integrate over x2 (the environment) and see what we can get. *)

ψγ[x1_] = (m w / (Pi ℏ)) ^ (1 / 4) Exp[-γ Conjugate[γ] / 2]
   Exp[(x1 Sqrt[m w / ℏ]) ^2 / 2] Exp[-(x1 Sqrt[m w / ℏ] - γ / Sqrt[2]) ^2];
ψδ[x2_] = (m w / (Pi ℏ)) ^ (1 / 4) Exp[-δ Conjugate[δ] / 2]
   Exp[(x2 Sqrt[m w / ℏ]) ^2 / 2] Exp[-(x2 Sqrt[m w / ℏ] - δ / Sqrt[2]) ^2];

(*since coherent states are not orthogonal we
 also have to multiply by their orthogonality condition *)

γα[t_] = Exp[1 / 2 (Conjugate[γ] Exp[αM1[t]] α - γ Conjugate[Exp[αM1[t]] α])]
    Exp[-1 / 2 (γ - Exp[αM1[t]] α) Conjugate[(γ - Exp[αM1[t]] α)]] /. {α → 0};
δβ[t_] = Exp[1 / 2 (Conjugate[δ] Exp[αM2[t]] β - δ Conjugate[Exp[αM2[t]] β])]
    Exp[-1 / 2 (δ - Exp[αM2[t]] β) Conjugate[(δ - Exp[αM2[t]] β)]] /.
   {β → X0 Sqrt[m w / (2 ℏ)]};

(*putting it all together we also carry
 a 1/Pi^2 from the completness relation with us *)

total1[t_, x1_, x2_] = ψγ[x1] ψδ[x2] u[t] γα[t] δβ[t] 1 / Pi^2;
(*Now we integrate over gamma and
 delta to get rid of them from our expression *)

X0 = 5.29 × 10^-11;
Integrate[total1[t, x1, x2], {γ, -Infinity, Infinity}, {δ, -Infinity, Infinity}]
```

3.04583 × 10^8

0.199306 Conjugate[InterpolatingFunction[Domain: {{0.,100.}} Output: scalar][t]] + 0.199306 InterpolatingFunction[

0.480138 + ½ e

∫_{-∞}^{∞} ∫_{-∞}^{∞} e

InterpolatingFunction[Domain: {{0.,100.}} Output: scalar][t]

0.199306 e Conjugate[γ] InterpolatingFunction[

e

+

InterpolatingFunction[Domain: {{0.,100.}} Output: scalar][t]

0.199306 e InterpolatingFunction[Domain: {{0. Output: scal

e

dδ

dγ

```
totalg[t_, x1_, x2_] = InfiniteComplexIntegral[total1[t, x1, x2], γ];
Error: γ Conjugate[γ]
Error: γ Conjugate[γ]
totalgd[t_, x1_, x2_] = InfiniteComplexIntegral[totalg[t, x1, x2], δ];
(*Now we have integrated over gamma and delta and need to partial trace over the
 x2 variable so we can observe what happened to x1 after interacting with x2*)
```



```
Error: δ Conjugate[δ]
Error: δ Conjugate[δ]
totalgdx2[t_, x1_] = InfiniteComplexIntegral[totalgd[90, x1, x2], x2]
```

Power::infy : Infinite expression $\frac{1}{0. + 0. i}$ encountered. ≫

Infinity::indet : Indeterminate expression (0. + 0. i) ComplexInfinity encountered. ≫

Power::infy : Infinite expression $\frac{1}{0. + 0. i}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0. + 0. i}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

Infinity::indet : Indeterminate expression (0. + 0. i) ComplexInfinity encountered. ≫

```
Indeterminate
```

```
totalgdx2b[t_, x1_] =
 Integrate[totalgd[t, x1, x2] /. {t → {0, 1, 2, 5, 10, 20, 30, 40, 50, 60, 70}},
  {x2, -Infinity, Infinity}]
```

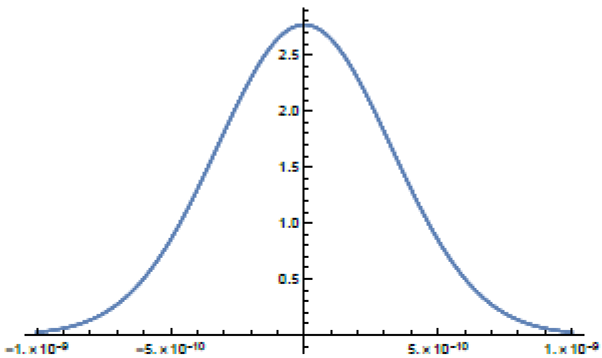
```
{(7.53726 - 1.44556 × 10^-16 i) e^(-1.41949×10^19 x1^2),
 (3.76863 - 1.35948 × 10^-18 i) e^(((6.55649×10^-61+1.87275×10^-62 i)-(1.41949×10^19+3.54018×10^-52 i) x1) x1) +
  (3.76863 - 1.35948 × 10^-18 i) e^(((1.30541×10^-60+3.74551×10^-62 i)-(1.41949×10^19+3.54018×10^-52 i) x1) x1),
 (3.76863 - 2.71896 × 10^-18 i) e^(((2.9543×10^-79+4.24445×10^-61 i)-(1.41949×10^19+8.02353×10^-51 i) x1) x1) +
  (3.76863 - 2.71896 × 10^-18 i) e^(((5.89956×10^-79+8.48889×10^-61 i)-(1.41949×10^19+8.02353×10^-51 i) x1) x1),
 (3.76863 - 6.79741 × 10^-18 i) e^(((2.93151×10^-59+1.68985×10^-41 i)-(1.41949×10^19+3.19443×10^-31 i) x1) x1) +
  (3.76863 - 6.79741 × 10^-18 i) e^(((5.86302×10^-59+3.37971×10^-41 i)-(1.41949×10^19+3.19443×10^-31 i) x1) x1),
 (3.76863 - 1.35948 × 10^-17 i) e^(((2.22059×10^-55+6.40023×10^-38 i)-(1.41949×10^19+1.20987×10^-27 i) x1) x1) +
  (3.76863 - 1.35948 × 10^-17 i) e^(((4.44117×10^-55+1.28005×10^-37 i)-(1.41949×10^19+1.20987×10^-27 i) x1) x1),
 (3.76863 - 2.71896 × 10^-17 i) e^(((5.15465×10^-42+7.42843×10^-25 i)-(1.41949×10^19+1.40424×10^-14 i) x1) x1) +
  (3.76863 - 2.71896 × 10^-17 i) e^(((1.03093×10^-41+1.48569×10^-24 i)-(1.41949×10^19+1.40424×10^-14 i) x1) x1),
 (3.76863 - 4.07845 × 10^-17 i) e^(((5.43233×10^-33+3.9143×10^-16 i)-(1.41949×10^19+7.39943×10^-6 i) x1) x1) +
  ((3.76863 - 4.07845 × 10^-17 i) e^(((9.50657×10^-33+7.82859×10^-16 i)-(1.41949×10^19+7.39943×10^-6 i) x1) x1)
     ((-0.000704191 - 1.01482 × 10^14 i) + 1. x1)) /
   ((-0.000704191 - 1.01482 × 10^14 i) + 1. x1),
 (3.76863 - 5.4427 × 10^-17 i) e^(((5.84927×10^-28+2.79395×10^-11 i)-(1.41949×10^19+0.528157 i) x1) x1) +
  ((3.76863 - 5.4455 × 10^-17 i) e^(((9.77982×10^-28+5.5879×10^-11 i)-(1.41949×10^19+0.528157 i) x1) x1)
     ((-9.64417 × 10^-9 - 1.42175 × 10^9 i) + 1. x1)) /
   ((-9.64417 × 10^-9 - 1.42175 × 10^9 i) + 1. x1),
 (3.76863 - 6.8165 × 10^-17 i) e^(((2.90252×10^-27+1.11752×10^-10 i)-(1.41949×10^19+2.11252 i) x1) x1) +
  (3.76863 - 6.82768 × 10^-17 i) e^(((4.89686×10^-27+2.23505×10^-10 i)-(1.41949×10^19+2.11252 i) x1) x1),
 (3.76863 - 8.20048 × 10^-17 i) e^(((8.87735×10^-27+3.21836×10^-10 i)-(1.41949×10^19+3.04192 i) x1) x1) +
  ((3.76863 - 8.18438 × 10^-17 i) e^(((5.41617×10^-27+1.60918×10^-10 i)-(1.41949×10^19+3.04192 i) x1) x1)
     ((-1.78296 × 10^-9 - 2.46853 × 10^8 i) + 1. x1)) /
   ((-1.78296 × 10^-9 - 2.46853 × 10^8 i) + 1. x1),
 ((3.76863 - 9.55734 × 10^-17 i) e^(((1.05657×10^-26+3.02487×10^-10 i)-(1.41949×10^19+2.85904 i) x1) x1)
     ((-9.07382 × 10^-10 - 2.62643 × 10^8 i) + 1. x1)) /
   ((-9.07382 × 10^-10 - 2.62643 × 10^8 i) + 1. x1) +
  ((3.76863 - 9.54221 × 10^-17 i) e^(((6.79319×10^-27+1.51243×10^-10 i)-(1.41949×10^19+2.85904 i) x1) x1)
     ((-7.74671 × 10^-10 - 2.62643 × 10^8 i) + 1. x1)) /
   ((-7.74671 × 10^-10 - 2.62643 × 10^8 i) + 1. x1)}
```

```
t1[x1_] = (3.768631996459896` - 1.3594819663996296`*^-18 i)
     e^((5.852350864444797`*^-104+1.6831735434109894`*^-85 i)-(1.4194860890458098`*^19+3.181802539529275`*^-75 i) x1) x
    + (3.768631996459896` - 1.3594819663996296`*^-18 i)
     e^((1.1692190857089022`*^-103+3.3663470868219764`*^-85 i)-(1.4194860890458098`*^19+3.181802539529275`*^-75 i) x1)
    ;
```

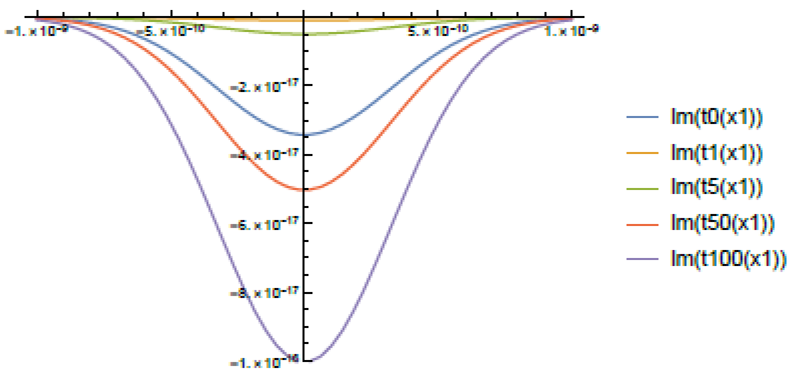
```
t1[0]
```

```
7.53726 - 2.71896 × 10^-18 i
```

```
Plot[{Re[totalgx2d[x1]]}, {x1, -1 × 10^-9, 1 × 10^-9}]
```

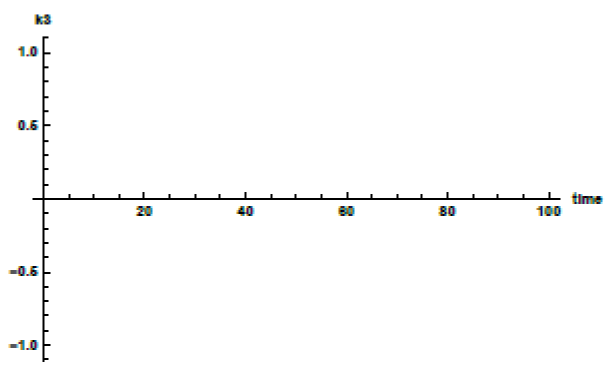


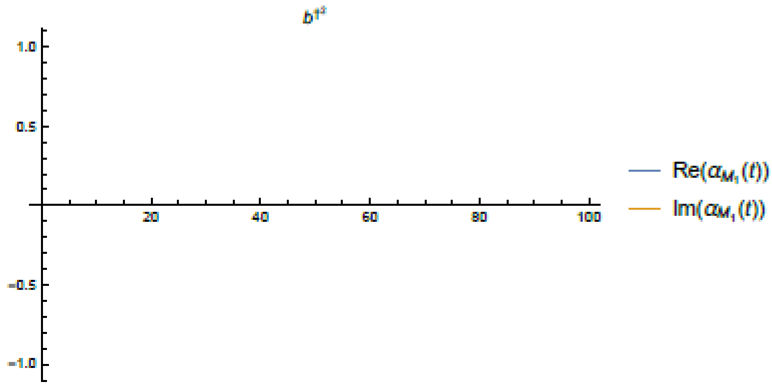

```
Plot[{Im[t0[x1]], Im[t1[x1]], Im[t5[x1]], Im[t50[x1]], Im[t100[x1]]},
 {x1, -1 × 10^-9, 1 × 10^-9}, PlotRange → {-1 × 10^-16, 0}, PlotLegends → "Expressions"]
```

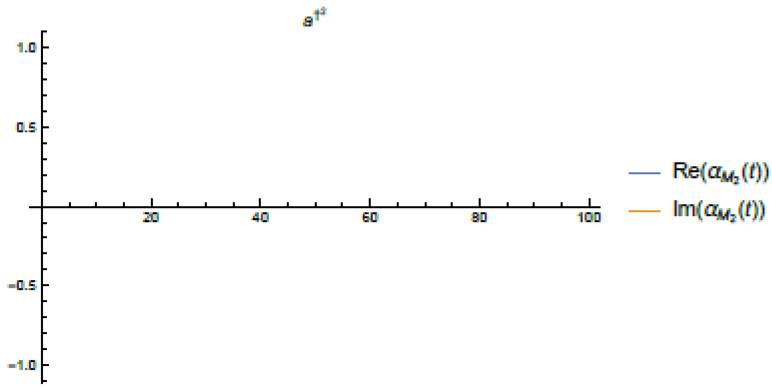


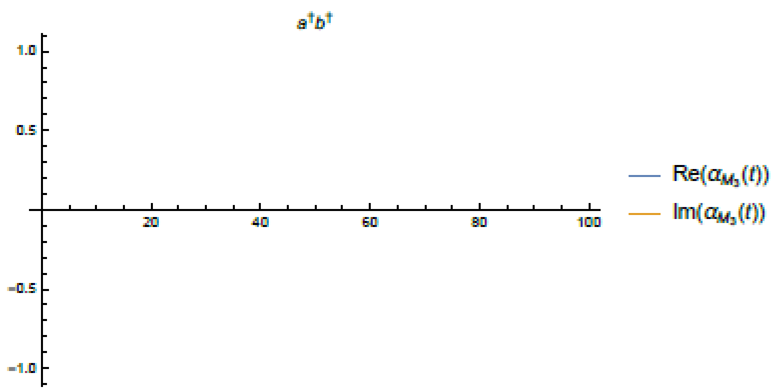
```
k3 = 0 (*no coupling*);
w = 3.29 × 10^15; m = 9.1 × 10^-31; ħ = 1.0545718 × 10^-34;
Plot[k3, {t, 0, 100}, PlotRange → All, AxesLabel → {"time", "k3"}, LabelStyle → {Bold}]
sol1 = NDSolve[
  {D[αM1[t], t] == -1/(4 m w) ⅈ ħ (4 k3 αM1[t]^2 + k3 (-1 + αM3[t])^2 + 4 αM1[t] (k3 + 2 m w^2 - k3 αM3[t])),
   D[αM2[t], t] == -1/(4 m w) ⅈ ħ (4 k3 αM2[t]^2 + k3 (-1 + αM3[t])^2 + 4 αM2[t] (k3 + 2 m w^2 - k3 αM3[t])),
   D[αM3[t], t] == 1/(2 m w) ⅈ ħ (k3 (1 + 2 αM1[t]) (1 + 2 αM2[t]) -
       2 (k3 + 2 m w^2 + k3 (αM1[t] + αM2[t])) αM3[t] + k3 αM3[t]^2), D[αM4[t], t] == -1/(2 m w) ⅈ k3 ħ
     (-1 + αM4[t]) (1 + 2 αM1[t] + (1 + 2 αM2[t]) αM4[t] - αM3[t] (1 + αM4[t])), D[αM5[t], t] ==
    1/(2 m w) ⅈ k3 ħ (1 + 2 αM2[t] - αM3[t] + 2 (αM1[t] - αM2[t] + (1 + 2 αM2[t] - αM3[t]) αM4[t]) αM5[t]),
   D[αM6[t], t] == -1/(2 m w) ⅈ ħ (k3 + 2 m w^2 + 2 k3 αM2[t] - k3 αM3[t] +
       k3 (-1 - 2 αM1[t] + αM3[t]) αM4[t]), D[αM7[t], t] ==
    -1/(2 m w) ⅈ ħ (k3 + 2 m w^2 + 2 k3 αM1[t] + k3 (1 + 2 αM2[t]) αM4[t] - k3 αM3[t] (1 + αM4[t])),
   D[αM8[t], t] == -(ⅈ e^(2 αM7[t]) k3 ħ (1 + (-1 + αM4[t]) αM5[t])^2)/(4 m w),
   D[αM9[t], t] == -(ⅈ e^(2 αM6[t]) k3 ħ (-1 + αM4[t])^2)/(4 m w),
   D[αM10[t], t] == -(ⅈ e^(αM6[t] + αM7[t]) k3 ħ (-1 + αM4[t]) (1 + (-1 + αM4[t]) αM5[t]))/(2 m w),
   D[αM11[t], t] == -(ⅈ ħ (k3 + 2 m w^2 + k3 (αM1[t] + αM2[t] - αM3[t])))/(2 m w), αM1[0] == 0, αM2[0] == 0,
   αM3[0] == 0, αM4[0] == 0, αM5[0] == 0, αM6[0] == 0, αM7[0] == 0, αM8[0] == 0, αM9[0] == 0,
   αM10[0] == 0, αM11[0] == 0}, {αM1[t], αM2[t], αM3[t], αM4[t], αM5[t], αM6[t],
   αM7[t], αM8[t], αM9[t], αM10[t], αM11[t]}, {t, 0, 100}, PrecisionGoal → 2000];
αM1[t_] = αM1[t] /. sol1[[1]]; αM2[t_] = αM2[t] /. sol1[[1]]; αM3[t_] = αM3[t] /. sol1[[1]];
αM4[t_] = αM4[t] /. sol1[[1]]; αM5[t_] = αM5[t] /. sol1[[1]]; αM6[t_] = αM6[t] /. sol1[[1]];
αM7[t_] = αM7[t] /. sol1[[1]]; αM8[t_] = αM8[t] /. sol1[[1]]; αM9[t_] = αM9[t] /. sol1[[1]];
αM10[t_] = αM10[t] /. sol1[[1]]; αM11[t_] = αM11[t] /. sol1[[1]];
Plot[{Re[αM1[t]], Im[αM1[t]]}, {t, 0, 100},
 PlotLabel → "b†²", PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM2[t]], Im[αM2[t]]}, {t, 0, 100}, PlotLabel → "a†²",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM3[t]], Im[αM3[t]]}, {t, 0, 100}, PlotLabel → "a†b†",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM4[t]], Im[αM4[t]]}, {t, 0, 100}, PlotLabel → "ab†",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM5[t]], Im[αM5[t]]}, {t, 0, 100},
 PlotLabel → "a†b", PlotLegends → "Expressions"]
Plot[{Re[αM6[t]], Im[αM6[t]]}, {t, 0, 100}, PlotLabel → "a†a",
```

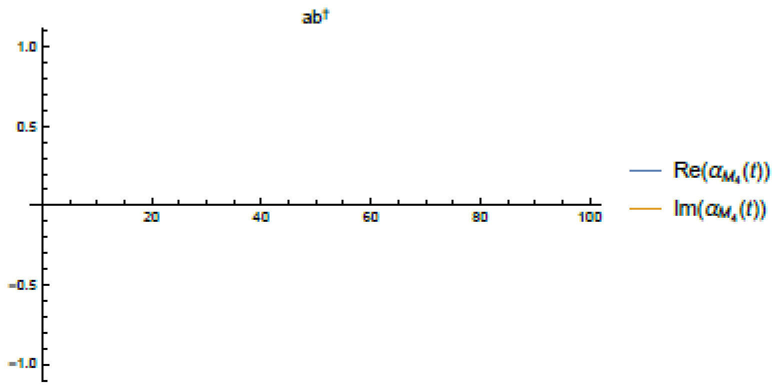
```
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM7[t]], Im[αM7[t]]}, {t, 0, 100},
 PlotLabel → "b†b", PlotLegends → "Expressions"]
Plot[{Re[αM8[t]], Im[αM8[t]]}, {t, 0, 100}, PlotLabel → "b²",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM6[t]], Im[αM6[t]]}, {t, 0, 100}, PlotLabel → "a²",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM10[t]], Im[αM10[t]]}, {t, 0, 100}, PlotLabel → "ab",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM11[t]], Im[αM11[t]]}, {t, 0, 100},
 PlotLabel → "1", PlotLegends → "Expressions"]
uapp[t_] = Exp[αM11[t]] Exp[αM10[t] α β] Exp[αM6[t] α^2]
   Exp[αM8[t] β^2] Exp[- (1/2) β Conjugate[β]] Exp[- (1/2) α Conjugate[α]]
   Exp[(1/2) Exp[αM7[t] β] Conjugate[Exp[αM7[t] β] ]]
   Exp[(1/2) Exp[αM5[t] α] Conjugate [Exp[αM5[t] α]] ]
   (Exp[αM4[t] Exp[αM5[t]] α Conjugate[δ]] Exp[αM3[t] Conjugate[γ] Exp[αM7[t]] β] +
     Exp[αM4[t] Conjugate[δ] ] Exp[αM3[t] Exp[αM7[t]] β])
   Exp[αM1[t] Conjugate[γ] Conjugate[δ]] Exp[αM2[t] Conjugate[γ] ^2]
   Exp[αM3[t] Conjugate[δ] ^2];
u[t_] = uapp[t] /. {α → 0, β → X0 Sqrt[m w / (2 ħ)]};
ψγ[x1_] = (m w / (Pi ħ)) ^ (1/4) Exp[-γ Conjugate[γ] / 2]
   Exp[(x1 Sqrt[m w / ħ]) ^2 / 2] Exp[- (x1 Sqrt[m w / ħ] - γ / Sqrt[2]) ^2];
ψδ[x2_] = (m w / (Pi ħ)) ^ (1/4) Exp[-δ Conjugate[δ] / 2]
   Exp[(x2 Sqrt[m w / ħ]) ^2 / 2] Exp[- (x2 Sqrt[m w / ħ] - δ / Sqrt[2]) ^2];
γα[t_] = Exp[1/2 (Conjugate[γ] Exp[αM5[t]] α - γ Conjugate[Exp[αM5[t]] α])]
    Exp[-1/2 (γ - Exp[αM5[t]] α) Conjugate[(γ - Exp[αM5[t]] α)]] /. {α → 0};
δβ[t_] = Exp[1/2 (Conjugate[δ] Exp[αM7[t]] β - δ Conjugate[Exp[αM7[t]] β])] Exp[
    -1/2 (δ - Exp[αM7[t]] β) Conjugate[(δ - Exp[αM7[t]] β)]] /. {β → X0 Sqrt[m w / (2 ħ)]};
total1[t_, x1_, x2_] = ψγ[x1] ψδ[x2] u[t] γα[t] δβ[t] 1/Pi^2;
X0 = 5.29 × 10^-11;
```

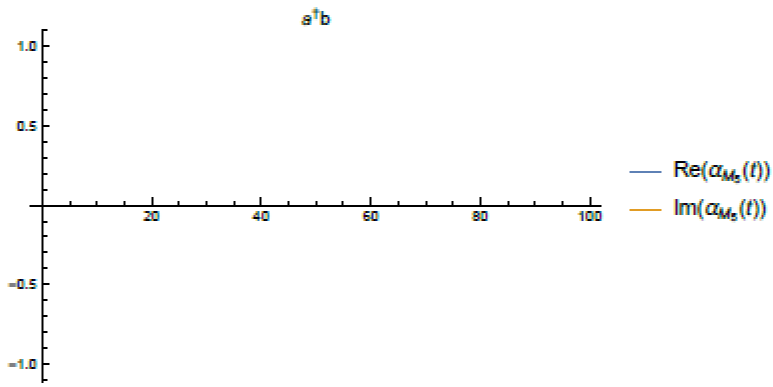


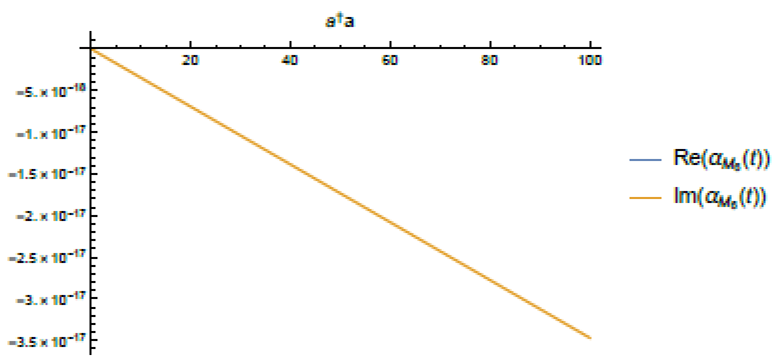

$b^{\dagger 2}$
1.0
0.5
-0.5
-1.0
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_1}(t))$
$\mathrm{Im}(\alpha_{M_1}(t))$

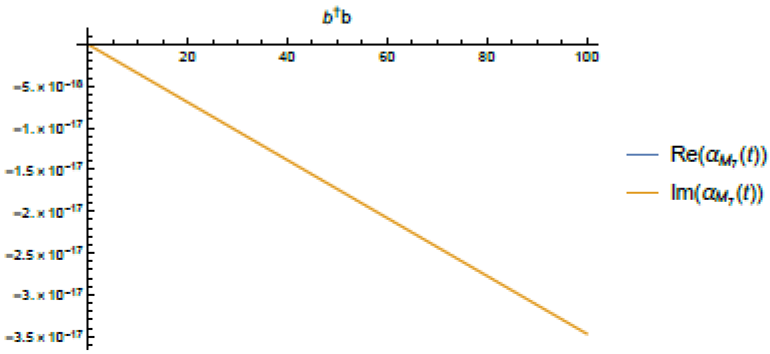

$a^{\dagger 2}$
1.0
0.5
-0.5
-1.0
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_2}(t))$
$\mathrm{Im}(\alpha_{M_2}(t))$

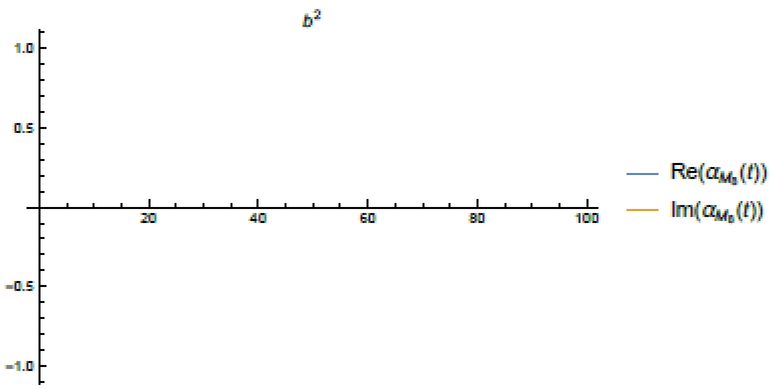

$a^{\dagger}b^{\dagger}$
1.0
0.5
-0.5
-1.0
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_3}(t))$
$\mathrm{Im}(\alpha_{M_3}(t))$

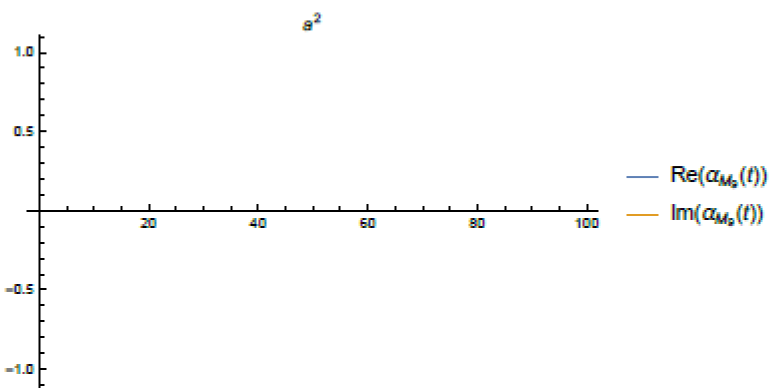

ab†
1.0
0.5
-0.5
-1.0
20
40
60
80
100
Re($\alpha_{M_4}(t)$)
Im($\alpha_{M_4}(t)$)

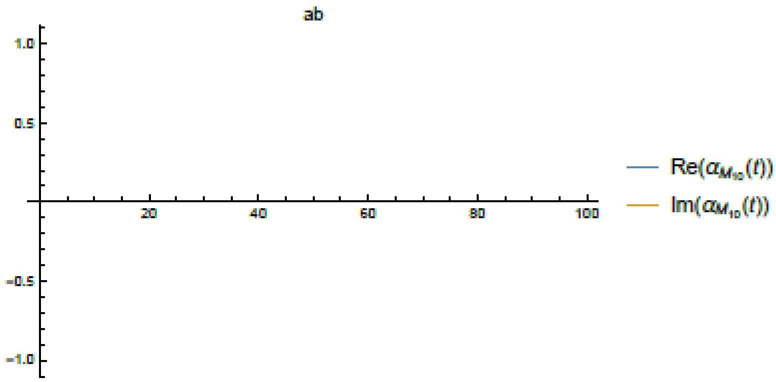

a†b
1.0
0.5
-0.5
-1.0
20
40
60
80
100
Re($\alpha_{M_5}(t)$)
Im($\alpha_{M_5}(t)$)

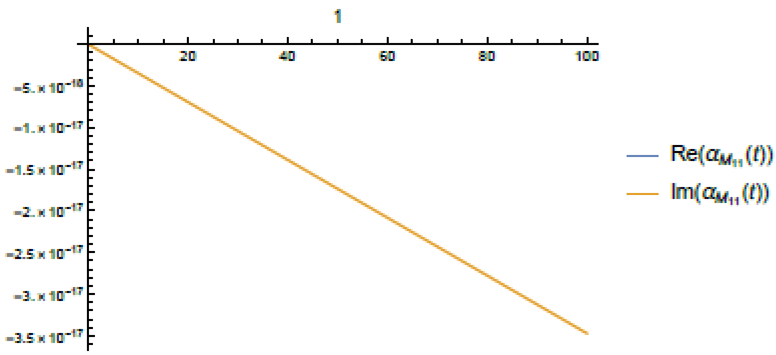

a†a
20
40
60
80
100
-5. × 10^-18
-1. × 10^-17
-1.5 × 10^-17
-2. × 10^-17
-2.5 × 10^-17
-3. × 10^-17
-3.5 × 10^-17
Re($\alpha_{M_6}(t)$)
Im($\alpha_{M_6}(t)$)


$b^{\dagger}b$
Re($\alpha_{M_7}(t)$)
Im($\alpha_{M_7}(t)$)


b^2
Re($\alpha_{M_8}(t)$)
Im($\alpha_{M_8}(t)$)


a^2
Re($\alpha_{M_9}(t)$)
Im($\alpha_{M_9}(t)$)





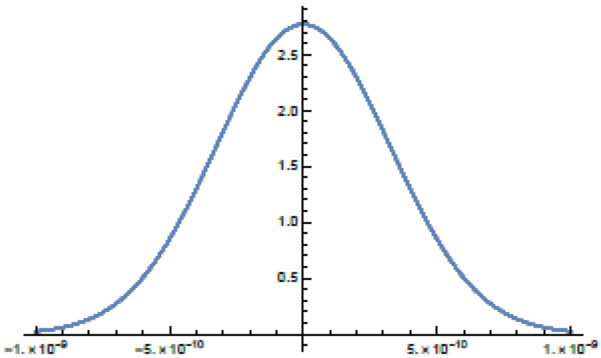

```
final0 =
 Integrate[total1[t, x1, x2] /. {t → {0, 1, 5, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100}},
  {γ, -Infinity, Infinity}, {δ, -Infinity, Infinity}]
```

$$\{3.35516\times10^{9}\, e^{-4.73162\times10^{18}\,x1^2+(5.00605\times10^{8}-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-1.1795\times10^{-9}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-1.73687\times10^{-10}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-5.89749\times10^{-9}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-8.68436\times10^{-10}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-1.1795\times10^{-8}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-1.73687\times10^{-9}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-2.359\times10^{-8}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-3.47374\times10^{-9}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-3.5385\times10^{-8}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-5.21061\times10^{-9}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-4.718\times10^{-8}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-6.94748\times10^{-9}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-5.89749\times10^{-8}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-8.68436\times10^{-9}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-7.07699\times10^{-8}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-1.04212\times10^{-8}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-8.25649\times10^{-8}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-1.21581\times10^{-8}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-9.43599\times10^{-8}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-1.3895\times10^{-8}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-1.06155\times10^{-7}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-1.56318\times10^{-8}\,i)-4.73162\times10^{18}\,x2)\,x2},$$
$$(3.35516\times10^{9}-1.1795\times10^{-7}\,i)\, e^{-4.73162\times10^{18}\,x1^2+((5.00605\times10^{8}-1.73687\times10^{-8}\,i)-4.73162\times10^{18}\,x2)\,x2}\}$$

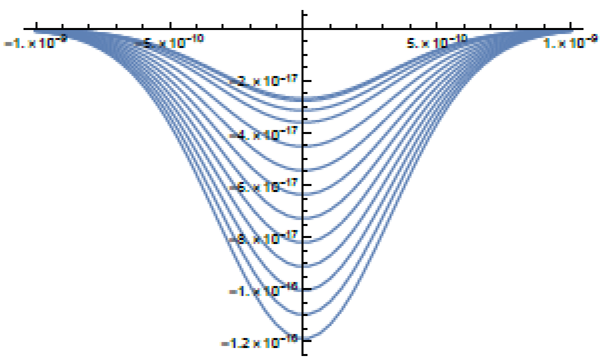
```
finalx20 = Integrate[final0, {x2, -Infinity, Infinity}]
```

$$\{(2.77034 - 2.6809\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},$$
$$(2.77034 - 2.77324\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},\ (2.77034 - 3.1426\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},$$
$$(2.77034 - 3.60431\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},\ (2.77034 - 4.52772\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},$$
$$(2.77034 - 5.45113\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},\ (2.77034 - 6.37454\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},$$
$$(2.77034 - 7.29795\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},\ (2.77034 - 8.22136\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},$$
$$(2.77034 - 9.14477\times10^{-17}\,i)\,e^{-4.73162\times10^{18}\,x1^2},\ (2.77034 - 1.00682\times10^{-16}\,i)\,e^{-4.73162\times10^{18}\,x1^2},$$
$$(2.77034 - 1.09916\times10^{-16}\,i)\,e^{-4.73162\times10^{18}\,x1^2},\ (2.77034 - 1.1915\times10^{-16}\,i)\,e^{-4.73162\times10^{18}\,x1^2}\}$$

```
Plot[Re[finalx20], {x1, -1 × 10^-9, 1 × 10^-9}]
```

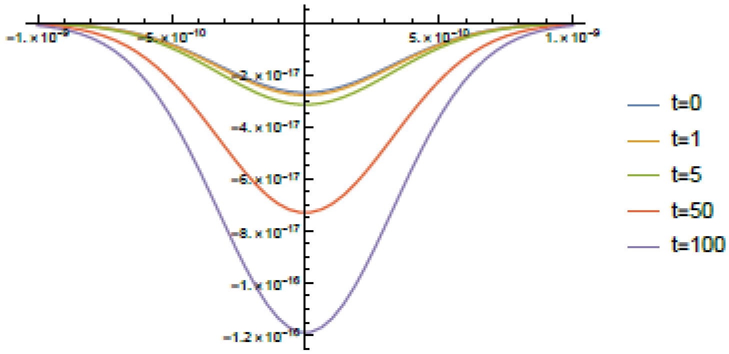



```
Plot[Im[finalx20], {x1, -1 × 10^-9, 1 × 10^-9}]
```



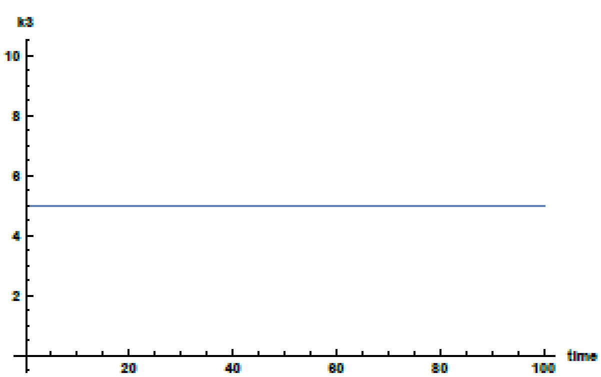

```
nonet0 = finalx20[[1]];
nonet1 = finalx20[[2]];
nonet5 = finalx20[[3]];
nonet50 = finalx20[[8]];
nonet100 = finalx20[[13]];
```

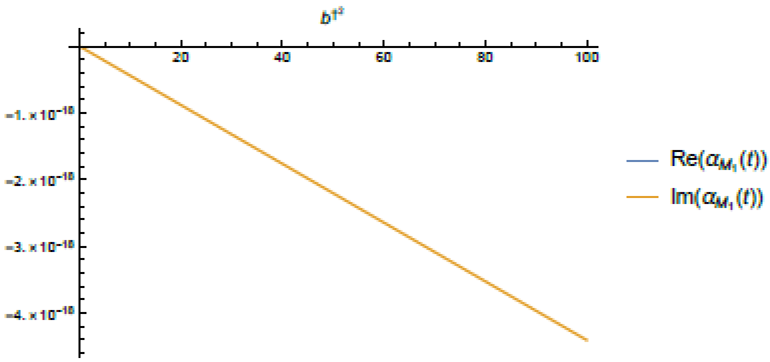
```
Plot[{Im[nonet0], Im[nonet1], Im[nonet5], Im[nonet50], Im[nonet100]},
 {x1, -1 × 10^-9, 1 × 10^-9}, PlotLegends → {"t=0", "t=1", "t=5", "t=50", "t=100"}]
```



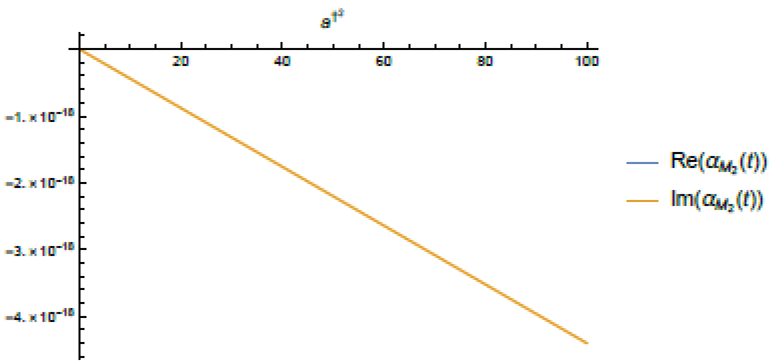
```
k3 = 5(*constant coupling*);
w = 3.29 × 10^15; m = 9.1 × 10^-31; ħ = 1.0545718 × 10^-34;
Plot[k3, {t, 0, 100}, PlotRange → All, AxesLabel → {"time", "k3"}, LabelStyle → {Bold}]
sol1 = NDSolve[
   {D[αM1[t], t] == -1/(4 m w) ⅈ ħ (4 k3 αM1[t]^2 + k3 (-1 + αM3[t])^2 + 4 αM1[t] (k3 + 2 m w^2 - k3 αM3[t])),
    D[αM2[t], t] == -1/(4 m w) ⅈ ħ (4 k3 αM2[t]^2 + k3 (-1 + αM3[t])^2 + 4 αM2[t] (k3 + 2 m w^2 - k3 αM3[t])),
    D[αM3[t], t] == 1/(2 m w) ⅈ ħ (k3 (1 + 2 αM1[t]) (1 + 2 αM2[t]) -
        2 (k3 + 2 m w^2 + k3 (αM1[t] + αM2[t])) αM3[t] + k3 αM3[t]^2), D[αM4[t], t] == -1/(2 m w) ⅈ k3 ħ
      (-1 + αM4[t]) (1 + 2 αM1[t] + (1 + 2 αM2[t]) αM4[t] - αM3[t] (1 + αM4[t])), D[αM5[t], t] ==
     1/(2 m w) ⅈ k3 ħ (1 + 2 αM2[t] - αM3[t] + 2 (αM1[t] - αM2[t] + (1 + 2 αM2[t] - αM3[t]) αM4[t]) αM5[t]),
    D[αM6[t], t] == -1/(2 m w) ⅈ ħ (k3 + 2 m w^2 + 2 k3 αM2[t] - k3 αM3[t] +
        k3 (-1 - 2 αM2[t] + αM3[t]) αM4[t]), D[αM7[t], t] ==
     -1/(2 m w) ⅈ ħ (k3 + 2 m w^2 + 2 k3 αM1[t] + k3 (1 + 2 αM2[t]) αM4[t] - k3 αM3[t] (1 + αM4[t])),
    D[αM8[t], t] == -(ⅈ e^(2 αM6[t]) k3 ħ (1 + (-1 + αM4[t]) αM5[t])^2)/(4 m w),
    D[αM9[t], t] == -(ⅈ e^(2 αM7[t]) k3 ħ (-1 + αM4[t])^2)/(4 m w),
    D[αM10[t], t] == -(ⅈ e^(αM6[t] + αM7[t]) k3 ħ (-1 + αM4[t]) (1 + (-1 + αM4[t]) αM5[t]))/(2 m w),
    D[αM11[t], t] == -(ⅈ ħ (k3 + 2 m w^2 + k3 (αM1[t] + αM2[t] - αM3[t])))/(2 m w), αM1[0] == 0, αM2[0] == 0,
    αM3[0] == 0, αM4[0] == 0, αM5[0] == 0, αM6[0] == 0, αM7[0] == 0, αM8[0] == 0, αM9[0] == 0,
    αM10[0] == 0, αM11[0] == 0}, {αM1[t], αM2[t], αM3[t], αM4[t], αM5[t], αM6[t],
    αM7[t], αM8[t], αM9[t], αM10[t], αM11[t]}, {t, 0, 100}, PrecisionGoal → 2000];
αM1[t_] = αM1[t] /. sol1[[1]]; αM2[t_] = αM2[t] /. sol1[[1]]; αM3[t_] = αM3[t] /. sol1[[1]];
αM4[t_] = αM4[t] /. sol1[[1]]; αM5[t_] = αM5[t] /. sol1[[1]]; αM6[t_] = αM6[t] /. sol1[[1]];
αM7[t_] = αM7[t] /. sol1[[1]]; αM8[t_] = αM8[t] /. sol1[[1]]; αM9[t_] = αM9[t] /. sol1[[1]];
αM10[t_] = αM10[t] /. sol1[[1]]; αM11[t_] = αM11[t] /. sol1[[1]];
Plot[{Re[αM1[t]], Im[αM1[t]]}, {t, 0, 100},
 PlotLabel → "b†²", PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM2[t]], Im[αM2[t]]}, {t, 0, 100}, PlotLabel → "a†²",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM3[t]], Im[αM3[t]]}, {t, 0, 100}, PlotLabel → "a†b†",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM4[t]], Im[αM4[t]]}, {t, 0, 100}, PlotLabel → "ab†",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM5[t]], Im[αM5[t]]}, {t, 0, 100},
 PlotLabel → "a†b", PlotLegends → "Expressions"]
Plot[{Re[αM6[t]], Im[αM6[t]]}, {t, 0, 100}, PlotLabel → "a†a",
```

```
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM7[t]], Im[αM7[t]]}, {t, 0, 100},
 PlotLabel → "b†b", PlotLegends → "Expressions"]
Plot[{Re[αM8[t]], Im[αM8[t]]}, {t, 0, 100}, PlotLabel → "b2",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM6[t]], Im[αM6[t]]}, {t, 0, 100}, PlotLabel → "a2",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM10[t]], Im[αM10[t]]}, {t, 0, 100}, PlotLabel → "ab",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM11[t]], Im[αM11[t]]}, {t, 0, 100},
 PlotLabel → "1", PlotLegends → "Expressions"]
uapp[t_] = Exp[αM11[t]] Exp[αM10[t] α β] Exp[αM6[t] α^2]
   Exp[αM8[t] β^2] Exp[- (1/2) β Conjugate[β]] Exp[- (1/2) α Conjugate[α]]
   Exp[(1/2) Exp[αM7[t] β] Conjugate[Exp[αM7[t] β] ]]
   Exp[(1/2) Exp[αM5[t] α] Conjugate [Exp[αM5[t] α]] ]
   (Exp[αM4[t] Exp[αM5[t]] α Conjugate[δ]] Exp[αM3[t] Conjugate[γ] Exp[αM7[t]] β] +
     Exp[αM4[t] Conjugate[δ] ] Exp[αM3[t] Exp[αM7[t]] β])
   Exp[αM2[t] Conjugate[γ] Conjugate[δ]] Exp[αM1[t] Conjugate[γ]^2]
   Exp[αM3[t] Conjugate[δ]^2];
u[t_] = uapp[t] /. {α → 0, β → X0 Sqrt[m w / (2 ℏ)]};
ψγ[x1_] = (m w / (Pi ℏ))^(1/4) Exp[-γ Conjugate[γ] / 2]
   Exp[(x1 Sqrt[m w / ℏ])^2/2] Exp[- (x1 Sqrt[m w / ℏ] - γ / Sqrt[2])^2];
ψδ[x2_] = (m w / (Pi ℏ))^(1/4) Exp[-δ Conjugate[δ] / 2]
   Exp[(x2 Sqrt[m w / ℏ])^2/2] Exp[- (x2 Sqrt[m w / ℏ] - δ / Sqrt[2])^2];
γα[t_] = Exp[1/2 (Conjugate[γ] Exp[αM5[t]] α - γ Conjugate[Exp[αM5[t]] α])]
    Exp[-1/2 (γ - Exp[αM5[t]] α) Conjugate[(γ - Exp[αM5[t]] α)]] /. {α → 0};
δβ[t_] = Exp[1/2 (Conjugate[δ] Exp[αM7[t]] β - δ Conjugate[Exp[αM7[t]] β])] Exp[
     -1/2 (δ - Exp[αM7[t]] β) Conjugate[(δ - Exp[αM7[t]] β)]] /. {β → X0 Sqrt[m w / (2 ℏ)]};
total1[t_, x1_, x2_] = ψγ[x1] ψδ[x2] u[t] γα[t] δβ[t] 1/Pi^2;
X0 = 5.29 × 10^-11;
```

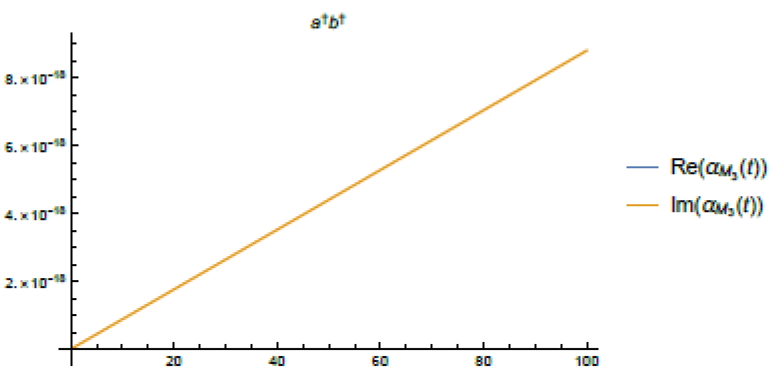




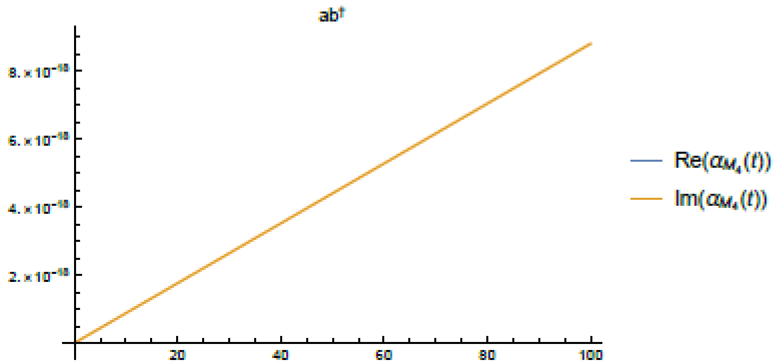
$b^{\dagger 2}$
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_1}(t))$
$\mathrm{Im}(\alpha_{M_1}(t))$



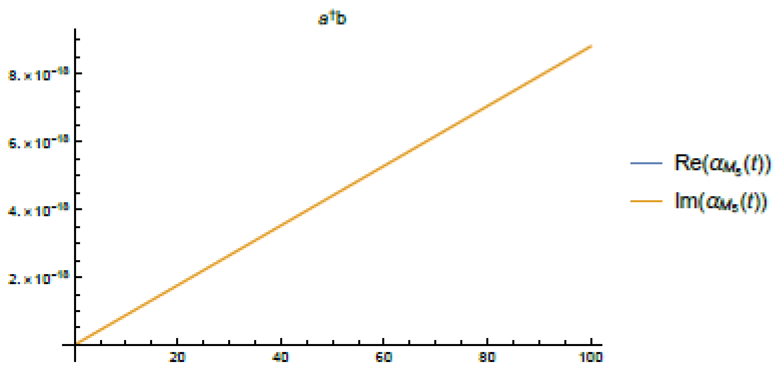
$a^{\dagger 2}$
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_2}(t))$
$\mathrm{Im}(\alpha_{M_2}(t))$



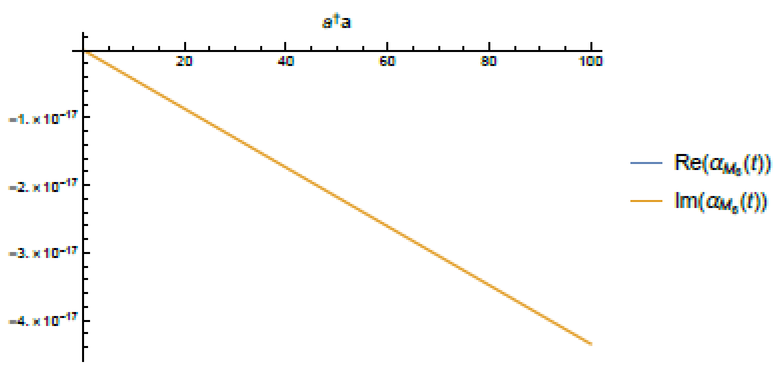
$a^{\dagger}b^{\dagger}$
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_3}(t))$
$\mathrm{Im}(\alpha_{M_3}(t))$



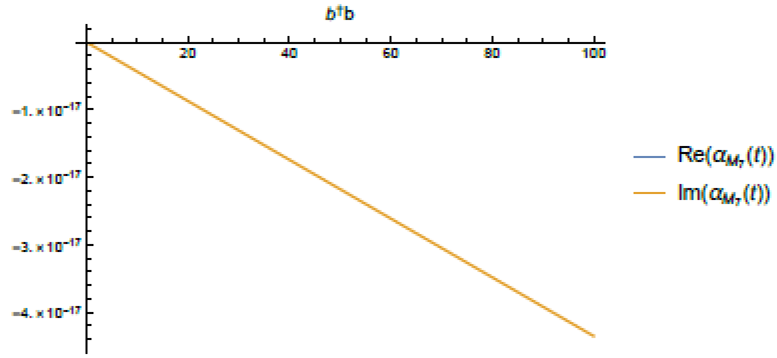
ab†
8. × 10^-18
6. × 10^-18
4. × 10^-18
2. × 10^-18
20
40
60
80
100
Re(α_{M_4}(t))
Im(α_{M_4}(t))

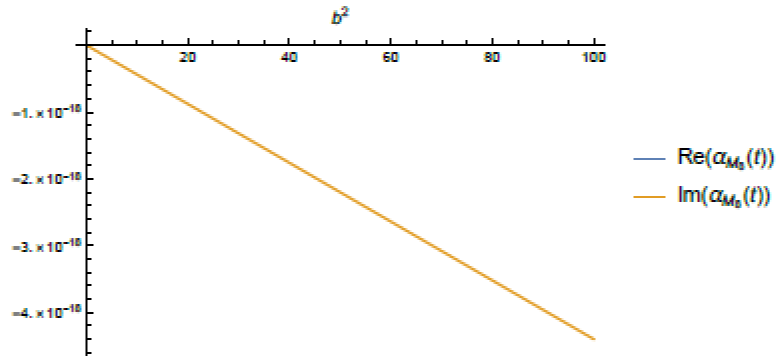


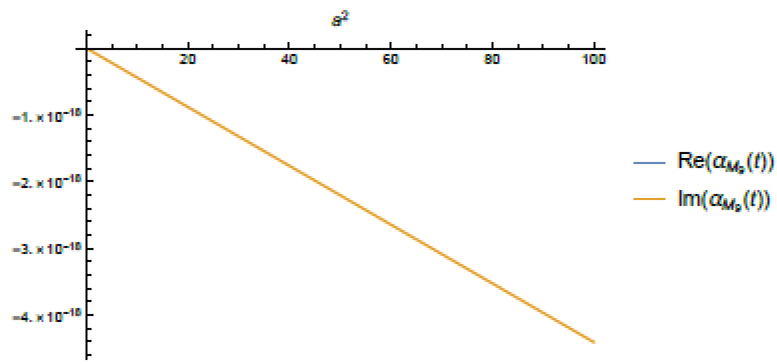
a†b
8. × 10^-18
6. × 10^-18
4. × 10^-18
2. × 10^-18
20
40
60
80
100
Re(α_{M_5}(t))
Im(α_{M_5}(t))

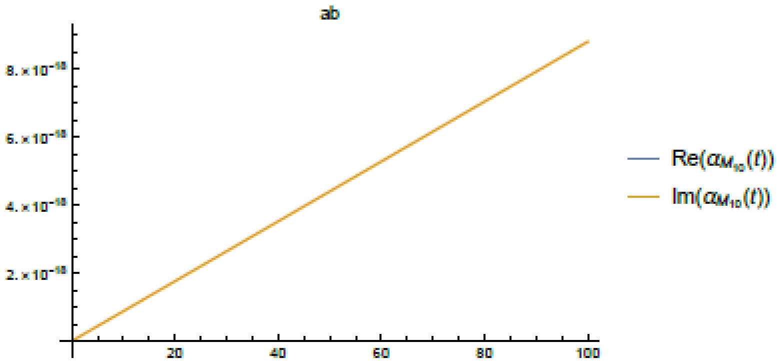


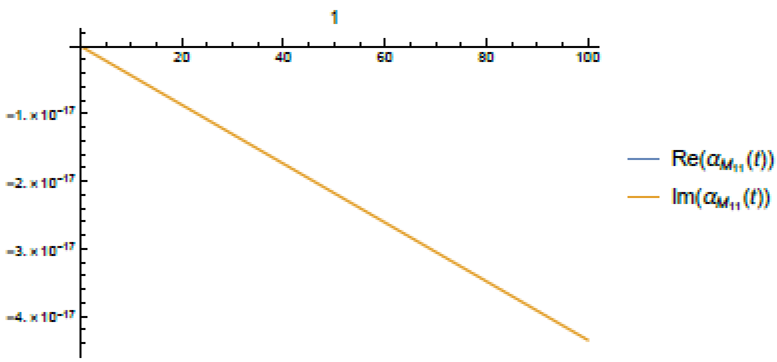
a†a
20
40
60
80
100
-1. × 10^-17
-2. × 10^-17
-3. × 10^-17
-4. × 10^-17
Re(α_{M_6}(t))
Im(α_{M_6}(t))


$b^{\dagger}b$
20
40
60
80
100
$-1. \times 10^{-17}$
$-2. \times 10^{-17}$
$-3. \times 10^{-17}$
$-4. \times 10^{-17}$
$\mathrm{Re}(\alpha_{M_7}(t))$
$\mathrm{Im}(\alpha_{M_7}(t))$


b^2
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_8}(t))$
$\mathrm{Im}(\alpha_{M_8}(t))$


a^2
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_9}(t))$
$\mathrm{Im}(\alpha_{M_9}(t))$

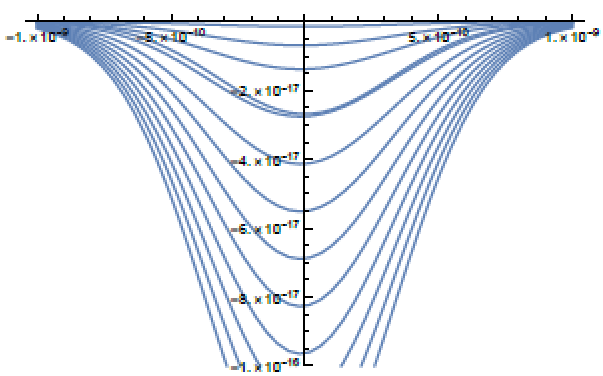





```
final = Integrate[
   total1[t, x1, x2] /. {t → {0, 1, 5, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100}},
   {γ, -Infinity, Infinity}, {δ, -Infinity, Infinity}];

finalx2 = Integrate[final, {x2, -Infinity, Infinity}];

Plot[Im[finalx2], {x1, -1 × 10^-9, 1 × 10^-9}, PlotRange → {-1 × 10^-16, 0}]
```

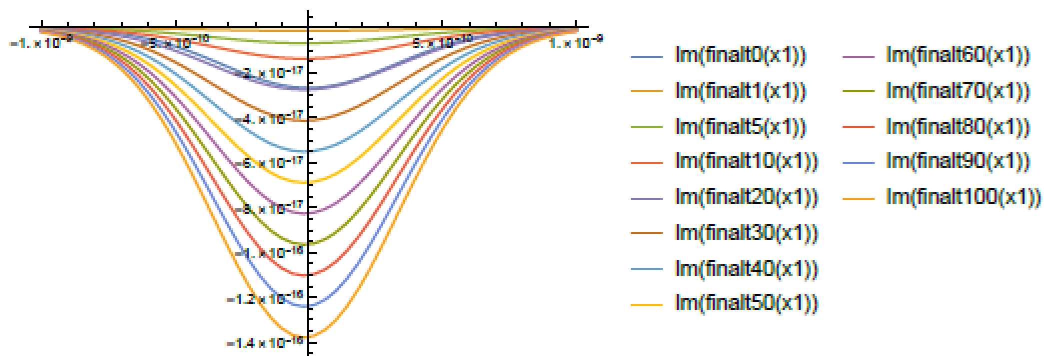


```
finalx2[[1]]
```

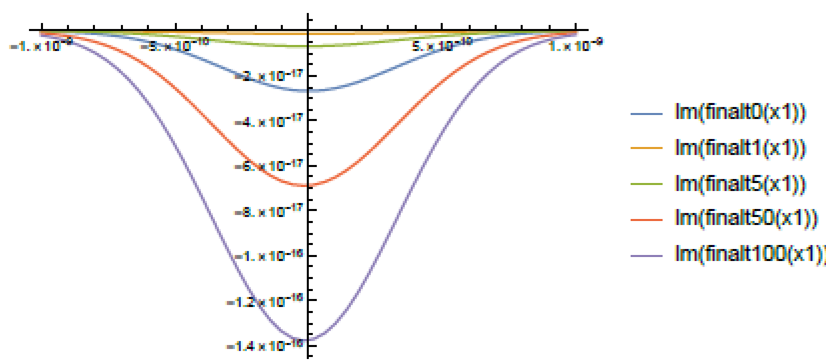
$\left(2.77034 - 2.6809 \times 10^{-17}\, i\right) e^{-4.73162\times 10^{18}\, x1^2}$

```
finalt0[x1_] = finalx2[[1]];
finalt1[x1_] = finalx2[[2]];
finalt5[x1_] = finalx2[[3]];
finalt10[x1_] = finalx2[[4]];
finalt20[x1_] = finalx2[[5]];
finalt30[x1_] = finalx2[[6]];
finalt40[x1_] = finalx2[[7]];
finalt50[x1_] = finalx2[[8]];
finalt60[x1_] = finalx2[[9]];
finalt70[x1_] = finalx2[[10]];
finalt80[x1_] = finalx2[[11]];
finalt90[x1_] = finalx2[[12]];
finalt100[x1_] = finalx2[[13]];
```

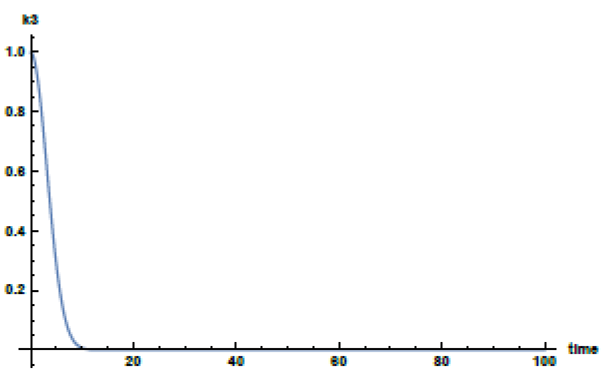
```
Plot[{Im[finalt0[x1]], Im[finalt1[x1]], Im[finalt5[x1]], Im[finalt10[x1]],
  Im[finalt20[x1]], Im[finalt30[x1]], Im[finalt40[x1]], Im[finalt50[x1]],
  Im[finalt60[x1]], Im[finalt70[x1]], Im[finalt80[x1]], Im[finalt90[x1]],
  Im[finalt100[x1]]}, {x1, -1 × 10^-9, 1 × 10^-9}, PlotLegends → "Expressions"]
```



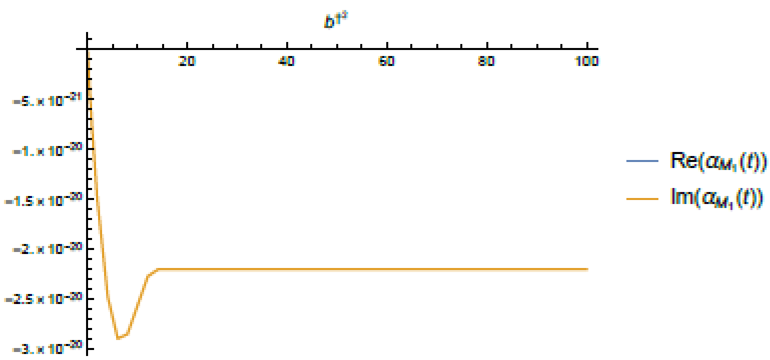

```
Plot[{Im[finalt0[x1]], Im[finalt1[x1]], Im[finalt5[x1]], Im[finalt50[x1]],
  Im[finalt100[x1]]}, {x1, -1 × 10^-9, 1 × 10^-9}, PlotLegends → "Expressions"]
```

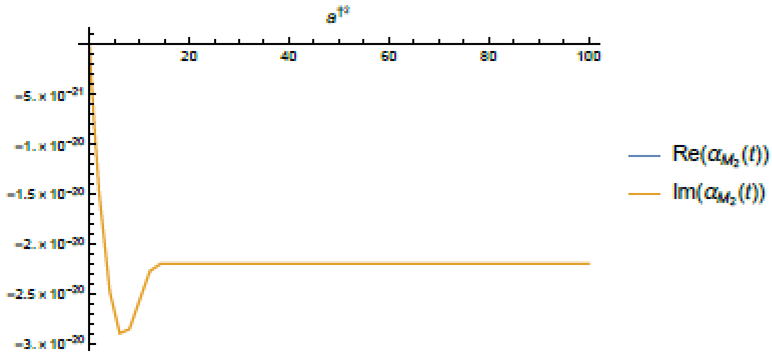


```
k3 = 2.5 / (Sqrt[2 Pi]) Exp[- (t) ^2 / (2 * 10)] (*Gaussian distribution with peak at 0*);
w = 3.29 × 10^15; m = 9.1 × 10^-31; ℏ = 1.0545718 × 10^-34;
Plot[k3, {t, 0, 100}, PlotRange → All, AxesLabel → {"time", "k3"}, LabelStyle → {Bold}]
sol1 = NDSolve[
  {D[αM1[t], t] == -1/(4 m w) ⅈ ℏ (4 k3 αM1[t]^2 + k3 (-1 + αM3[t])^2 + 4 αM1[t] (k3 + 2 m w^2 - k3 αM3[t])),
   D[αM2[t], t] == -1/(4 m w) ⅈ ℏ (4 k3 αM2[t]^2 + k3 (-1 + αM3[t])^2 + 4 αM2[t] (k3 + 2 m w^2 - k3 αM3[t])),
   D[αM3[t], t] == 1/(2 m w) ⅈ ℏ (k3 (1 + 2 αM1[t]) (1 + 2 αM2[t]) -
       2 (k3 + 2 m w^2 + k3 (αM1[t] + αM2[t])) αM3[t] + k3 αM3[t]^2), D[αM4[t], t] == -1/(2 m w) ⅈ k3 ℏ
     (-1 + αM4[t]) (1 + 2 αM1[t] + (1 + 2 αM2[t]) αM4[t] - αM3[t] (1 + αM4[t])), D[αM5[t], t] ==
    1/(2 m w) ⅈ k3 ℏ (1 + 2 αM2[t] - αM3[t] + 2 (αM1[t] - αM2[t] + (1 + 2 αM2[t] - αM3[t]) αM4[t]) αM5[t]),
   D[αM6[t], t] == -1/(2 m w) ⅈ ℏ (k3 + 2 m w^2 + 2 k3 αM2[t] - k3 αM3[t] +
       k3 (-1 - 2 αM2[t] + αM3[t]) αM4[t]), D[αM7[t], t] ==
    -1/(2 m w) ⅈ ℏ (k3 + 2 m w^2 + 2 k3 αM1[t] + k3 (1 + 2 αM2[t]) αM4[t] - k3 αM3[t] (1 + αM4[t])),
   D[αM8[t], t] == -(ⅈ e^(2 αM7[t]) k3 ℏ (1 + (-1 + αM4[t]) αM5[t])^2)/(4 m w),
   D[αM9[t], t] == -(ⅈ e^(2 αM6[t]) k3 ℏ (-1 + αM4[t])^2)/(4 m w),
   D[αM10[t], t] == -(ⅈ e^(αM6[t] + αM7[t]) k3 ℏ (-1 + αM4[t]) (1 + (-1 + αM4[t]) αM5[t]))/(2 m w),
   D[αM11[t], t] == -(ⅈ ℏ (k3 + 2 m w^2 + k3 (αM1[t] + αM2[t] - αM3[t])))/(2 m w), αM1[0] == 0, αM2[0] == 0,
   αM3[0] == 0, αM4[0] == 0, αM5[0] == 0, αM6[0] == 0, αM7[0] == 0, αM8[0] == 0, αM9[0] == 0,
   αM10[0] == 0, αM11[0] == 0}, {αM1[t], αM2[t], αM3[t], αM4[t], αM5[t], αM6[t],
   αM7[t], αM8[t], αM9[t], αM10[t], αM11[t]}, {t, 0, 100}, PrecisionGoal → 2000];
```

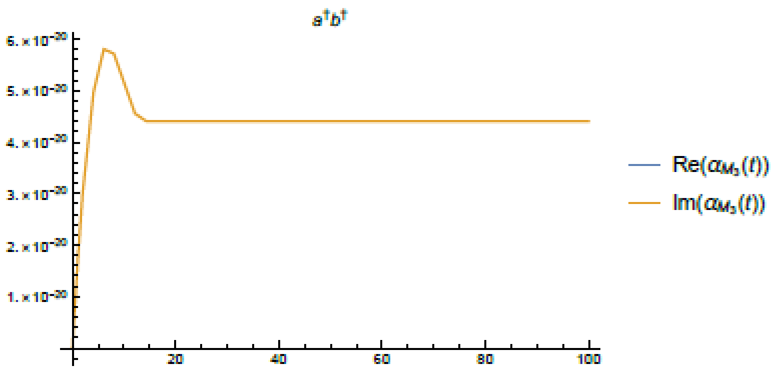


```
αM1[t_] = αM1[t] /. sol1[[1]]; αM2[t_] = αM2[t] /. sol1[[1]];
αM3[t_] = αM3[t] /. sol1[[1]]; αM4[t_] = αM4[t] /. sol1[[1]];
αM5[t_] = αM5[t] /. sol1[[1]]; αM6[t_] = αM6[t] /. sol1[[1]];
αM7[t_] = αM7[t] /. sol1[[1]]; αM8[t_] = αM8[t] /. sol1[[1]]; αM9[t_] = αM9[t] /. sol1[[1]];
αM10[t_] = αM10[t] /. sol1[[1]];  αM11[t_] = αM11[t] /. sol1[[1]];
Plot[{Re[αM1[t]], Im[αM1[t]]}, {t, 0, 100},
 PlotLabel → "b†2", PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM2[t]], Im[αM2[t]]}, {t, 0, 100}, PlotLabel → "a†2",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM3[t]], Im[αM3[t]]}, {t, 0, 100}, PlotLabel → "a†b†",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM4[t]], Im[αM4[t]]}, {t, 0, 100}, PlotLabel → "ab†",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM5[t]], Im[αM5[t]]}, {t, 0, 100},
 PlotLabel → "a†b", PlotLegends → "Expressions"]
Plot[{Re[αM6[t]], Im[αM6[t]]}, {t, 0, 100}, PlotLabel → "a†a",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM7[t]], Im[αM7[t]]}, {t, 0, 100},
 PlotLabel → "b†b", PlotLegends → "Expressions"]
Plot[{Re[αM8[t]], Im[αM8[t]]}, {t, 0, 100}, PlotLabel → "b2",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM9[t]], Im[αM9[t]]}, {t, 0, 100}, PlotLabel → "a2",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM10[t]], Im[αM10[t]]}, {t, 0, 100}, PlotLabel → "ab",
 PlotLegends → "Expressions", PlotRange → All]
Plot[{Re[αM11[t]], Im[αM11[t]]}, {t, 0, 100}, PlotLabel → "1", PlotLegends → "Expressions"]
```

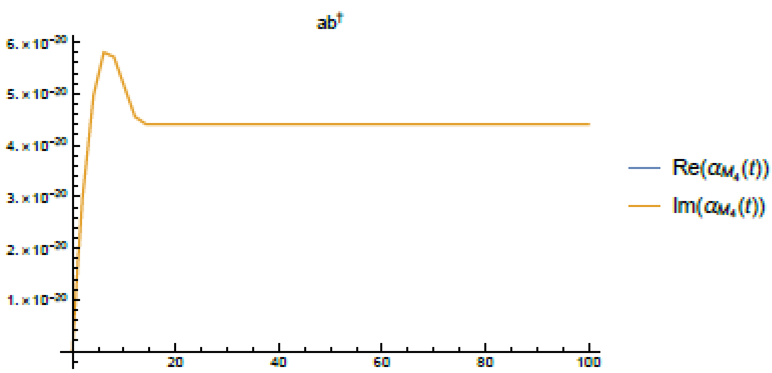




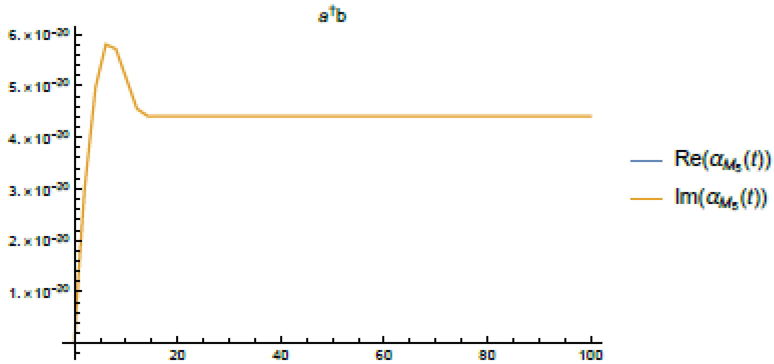
$a^{\dagger 2}$
20
40
60
80
100
-5. × 10^-21
-1. × 10^-20
-1.5 × 10^-20
-2. × 10^-20
-2.5 × 10^-20
-3. × 10^-20
$\mathrm{Re}(\alpha_{M_2}(t))$
$\mathrm{Im}(\alpha_{M_2}(t))$

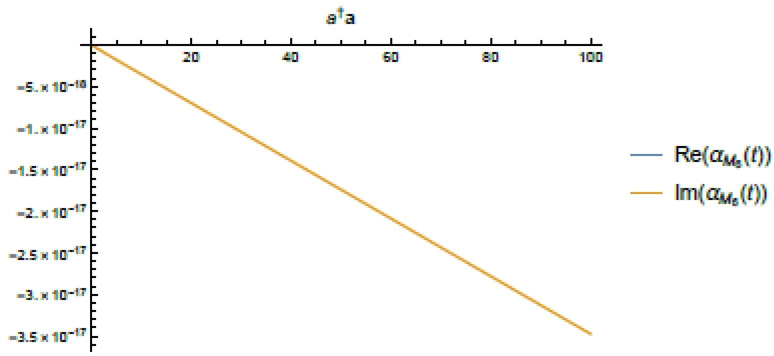


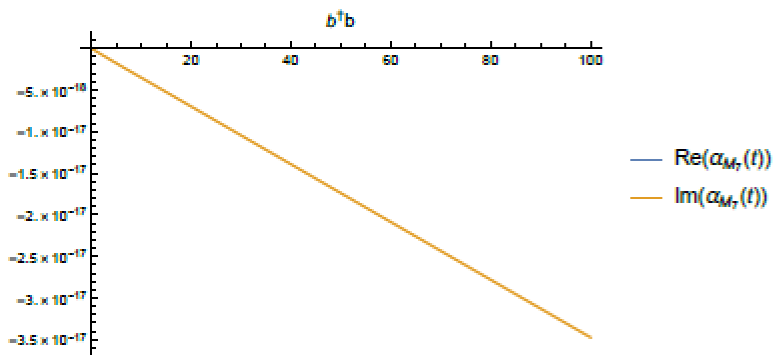
$a^{\dagger}b^{\dagger}$
6. × 10^-20
5. × 10^-20
4. × 10^-20
3. × 10^-20
2. × 10^-20
1. × 10^-20
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_3}(t))$
$\mathrm{Im}(\alpha_{M_3}(t))$

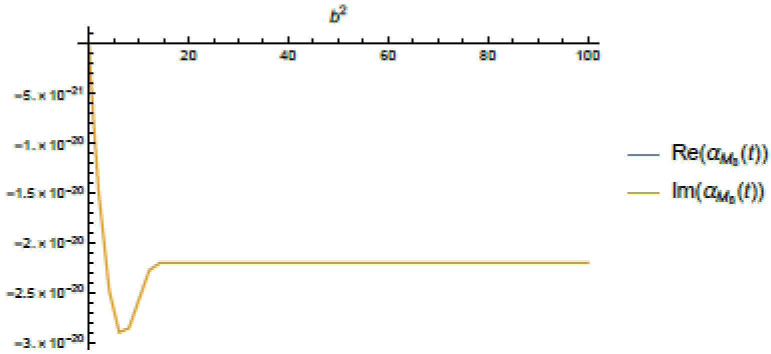


$ab^{\dagger}$
6. × 10^-20
5. × 10^-20
4. × 10^-20
3. × 10^-20
2. × 10^-20
1. × 10^-20
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_4}(t))$
$\mathrm{Im}(\alpha_{M_4}(t))$

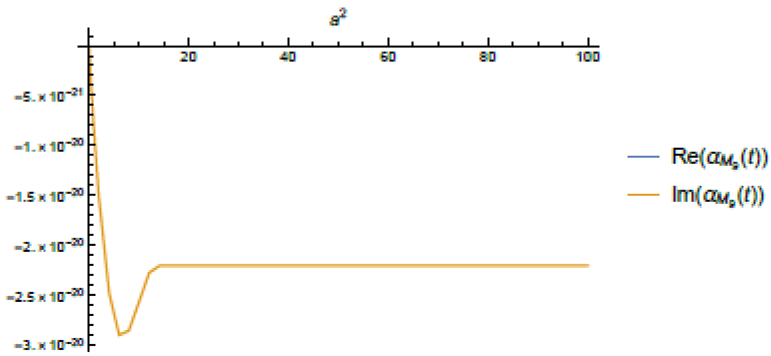

a†b
6.×10^-20
5.×10^-20
4.×10^-20
3.×10^-20
2.×10^-20
1.×10^-20
20
40
60
80
100
Re(α_M5(t))
Im(α_M5(t))


a†a
20
40
60
80
100
-5.×10^-18
-1.×10^-17
-1.5×10^-17
-2.×10^-17
-2.5×10^-17
-3.×10^-17
-3.5×10^-17
Re(α_M6(t))
Im(α_M6(t))

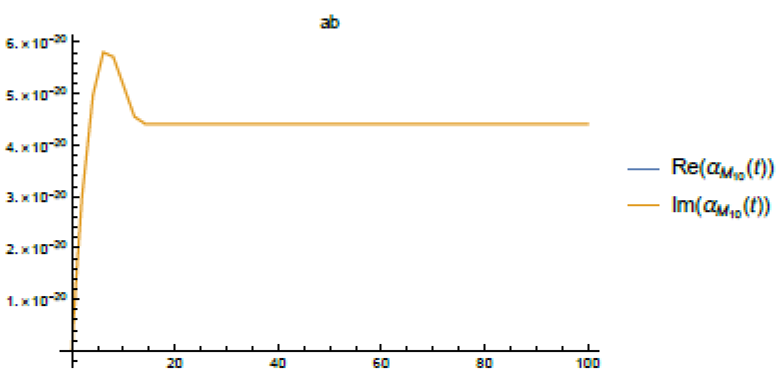

b†b
20
40
60
80
100
-5.×10^-18
-1.×10^-17
-1.5×10^-17
-2.×10^-17
-2.5×10^-17
-3.×10^-17
-3.5×10^-17
Re(α_M7(t))
Im(α_M7(t))



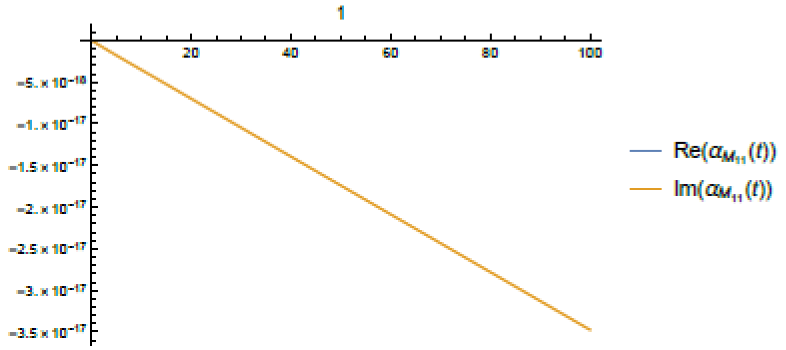
b^2
20
40
60
80
100
$-5.\times10^{-21}$
$-1.\times10^{-20}$
-1.5×10^{-20}
$-2.\times10^{-20}$
-2.5×10^{-20}
$-3.\times10^{-20}$
$\mathrm{Re}(\alpha_{M_8}(t))$
$\mathrm{Im}(\alpha_{M_8}(t))$



a^2
20
40
60
80
100
$-5.\times10^{-21}$
$-1.\times10^{-20}$
-1.5×10^{-20}
$-2.\times10^{-20}$
-2.5×10^{-20}
$-3.\times10^{-20}$
$\mathrm{Re}(\alpha_{M_9}(t))$
$\mathrm{Im}(\alpha_{M_9}(t))$



ab
$6.\times10^{-20}$
$5.\times10^{-20}$
$4.\times10^{-20}$
$3.\times10^{-20}$
$2.\times10^{-20}$
$1.\times10^{-20}$
20
40
60
80
100
$\mathrm{Re}(\alpha_{M_{10}}(t))$
$\mathrm{Im}(\alpha_{M_{10}}(t))$




```
uapp[t_] =
  Exp[αM11[t]] Exp[αM10[t] α β] Exp[αM6[t] α^2] Exp[αM8[t] β^2] Exp[- (1/2) β Conjugate[β]]
   Exp[- (1/2) α Conjugate[α]] Exp[(1/2) Exp[αM7[t] β] Conjugate[Exp[αM7[t] β] ]]
   Exp[(1/2) Exp[αM6[t] α] Conjugate [Exp[αM6[t] α]] ]
   (Exp[αM4[t] Exp[αM6[t]] α Conjugate[δ]] Exp[αM5[t] Conjugate[γ] Exp[αM7[t]] β] +
     Exp[αM4[t] Conjugate[δ] ] Exp[αM5[t] Exp[αM7[t]] β])
   Exp[αM3[t] Conjugate[γ] Conjugate[δ]] Exp[αM2[t] Conjugate[γ]^2]
   Exp[αM1[t] Conjugate[δ]^2];
u[t_] = uapp[t] /. {α → 0, β → X0 Sqrt[m w/ (2 ħ)]};
ψγ[x1_] = (m w/ (Pi ħ))^(1/4) Exp[-γ Conjugate[γ] /2]
   Exp[(x1 Sqrt[m w / ħ])^2/2] Exp[- (x1 Sqrt[m w / ħ] - γ/Sqrt[2])^2];
ψδ[x2_] = (m w/ (Pi ħ))^(1/4) Exp[-δ Conjugate[δ] /2]
   Exp[(x2 Sqrt[m w / ħ])^2/2] Exp[- (x2 Sqrt[m w / ħ] - δ/Sqrt[2])^2];
γα[t_] = Exp[1/2 (Conjugate[γ] Exp[αM6[t]] α - γ Conjugate[Exp[αM6[t]] α])]
    Exp[-1/2 (γ - Exp[αM6[t]] α) Conjugate[(γ - Exp[αM6[t]] α)]] /. {α → 0};
δβ[t_] = Exp[1/2 (Conjugate[δ] Exp[αM7[t]] β - δ Conjugate[Exp[αM7[t]] β])] Exp[
     -1/2 (δ - Exp[αM7[t]] β) Conjugate[(δ - Exp[αM7[t]] β)]] /. {β → X0 Sqrt[m w / (2 ħ)]};
total1[t_, x1_, x2_] = ψγ[x1] ψδ[x2] u[t] γα[t] δβ[t] 1/Pi^2;
X0 = 5.29 × 10^-11;
```

```
totalg[x1_, x2_] =
 Integrate[total1[t, x1, x2] /. {t → {0, 1, 5, 50, 100}}, {γ, -Infinity, Infinity}]
{2.30308 × 10^9 e^(-4.73162×10^18 x1^2 - 1.41949×10^19 x2^2 + 7.53521×10^9 x2 δ - 0.5 δ^2 + (0.199306 - 1. δ) Conjugate[δ]),
 e^((-4.73162×10^18 - 0.0520331 i) x1^2 - 1.41949×10^19 x2^2 + (7.53521×10^9 - 6.46235×10^-27 i) x2 δ - (0.5 - 1.50463×10^-36 i) δ^2 - (5.07836×10^-39 + 8.2476
   (((1.15154 × 10^9 - 4.22069 × 10^-10 i)
       e^((3.25092×10^-30 + 8.25766×10^-12 i) x1 + ((0.199306 - 7.24378×10^-30 i) + (2.5511×10^-29 + 4.1432×10^-11 i) x1 - (1. - 3.00927×10^-36 i) δ) Co
       ((1.69367 × 10^-49 + 4.36302 × 10^-31 i) + 1. x1 + (1.33586 × 10^-48 + 2.1891 × 10^-30 i)
         Conjugate[δ])) / ((1.69367 × 10^-49 + 4.36302 × 10^-31 i) +
       1. x1 + (1.33586 × 10^-48 + 2.1891 × 10^-30 i) Conjugate[δ]) +
    (e^(((0.199306 - 5.59425×10^-20 i) + (2.5511×10^-29 + 4.1432×10^-11 i) x1 - (1. - 3.00927×10^-36 i) δ) Conjugate[δ])
       ((1.15154 × 10^9 - 4.18283 × 10^-10 i) x1 +
         (2.45396 × 10^-39 + 2.52083 × 10^-21 i) Conjugate[δ])) /
     (1. x1 + (1.33586 × 10^-48 + 2.1891 × 10^-30 i) Conjugate[δ])),
 e^((-4.73162×10^18 - 0.173416 i) x1^2 - (1.41949×10^19 + 1.66533×10^-16 i) x2^2 + 7.53521×10^9 x2 δ - 0.5 δ^2 - (6.6795×10^-38 + 2.74879×10^-20 i) Conjugate[
   ((1.15154 × 10^9 - 2.06015 × 10^-9 i)
      e^(((0.199306 - 3.01732×10^-19 i) + (3.35543×10^-28 + 1.38885×10^-10 i) x1 - 1. δ) Conjugate[δ]) +
    (1.15154 × 10^9 - 2.07277 × 10^-9 i)
      e^((5.15553×10^-29 + 2.75212×10^-11 i) x1 + ((0.199306 - 3.56708×10^-19 i) + (3.35543×10^-28 + 1.38885×10^-10 i) x1 - 1. δ) Conjugate[δ])),
 e^((-4.73162×10^18 - 0.139083 i) x1^2 - 1.41949×10^19 x2^2 + (7.53521×10^9 + 1.29247×10^-26 i) x2 δ - (0.5 - 1.50463×10^-36 i) δ^2 - (7.54062×10^-37 + 2.20458
   ((1.15154 × 10^9 - 2.00267 × 10^-8 i)
      e^(((0.199306 - 3.42221×10^-18 i) + (3.78801×10^-27 + 1.10746×10^-10 i) x1 - (1. - 3.00927×10^-36 i) δ) Conjugate[δ]) +
    (1.15154 × 10^9 - 2.00368 × 10^-8 i)
      e^((3.85519×10^-28 + 2.20725×10^-11 i) x1 + ((0.199306 - 3.4663×10^-18 i) + (3.78801×10^-27 + 1.10746×10^-10 i) x1 - (1. - 3.00927×10^-36 i) δ) Conj
      ),
 e^((-4.73162×10^18 - 0.139083 i) x1^2 - 1.41949×10^19 x2^2 + (7.53521×10^9 + 1.29247×10^-26 i) x2 δ - (0.5 - 1.50463×10^-36 i) δ^2 - (1.51895×10^-36 + 2.20458
   ((1.15154 × 10^9 - 4.00032 × 10^-8 i)
      e^(((0.199306 - 6.87971×10^-18 i) + (7.6304×10^-27 + 1.10746×10^-10 i) x1 - (1. - 3.00927×10^-36 i) δ) Conjugate[δ]) +
    (1.15154 × 10^9 - 4.00133 × 10^-8 i)
      e^((7.68425×10^-28 + 2.20725×10^-11 i) x1 + ((0.199306 - 6.92381×10^-18 i) + (7.6304×10^-27 + 1.10746×10^-10 i) x1 - (1. - 3.00927×10^-36 i) δ) Conj
      )}
```

```
totalgx2[x1_] = Integrate[totalg[x1, x2], {x2, -Infinity, Infinity}]
```

```
{1.08347 e^(-4.73162×10^18 x1^2+0.5 δ^2+(0.199306-1. δ) Conjugate[δ]), (4.70446 × 10^-10 - 7.77877 × 10^-62 i)
  e^((-4.73162×10^18-0.0520331 i) x1^2+(0.5-2.10607×10^-37 i) δ^2-(5.07836×10^-39+8.24767×10^-21 i) Conjugate[δ]^2)
  (((1.15154 × 10^9 - 4.22069 × 10^-10 i)
        e^((3.25092×10^-30+8.25766×10^-12 i) x1+((0.199306-7.24378×10^-20 i)+(2.5511×10^-29+4.1432×10^-11 i) x1-(1.-3.00927×10^-36 i) δ) Co
        ((1.69367 × 10^-49 + 4.36302 × 10^-31 i) + 1. x1 + (1.33586 × 10^-48 + 2.1891 × 10^-30 i)
          Conjugate[δ])) / ((1.69367 × 10^-49 + 4.36302 × 10^-31 i) +
        1. x1 + (1.33586 × 10^-48 + 2.1891 × 10^-30 i) Conjugate[δ]) +
     (e^((0.199306-5.59425×10^-20 i)+(2.5511×10^-29+4.1432×10^-11 i) x1-(1.-3.00927×10^-36 i) δ) Conjugate[δ]
        ((1.15154 × 10^9 - 4.18283 × 10^-10 i) x1 +
          (2.45396 × 10^-39 + 2.52083 × 10^-21 i) Conjugate[δ])) /
      (1. x1 + (1.33586 × 10^-48 + 2.1891 × 10^-30 i) Conjugate[δ])),
 e^((-4.73162×10^18-0.173416 i) x1^2+(0.5-1.1732×10^-35 i) δ^2-(6.6795×10^-38+2.74879×10^-20 i) Conjugate[δ]^2)
  ((0.541736 - 9.69189 × 10^-19 i) e^((0.199306-3.01732×10^-19 i)+(3.35543×10^-28+1.38085×10^-10 i) x1-1. δ) Conjugate[δ] +
    (0.541736 - 9.75125 × 10^-19 i)
     e^((5.15553×10^-29+2.75212×10^-11 i) x1+((0.199306-3.56708×10^-19 i)+(3.35543×10^-28+1.38085×10^-10 i) x1-1. δ) Conjugate[δ])),
 e^((-4.73162×10^18-0.139083 i) x1^2+(0.5+4.93511×10^-36 i) δ^2-(7.54062×10^-37+2.20458×10^-20 i) Conjugate[δ]^2)
  ((0.541736 - 9.42145 × 10^-18 i)
     e^((0.199306-3.42221×10^-18 i)+(3.78801×10^-27+1.10746×10^-10 i) x1-(1.-3.00927×10^-36 i) δ) Conjugate[δ] +
    (0.541736 - 9.42621 × 10^-18 i)
     e^((3.85519×10^-28+2.20725×10^-11 i) x1+((0.199306-3.4663×10^-18 i)+(3.78801×10^-27+1.10746×10^-10 i) x1-(1.-3.00927×10^-36 i) δ) Conj
    ), e^((-4.73162×10^18-0.139083 i) x1^2+(0.5+4.93511×10^-36 i) δ^2-(1.51895×10^-36+2.20458×10^-20 i) Conjugate[δ]^2)
  ((0.541736 - 1.88193 × 10^-17 i)
     e^((0.199306-6.87971×10^-18 i)+(7.6304×10^-27+1.10746×10^-10 i) x1-(1.-3.00927×10^-36 i) δ) Conjugate[δ] +
    (0.541736 - 1.88241 × 10^-17 i)
     e^((7.68425×10^-28+2.20725×10^-11 i) x1+((0.199306-6.92381×10^-18 i)+(7.6304×10^-27+1.10746×10^-10 i) x1-(1.-3.00927×10^-36 i) δ) Conj
    )}
```

```
totalgx2d[x1_] = Integrate[totalgx2[x1], {δ, -Infinity, Infinity}]
```

$$\Big\{\left(2.77034 - 3.42483\times10^{-17}\,i\right) e^{-4.73162\times10^{18}\,x1^2},$$

$$\frac{1}{x1^2} e^{\left(-4.73162\times10^{18}-0.0520331\,i\right) x1^2} \Big(\left(-7.16533\times10^{-60} + 1.12892\times10^{-77}\,i\right) + \left(1.63077\times10^{-48} + 1.81306\times10^{-30}\,i\right) x1 + \left(1.38517 - 5.03148\times10^{-19}\,i\right) x1^2 + e^{\left(3.25092\times10^{-30}+8.25766\times10^{-12}\,i\right) x1} \Big(\left(-7.16533\times10^{-60} + 1.13128\times10^{-77}\,i\right) + \left(1.63673\times10^{-48} + 1.81306\times10^{-30}\,i\right) x1 + \left(1.38517 - 5.07701\times10^{-19}\,i\right) x1^2\Big)\Big),$$

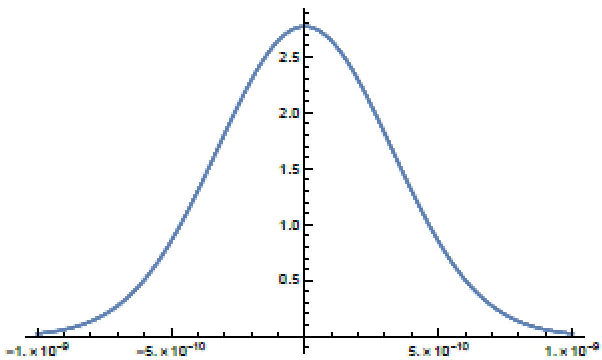
$$\left(1.38517 - 2.60102\times10^{-18}\,i\right) e^{\left(\left(1.10054\times10^{-28}+2.75212\times10^{-11}\,i\right)-\left(4.73162\times10^{18}+0.173416\,i\right) x1\right) x1} + \Big(\left(1.38517 - 2.63137\times10^{-18}\,i\right) e^{\left(\left(1.692\times10^{-28}+5.50424\times10^{-11}\,i\right)-\left(4.73162\times10^{18}+0.173416\,i\right) x1\right) x1} \left(\left(9.24088\times10^{-10} - 1.44336\times10^{9}\,i\right) + 1.\,x1\right)\Big) \Big/ \left(\left(9.24088\times10^{-10} - 1.44336\times10^{9}\,i\right) + 1.\,x1\right),$$

$$\left(1.38517 - 2.50664\times10^{-17}\,i\right) e^{\left(\left(1.13495\times10^{-27}+2.20725\times10^{-11}\,i\right)-\left(4.73162\times10^{18}+0.139083\,i\right) x1\right) x1} + \left(1.38517 - 2.50907\times10^{-17}\,i\right) e^{\left(\left(1.52535\times10^{-27}+4.41449\times10^{-11}\,i\right)-\left(4.73162\times10^{18}+0.139083\,i\right) x1\right) x1},$$

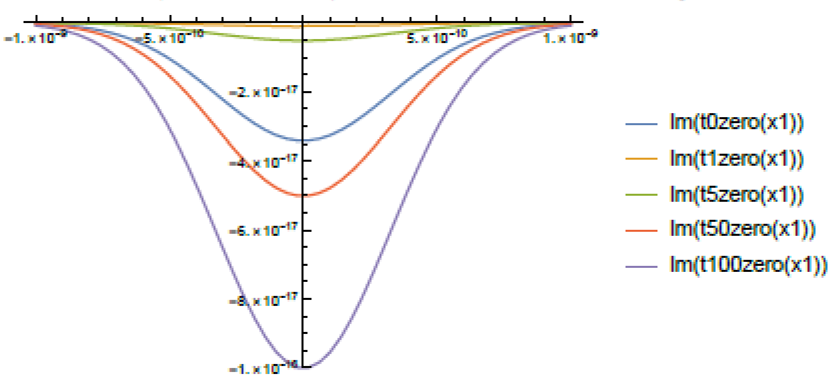
$$\left(1.38517 - 5.00504\times10^{-17}\,i\right) e^{\left(\left(2.28366\times10^{-27}+2.20725\times10^{-11}\,i\right)-\left(4.73162\times10^{18}+0.139083\,i\right) x1\right) x1} + \left(1.38517 - 5.00748\times10^{-17}\,i\right) e^{\left(\left(3.05697\times10^{-27}+4.41449\times10^{-11}\,i\right)-\left(4.73162\times10^{18}+0.139083\,i\right) x1\right) x1}\Big\}$$

```
t0zero[x1_] = totalgx2d[x1][[1]];
t1zero[x1_] = totalgx2d[x1][[2]];
t5zero[x1_] = totalgx2d[x1][[3]];
t50zero[x1_] = totalgx2d[x1][[4]];
t100zero[x1_] = totalgx2d[x1][[5]];
```

```
Plot[{Re[totalgx2d[x1]]}, {x1, -1×10^-9, 1×10^-9}]
```




```
Plot[{Im[t0zero[x1]], Im[t1zero[x1]], Im[t5zero[x1]],
  Im[t50zero[x1]], Im[t100zero[x1]]}, {x1, -1×10^-9, 1×10^-9},
 PlotRange → {-1×10^-16, 0}, PlotLegends → "Expressions"]
```



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