EXPLORATIONS IN QUANTUM RELATIVITY

by

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ABSTRACT

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We review some of the difficulties in merging quantum theory with relativity. In particular, we discuss the issue of localization in quantum mechanics. We introduce the conformal group, a supergroup of the Poincaré group and give its generators and corresponding algebra. We then illustrate how this allows us to construct a space-time localization operator that is consistent with special relativity and quantum theory.

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Chapter 1

Introduction

The goal of physics is to provide an elegant and accurate description of the physical world. This work attempts to contribute to this effort. In particular, we address the issue of merging the relativistic and quantum worlds. First we argue in favor of a unified description of physical phenomena. We then review some difficulties in merging quantum theory with special relativity. As a tool for achieving unification, we then introduce the transformation groups in physics. The chapter concludes with a summary of the thesis.

1.1 Motivation: Unification

In theoretical physics we attempt to describe all phenomena using a consistent theory. While there are aesthetic reasons for this (ie., a theory based on one equation is more elegant than one based on multiple equations) a unified theory can do much more than the two parent theories. As an example, consider the advent of electromagnetism. In the 19th century James Clerk Maxwell unified the theories of electricity and magnetism into one field theory, electromagnetism. Optics was a direct application of

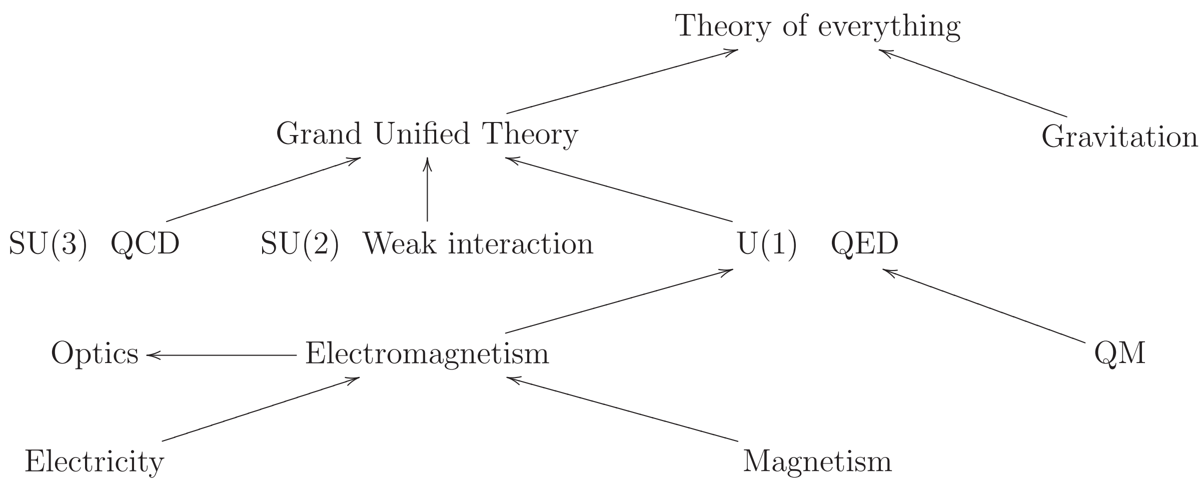

Figure 1.1 The unification of the interactions.

electromagnetism, which was previously a separate discipline. In 1905 Einstein unified electrodynamics with mechanics (in particular, the principle of relativity). Not all attempts at unification were successful, however. In 1915-1916 Albert Einstein published his general theory of relativity, a geometric theory of gravitation. Much of his research through the rest of his life dealt with combining his general theory with Maxwell's electromagnetism. At the time, these were the only fundamental forces that were known. Einstein had little success. Since the discovery of the weak and strong nuclear interactions, physicists have been searching for grand unified theories (GUTs), without validated results. Quantum field theory unifies electromagnetism and the weak interaction in what is known as the electro-weak theory. Further unification is the goal of much of today's research in theoretical physics.

1.2 Subject: Relativity versus quantum theory

Relativity (both special and general) and quantum theory are two well-established theories. Einstein formulated relativity in the early 20th century to further understand electromagnetism in different frames of reference. In this same period quantum theory explained some unexpected experimental results concerning the interaction of light and matter. These theories have been tested to very high precision. Despite this, there are reasons to believe that these theories do not give an exact description of reality. A complete theory of the universe should describe things at all scales, which is not currently the case for quantum theory and relativity. Quantum theory applies on the microscopic scale, whereas relativity has most of its application on a larger scale. There are further incompatibilities between both theories. One incompatibility is localization, the process of determining exactly where a particle is: quantum theory does not allow exact localization (as illustrated by the need for uncertainty relations), which is not the case in relativity. Another incompatibility is the interpretation of time, time in quantum mechanics is absolute, just as in Newtonian physics, contrasting time in relativity.

1.3 Tool: Transformation groups in physics

There are many transformation groups in physics. Many of these groups originate in the invariance properties of physical laws such as electrodynamics. They are often associated with a relativity principle. A property that is invariant under a group is a property that does not change under the action of the group.

The Lorentz group, or the orthogonal group, is the group of all origin-fixing isometries such as rotations. An isometry is an hyper-length preserving transformation. Mass is one example of a Lorentz invariant, the mass of the particle is unaffected from one frame to another.

The Poincaré group is a supergroup of the Lorentz group that also includes translations (both in time and in space). Mass is also an invariant in the Poincaré group.

The group that will be studied in detail in this thesis, the conformal group, was first described by Bateman and Cunningham [1] [2]. It has been shown to preserve Maxwell's equations. This group consists of all angle preserving transformations. One should note that angle is defined in a 4-dimensional space by the definition of the trigonometric functions as the generalized dot product. Mass is no longer an invariant in the conformal group, at least not in the usual sense.

1.4 Summary of thesis

In chapter 2, I state my thesis. Chapter 3 is devoted to the general mathematical and conceptual framework necessary to construct quantum relativity. It begins with a discussion of conformal symmetry and the conformal group. It then discusses the invariance properties of the conformal group. After that it presents the algebra of the conformal group. Then we discuss the problems of localization in quantum theory. In Appendix A the conventions used throughout the thesis are enumerated. In Appendix B we discuss the theory of groups. Appendix C collects some explicit calculations that have been omitted from the actual text to avoid distraction.

Chapter 2

Statement of the problem

Quantum theory and relativity are shown to be inconsistent, particularly in their treatment of localization. In this thesis we use symmetry and group theory to define localization in quantum theory. More specifically, we show that the conformal group, the largest group under which Maxwell's equations are invariant, can be used to define a consistent space and time localization operator.

Chapter 3

General background

In this chapter we have some important concepts needed to obtain the results given in chapter 4. We first discuss conformal symmetry and the associated algebra. The chapter concludes with discussions on Dirac theory and Zitterbewegung. Most of this material can be found separately in the literature.

3.1 Conformal symmetry

The word conformal means angle-preserving. Conformal transformations are transformations of space-time that preserve angles. Conformal symmetry characterizes systems that remain invariant under conformal transformations. Angles between vectors in this 4-dimensional space are defined as

$$\cos\theta = \frac{a^\mu b_\mu}{\|a\|\|b\|},$$

where $\|a\|^2 = a^\mu a_\mu$. There is an ambiguity in the sign of the angle. Conformal transformations that reverse the sign of the angle are improper[1], and cannot be

[1]The determinant of the Jacobian is negative, this terminology is in analogy with consideration of the Lorentz group, where transformations like time reversal are considered improper.

obtained continuously from the identity and we are not going to consider them any further in this thesis.

Spatial rotations, Lorentz boosts, and translations in space-time are all conformal transformations. Further conformal transformations include dilatations and so-called special conformal transformations. Dilatations are transformations of the form

$$x_\mu \mapsto \bar{x}_\mu = \rho x_\mu,$$

where ρ is any non-zero real number. Special conformal transformations are of the form

$$x_\mu \mapsto \bar{x}_\mu = \frac{x^\mu - c^\mu x^2}{1 - 2c^\rho x_\rho + c^2 x^2}$$

where c_μ are constant coefficients.

Dilatations and special conformal transformations have a physical realization. Dilatations are scale transformations, transformations that stretch space and time by some common factor. Special conformal transformations can be viewed as transformations to accelerated frames.

We can show that the physical world is dilatationally-invariant: if the whole universe were dilated, no experiment could be performed to show it. If lengths are all multiplied by some factor,

$$L \mapsto \lambda L,$$

masses also transform,

$$m^2 \mapsto \lambda^{-2} m^2,$$

as is shown in Sec. (3.5). If lengths all change by a common factor, the ruler I use for length measurement also changes by this same factor and I will obtain the same numerical value. Likewise, the mass scale will measure the same mass in the old and transformed systems. Thus dilatationally-invariant theories make sense, because measurement consists in taking the ratio of values to be measured with standard values.

3.2 The conformal group

The set of all conformal transformations forms a group, which we call $\mathrm{SO}(4,2)$. This notation indicates that one may represent the conformal group using 6×6 special orthogonal matrices with signature (4,2). The idea is that space-time can be represented by hexaspherical[2] coordinates forming a 6-dimensional space, which represent conformal transformations as rotations in this space. The metric for this space can be diagonalized with 4 positive and 2 negative entries.

The conformal group, $\mathrm{SO}(4,2)$, is the largest group under which Maxwell's equations are covariant. It consists of (global) dilatations, special conformal transformations (local dilatations), Poincaré transformations, and other non-restricted transformations. The different groups are represented in Fig. (3.1).

Dilatations together with the Poincaré group form the so-called similitude group. It is now shown that all members of the similitude group are angle preserving (and thus are members of the conformal group). To this end, I now state the following

[2] The hexaspherical coordinates, (y_μ, y_+, y_-) are related to the standard coordinates x_μ by the equations:

$$\begin{aligned} y_- + y_+ &= -\lambda \\ y_\mu &= \lambda x_\mu \\ y_+ - y_- &= \lambda x^2, \end{aligned}$$

where λ is called the conformal factor. λ is in general a function of x_μ and takes on particular values for specific types of transformations, e.g., $\lambda = 1-2x_\mu c^\mu + c^2 x^2$ for a special conformal transformation with parameters c_μ.

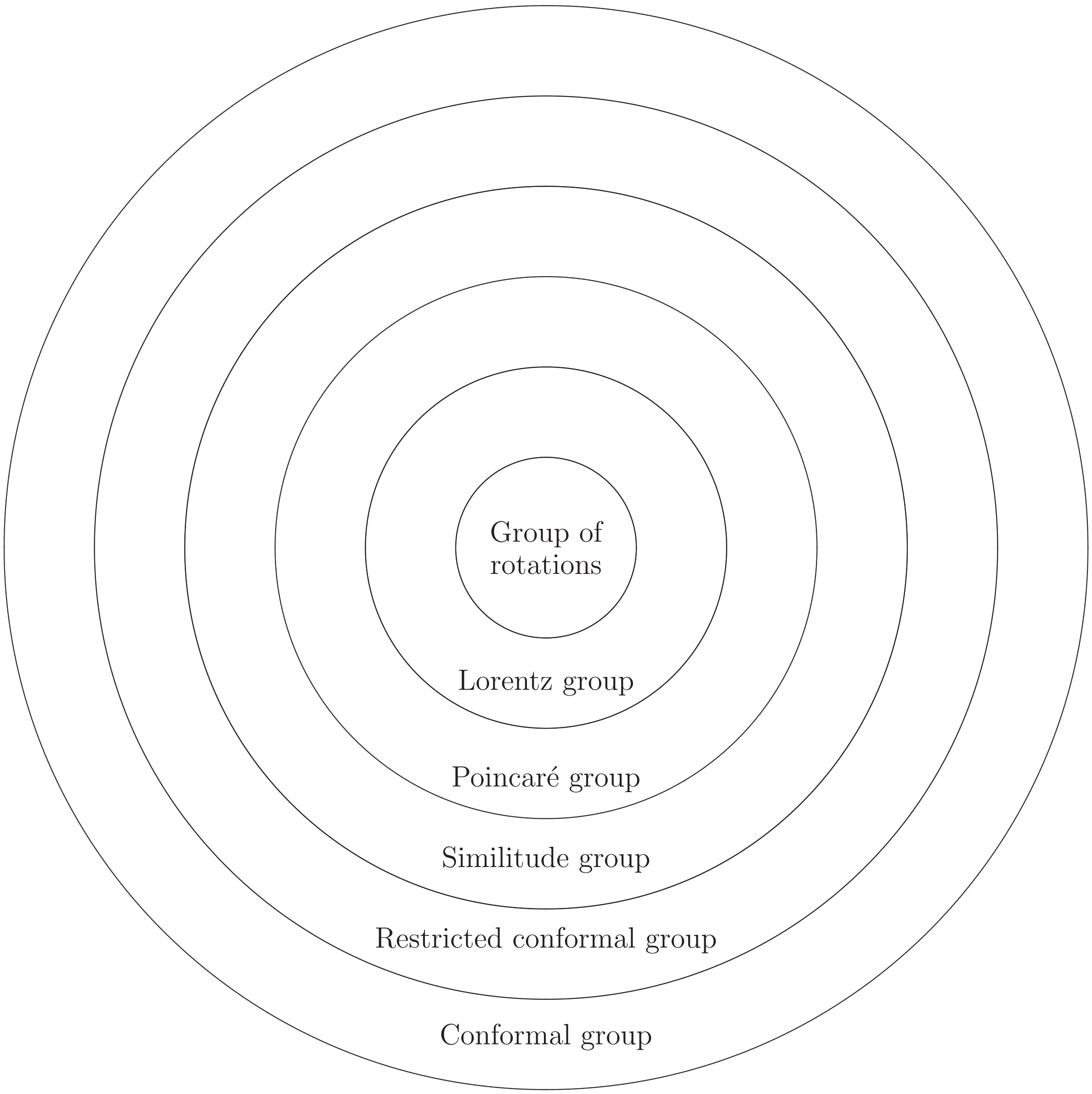

Figure 3.1 Subgroups of the conformal group.

proposition.

Proposition 3.2.1 *Let M(1,3) be the Minkowski space with signature -2. Let* ϕ : $\mathrm{M}(1,3) \mapsto \mathrm{M}(1,3)$. *Then* ϕ *is conformal (preserves angles) if the metric is transformed as* $\eta_{\mu\nu} \mapsto \bar{\eta}_{\mu\nu} = \Omega(\mathbf{x})\eta_{\mu\nu}$, *where* $\Omega(\mathbf{x})$ *is a scalar factor depending on* x.

The elements of the similitude group multiply the metric with a scalar factor. This shows that the similitude group is a subgroup of the conformal group.

3.3 Conformal generators

In this section we present the generators of the conformal group. They are explicitly derived in Appendix C up to an arbitrary scale factor. The generators of the Lorentz group, which includes rotations and boosts are the $J_{\mu\nu}$, the angular momentum operators,

$$J_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu). \tag{3.1}$$

The equality between the generators of the Lorentz group and the angular momentum operators is discussed in most quantum mechanics textbooks [3].

The generators of translations and the momentum operators, P_μ, are given by

$$P_\mu = -i\partial_\mu.$$

That the generators of translation are the momentum operators is again discussed in quantum texts. Together these ten generators form a basis for the Poincaré algebra and generate the Poincaré group. We now give the additional generators needed to extend the Poincaré group to the restricted conformal group. Dilatations are generated by D where

$$D = -x^\nu \partial_\nu, \tag{3.2}$$

whereas the four generators of the special conformal transformations are

$$C_\mu = -i(x^2\partial_\mu - 2x_\mu x_\nu \partial^\nu). \tag{3.3}$$

We now have the 15 generators of the restricted conformal group $J^{\mu\nu}, P^\mu, C^\mu$, and D.

3.4 Conformal algebra

The conformal algebra, so$(4, 2)$, consists of all linear combinations of the generators given in the previous section. The structure of the algebra is specified by it's commutation relations, which are presented here.

We begin with the Poincaré algebra

$$(P^\mu, P^\nu) = 0 \tag{3.4}$$

$$(J_{\mu\nu}, P_\rho) = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \tag{3.5}$$

$$(J_{\mu\nu}, J_{\sigma\rho}) = \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho}. \tag{3.6}$$

We now add the remaining generators of the restricted conformal group: D and C_μ.

$$\begin{aligned}
(D, D) &= 0 \\
(D, P^\nu) &= P^\nu \\
(D, J_{\mu\nu}) &= 0 \\
(D, C_\mu) &= -C_\mu \\
(C_\mu, C_\nu) &= 0 \\
(J_{\mu\nu}, C_\rho) &= \eta_{\nu\rho} C_\mu - \eta_{\mu\rho} C_\nu \\
(P_\mu, C_\nu) &= -2\eta_{\mu\nu} D - 2J_{\mu\nu}
\end{aligned} \tag{3.7}$$

We see from these commutation relations that the algebra satisfies closure.

3.5 Invariance

In this section we discuss many of the invariants of the conformal group. We also discuss mass non-invariance under conformal transformations.

Properties that identify particles such as charge, color charge, etc. are some few examples of invariants of the conformal group. Elements of the conformal group are said to satisfy Weyl invariance, ie.,

$$\eta^{\mu\nu} \mapsto \bar{\eta}^{\mu\nu} = \Omega(\mathbf{x})\eta^{\mu\nu},$$

which, as has been stated, is equivalent to conformal invariance.

To understand how mass transforms under the action of the conformal group we first need the Baker-Campbell-Hausdorff relation [4].

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] \ldots \tag{3.8}$$

First, we must show how mass transforms under a dilatation ,

$$x^{\mu} \mapsto \bar{x}^{\mu} = \lambda x^{\mu}.$$

We know that $P_{\mu} = i\frac{\partial}{\partial x^{\mu}}$ transforms to $\lambda^{-1}P_{\mu}$. Using $P_{\mu}P^{\mu} = m^2$ yields

$$m^2 \mapsto \lambda^{-2}m^2.$$

This approach gets messy when we apply it to special conformal transformations. Instead we use the Campbell-Hausdorff relation with $A = -i\alpha D$, and $B = P^{\mu}P_{\mu}$. From Eq. (3.7) we know that

$$[B, A] = 2\alpha B.$$

Therefore,

$$e^{i\alpha D}P_{\mu}P^{\mu}e^{-i\alpha D} = P^{\mu}P_{\mu}\left(1 + 2\alpha + \frac{1}{2!}(2\alpha)^2 + \ldots\right) = e^{2\alpha}P_{\mu}P^{\mu} = \lambda^{-2}m^2.$$

We can derive a similar expression for the special conformal transformations,

$$m^2 \mapsto \sigma(x)^2 m^2,$$

where $\sigma(x)$ is given by $1 - 2c_\mu x^\mu + c^2 x^2$.

3.6 Dirac's relativistic electron theory

In this section we derive the Dirac equation and we discuss some of its properties. We write a Dirac-like equation using quantum relativity in the next chapter and we use the results derived here to analyze the new equation. To obtain the Dirac equation we start with Einstein's expression for classical momenta, p_μ,

$$p^\mu p_\mu = m^2. \tag{3.9}$$

Now, in quantum theory the momentum operators, P_μ, can be represented as

$$P^\mu = i\partial^\mu, \tag{3.10}$$

which we insert into the quantum version of Eq. (3.9) and operate on a wavefunction to obtain the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\psi(x) = 0. \tag{3.11}$$

This equation is useful for describing scalar bosons, but it is not sufficient for particles with nonzero spin. Dirac takes quite a different approach. He begins with Eq. (3.9), quantizes, and assumes the factorization:

$$(\gamma_\mu P^\mu - m)(\alpha^\nu P_\nu + m) = 0. \tag{3.12}$$

Where γ^μ and α^μ have yet to be determined. Expanding Eq. (3.12) we obtain

$$\gamma_\mu \alpha^\nu P^\mu P_\nu + m(\gamma_\sigma P^\sigma - \alpha^\sigma P_\sigma) - m^2 = P^\mu P_\mu - m^2,$$

which implies that $\gamma^\nu = \alpha^\nu$, and thus since $\gamma_\mu \gamma^\nu P^\mu P_\nu = P^\mu P_\mu$, we get

$$\gamma_\mu \cdot \gamma^\nu = g_\mu^\nu. \tag{3.13}$$

Combining Eq. (3.10) with Eq. (3.12) we obtain

$$(i\gamma^\mu \partial_\mu - m)(i\gamma^\mu \partial_\mu + m)\psi = 0. \tag{3.14}$$

Eq. (3.13) implies that there are 4 gamma symbols. Appendix A gives an explicit representation of the gamma symbols as 4×4 matrices.

Factorization of a quadratic expression as $(x + a)(x + b) = 0$, has two solutions $x = -a$ and $x = -b$, which are also solutions of $(x + a)$, and $(x + b)$ respectively. Similarly, Eq. (3.14) has two classes of solutions, solutions that satisfy

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

and ones that satisfy

$$(i\gamma^\mu \partial_\mu + m)\psi = 0.$$

States that satisfy the first expression are interpreted to be particles, those that satisfy the second, antiparticles. So for particles the Dirac equation becomes,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{3.15}$$

3.7 Zitterbewegung

In this section we derive an effect first predicted by Schrödinger in 1930 [5]. This effect, called Zitterbewegung, predicts that the free electron oscillates in a microscopic helix.

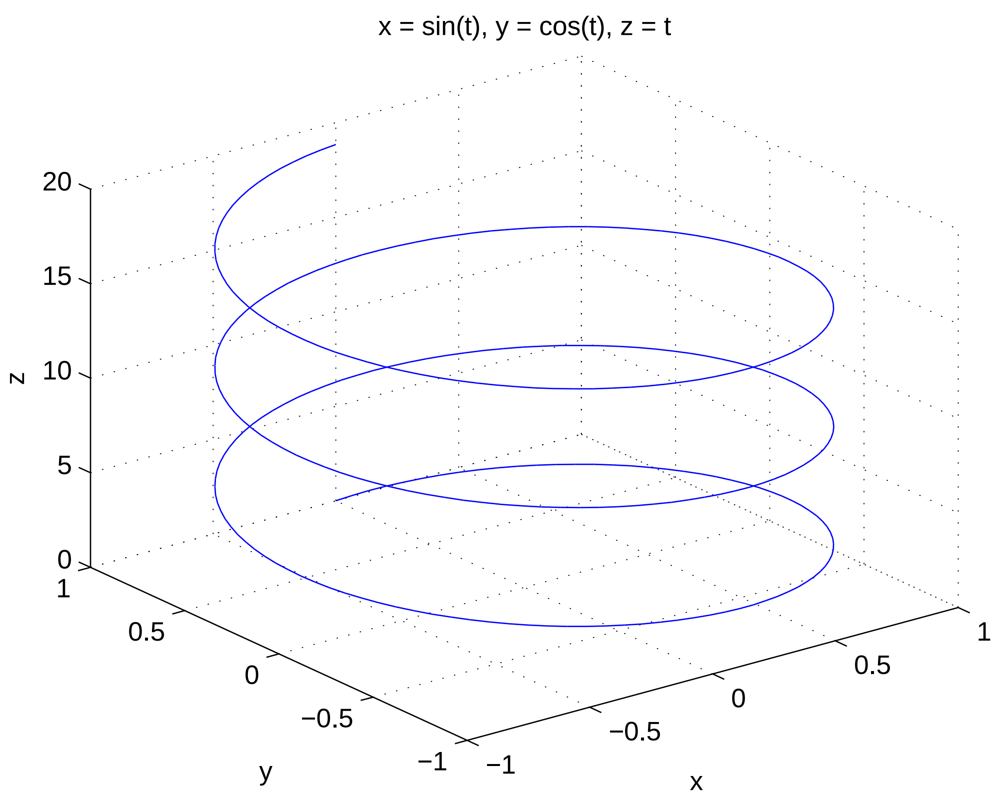

Figure 3.2 Helical motion in Zitterbewegung.

We use standard Dirac theory, and begin with the free Dirac equation written in a form similar to the Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = \mathcal{H}\psi,$$

with $\mathcal{H} = \gamma^0 m + \alpha_i p^i$, and $\alpha_k = \gamma_0\gamma_k$. We cite the Heisenberg equation for the evolution of an operator in the Heisenberg picture,

$$-i\frac{dA}{dt} = [\mathcal{H}, A]. \tag{3.16}$$

First of all[3], $[x_i, p_k] = i\delta_{ik}$ gives the shift of position with the Hamiltonian,

$$[\mathcal{H}, x_k] = -i\alpha_k,$$

which implies

$$\frac{dx_k}{dt} = \alpha_k. \tag{3.17}$$

In particular, this gives

$$\left(\frac{dx_k}{dt}\right)^2 = 1$$

which implies that at every instant an electron will be measured to travel at the speed of light. The expectation value of the velocity of the electron will however be less, and will not violate special relativity. Now, it is straightforward to verify from our definition of the Hamiltonian that

$$\mathcal{H}\alpha_k + \alpha_k\mathcal{H} = 2p_k.$$

Using this expression and Eq. (3.16) we get

$$-i\frac{d\alpha_k}{dt} = \mathcal{H}\alpha_k - \alpha_k\mathcal{H} = 2(p_k - \alpha_k\mathcal{H}) = 2(\mathcal{H}\alpha_k - p_k). \tag{3.18}$$

Making the substitution,

$$\eta_k = \alpha_k - \mathcal{H}^{-1}p_k, \tag{3.19}$$

[3] In this section we drop the condition that observables are written as capital letters

we obtain the differential equation

$$-i\frac{d\eta_k}{dt} = -2\eta_k\mathcal{H} = 2\mathcal{H}\eta_k$$

(since $\mathcal{H}$ and p_k are constants of the motion). Now solving for η_k,

$$\eta_k = \eta_k^o e^{2i\mathcal{H}t} = e^{-2i\mathcal{H}t}\eta_k^o. \tag{3.20}$$

Where η_k^o, is the exponentiated integration constant[4]. Combining Eq. (3.17), Eq. (3.19), and Eq. (3.20) we obtain the following differential equation,

$$\frac{dx_k}{dt} = \mathcal{H}^{-1}p_k + \eta_k^o e^{2i\mathcal{H}t}$$

which gives the solution,

$$x_k = x_k^0 + \mathcal{H}^{-1}p_k t - \frac{i}{2}\eta_k^o\mathcal{H}^{-1}e^{2i\mathcal{H}t},$$

which we decompose in to two parts: $\tilde{x}_k$ and ξ_k.

$$\begin{aligned} x_k &= \tilde{x}_k + \xi_k \\ \tilde{x}_k &= \alpha_k + \mathcal{H}^{-1}p_k t \\ \xi_k &= -\frac{i}{2}\eta_k^o\mathcal{H}^{-1}e^{2i\mathcal{H}t} = -\frac{i}{2}\eta\mathcal{H}^{-1} = \frac{i}{2}\mathcal{H}^{-1}\eta_k. \end{aligned} \tag{3.21}$$

$\tilde{x}_k$ corresponds to the classical expression for a free particle,

$$x(t) = x_0 + vt,$$

whereas ξ_k is an unexpected term, the Zitterbewegung.

Eq. (3.21) predicts that the free electron moves in a helical trajectory with angular frequency

$$\omega_0 = \frac{2E_0}{\hbar} = \frac{2mc^2}{\hbar} \approx 1.55 \times 10^{21}\text{s}^{-1},$$

[4] This constant is an operator!

and with radius

$$\lambda_0 = c/\omega_0 = \frac{\hbar}{2mc} \approx 1.93 \times 10^{-13}\mathrm{m}$$

(compare with Compton wavelength). This result of relativistic electron theory illustrates the difficulty of localizing an electron within the Compton wavelength. An illustration of Zitterbewegung is given in Fig. (3.2).

Chapter 4

Results and conclusions

This chapter contains, a derivation of the space-time localization operator X_ν and some possible applications of quantum relativity.

4.1 Space-time localization in M(1,1)

We begin by deriving an expression for the space-time localization operator. The idea in this chapter is to intersect multiple propagating photons. The directions that the photons propogate in satisfy certain conditions which have yet to be determined. We define position in space-time as the point of intersection. By the invariance of the speed of light we can define light-cone coordinates that are invariant for each photon.

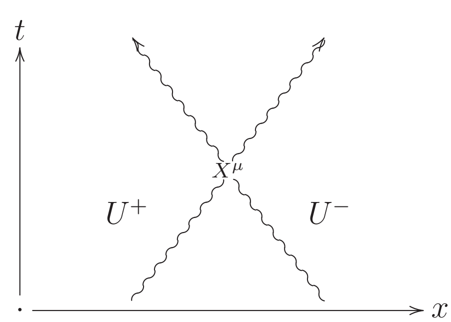

Figure 4.1 Light-cone coordinates defining location.

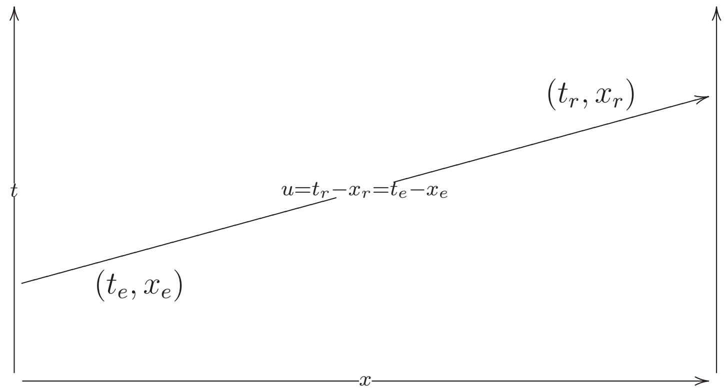

Figure 4.2 The light cone coordinate for one photon

For example, if the photon is propagating in the positive x direction, the coordinate $t-x$ is invariant on it's worldline.

We begin with Minkowski 2-space, M(1,1), with one spatial dimension and one temporal dimension. In this space we characterize position through the intersection of two photons. We can construct the light-cone coordinate, $u = t - x$. So say we have a light source that emits at time t_e and at a position x_e, and say we detect the light at time t_r and position x_r respectively. We can see that this coordinate is equal for both receiver and emitter:

$$u = t_e - x_e = t_r - x_r.$$

Now we may define position in this two-dimensional space-time in the following way. Let two rays of light, forward and reverse propagating beams, intersect at a point in space-time. Let u^+ and u^- be their respective light-cone coordinates:

$$u^+ = t - x, \quad u^- = t + x.$$

Then we have,

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \frac{u^+ + u^-}{2} \\ \frac{u^- - u^+}{2} \end{pmatrix}.$$

In our analysis thus far, we have not used any quantum notion. From quantum field theory elements of the conformal algebra can be defined in terms of integrals [6]:

$$E = \int du\ e(u) \tag{4.1}$$
$$D = \int du\ u\ e(u) \tag{4.2}$$
$$C = \int du\ u^2 e(u) \tag{4.3}$$

where $e(u)$ is the energy density. Looking at Eq. (4.1) and Eq. (4.2) we define the operator[1],

$$U \equiv D \cdot \frac{1}{E},$$

in analogy to u.

We can now define two collections of conformal generators, ie., $E^+, D^+, \ldots,$ and $E^-, D^-, \ldots$ which give two light-cone coordinates:

$$U^+ = \frac{D^+}{E^+} \tag{4.4}$$
$$U^- = \frac{D^-}{E^-} \tag{4.5}$$

Let us derive relations between all these operators. In particular, we must connect these operators that act on the single photon field states (labeled with $+$ and $-$) with their more general counterparts. The energy operator is[2]

$$E = \frac{1}{2}(E^+ + E^-)$$

[1]From this point through Sec. 4.2 all products are symmetrized.

[2]One should note that the operators which act on the forward propagating photon and those that act on the reverse operating photon are in different Hilbert spaces. The notation used here is slightly sloppy, we should for example write $E = \frac{1}{2}(E^+ \otimes \mathbf{1} + \mathbf{1} \otimes E^-)$. The notation, however, does not lead to inconsistencies.

the average of the energy operators of the forward and reverse propagating photons. We know that the momenta of both photons are $(P^+, P^-) = (E^+, -E^-)$, and the momentum operator for the whole field state is the average of these operators,

$$P = \frac{1}{2}(E^+ - E^-).$$

In like fashion,

$$D = \frac{1}{2}(D^+ + D^-).$$

Determining the form of the generator of boosts requires Eq. (3.1)

$$J^{01} = X^0 {\cdot} P^1 - X^1 {\cdot} P^0 = X^0 {\cdot} P - X^1 {\cdot} E = \frac{1}{4}[(U^- + U^-) {\cdot} (E^+ - E^-) - (U^- - U^+) {\cdot} (E^+ + E^-)]$$

$$= \frac{1}{2}(U^+ E^+ - U^- E^-) = \frac{D^+ - D^-}{2} \equiv J.$$

$$J = \frac{D^+ - D^-}{2} \tag{4.6}$$

Furthermore,

$$E^2 - P^2 = \frac{1}{4}[(E^+)^2 + 2E^+E^- + (E^-)^2 - ((E^+)^2 - 2E^+E^- + (E^-)^2)] = E^+E^-.$$

Finally, we can define a quantum space-time position operator for this toy model:

$$\begin{pmatrix} X^0 \\ X^1 \end{pmatrix} = \begin{pmatrix} \frac{D^+E^- + D^-E^+}{2E^+E^-} \\ \frac{D^+E^- - D^-E^+}{2E^+E^-} \end{pmatrix}.$$

This can be factorized as:

$$\begin{pmatrix} \frac{1}{4E^+E^-}[(D^+ + D^-)(E^+ + E^-) - (E^+ - E^-)(D^+ - D^-)] \\ \frac{1}{4E^+E^-}[(D^+ + D^-)(-E^+ + E^-) + (E^+ + E^-)(D^+ - D^-)] \end{pmatrix}$$

Inserting the definitions for the total operators, we obtain

$$\begin{pmatrix} \frac{DE - PJ}{E^2 - P^2} \\ \frac{D(-P) - (P)(-J)}{E^2 - P^2} \end{pmatrix}$$

A M(1,3) generalization is given in Eq. (4.20).

The relationship Eq. (4.6) may seem puzzling. The following geometric description may clear it up. Let the parameter α be taken to be $\log\lambda$, where λ is any positive real number. Any arbitrary position is transformed by J in the following way.

$$\begin{aligned} t-x &\mapsto \lambda(t-x) && (4.7)\\ t+x &\mapsto \frac{1}{\lambda}(t+x) && (4.8)\\ & && (4.9)\end{aligned}$$

From this we obtain

$$\begin{array}{ccccc} t & \mapsto & \bar{t} & = & \frac{1+\lambda^2}{2\lambda}t+\frac{1-\lambda^2}{2\lambda}x \\ x & \mapsto & \bar{x} & = & \frac{1-\lambda^2}{2\lambda}t+\frac{1+\lambda^2}{2\lambda}x \end{array}.$$

We see that the rapidity, ϑ, satisfies the following identities,

$$\begin{aligned} \cosh\vartheta &= \frac{1+\lambda^2}{2\lambda} && (4.10)\\ \sinh\vartheta &= \frac{1-\lambda^2}{2\lambda}, && (4.11)\end{aligned}$$

and

$$\cosh^2\vartheta-\sinh^2\vartheta=1 \qquad (4.12)$$

We can also see that the expression t^2-x^2 is invariant,

$$\bar{t}^2-\bar{x}^2=t^2-x^2. \qquad (4.13)$$

which implies that J generates the orthogonal group $\mathrm{O}(1,1)$.

4.2 Sketch of localization in M(1,2) and M(1,3)

The problem of generalizing quantum relativity to M(1,2) and M(1,3) is non-trivial. We show only a sketch of how we think it should be done.

We first examine the case of M(1,2). We see that in order to have a solution to a system of equations, we need at least three equations (and thus three photons[3]). Also, the third equation cannot be obtained from a linear combination of the other two. We make the following choice of light-cone coordinates.

$$\begin{pmatrix} \sigma^+ \\ \sigma^- \\ \sigma^* \end{pmatrix} = \begin{pmatrix} x^0 - x^1 \\ x^0 + x^1 \\ x^0 - x^2 \end{pmatrix} \tag{4.14}$$

This equation has solution

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sigma^+ + \sigma^- \\ \sigma^- - \sigma^+ \\ \sigma^+ + \sigma^- - 2\sigma^* \end{pmatrix} \tag{4.15}$$

As before, we define the light-cone operator

$$\Sigma = \frac{D}{E}.$$

We define the position operators in analogy to Eq. (4.15).

$$\begin{pmatrix} X^0 \\ X^1 \\ X^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Sigma^+ + \Sigma^- \\ \Sigma^- - \Sigma^+ \\ \Sigma^+ + \Sigma^- - 2\Sigma^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{D^+E^- + D^-E^+}{E^-E^+} \\ \frac{D^-E^+ - D^+E^-}{E^-E^+} \\ \frac{D^+E^- + D^-E^+}{E^+E^-} - 2\frac{D^*E^+E^-}{E^*} \end{pmatrix} \tag{4.16}$$

As before, we need to find relations between these operators.

$$P^0 = E = \frac{1}{2}(E^+ + E^-) = \frac{1}{2}(\tilde{E} + E^*),$$

where we have associated the symbol $\tilde{E}$ with the energy of the reverse propagating photon in the x^2 direction.

$$P^1 = \frac{1}{2}(E^+ - E^-)$$

[3]It is unimportant to us how the photons were made to intersect.

$$P^2 = \frac{E^* - \tilde{E}}{2} = \frac{2E^* - E^+ - E^-}{2}$$
$$D = \frac{D^+ + D^-}{2} = \frac{\tilde{D} + D^*}{2}.$$

To determine the generators of the Lorentz group, $\mathrm{O}(2,1)$, we proceed as before:

$$J^{01} = J = \frac{D^+ - D^-}{2}$$

$$J^{02} = X^0 P^2 - P^0 X^2 = \frac{1}{4}[(\Sigma^+ + \Sigma^-)(2E^* - E^+ - E^-) - (E^+ + E^-)(\Sigma^+ + \Sigma^- - 2\Sigma^*)]$$
$$= \frac{1}{4}(2E^*(\Sigma^+ + \Sigma^-)(E^+ + E^-) + 2\Sigma^*(E^+ + E^-))$$

$$J^{12} = X^1 P^2 - X^2 P^1$$

$$= (\frac{D^- E^+ - D^+ E^-}{E^- E^+})(\frac{2E^* - E^+ - E^-}{2}) - (\frac{D^+ E^- + D^- E^+}{E^+ E^-} - 2\frac{D^* E^+ E^-}{E^*})(\frac{1}{2}(E^+ - E^-))$$

etc. The challenge is to simplify these expressions. We will not examine M(1,2) any further in this work.

Now we move to the M(1,3), the standard Minkowski space-time. In order to define position in this four dimensional space, we need at least four equations, which come from four light-cone coordinates. We therefore must have four non-parallel photons. These photons could travel in the +z,-z,+x,+y directions respectively. They admit the following variables,

$$\begin{pmatrix} \sigma^+ \\ \sigma^- \\ \sigma^1 \\ \sigma^2 \end{pmatrix} = \begin{pmatrix} x^0 - x^3 \\ x^0 + x^3 \\ x^0 - x^1 \\ x^0 - x^2 \end{pmatrix} \tag{4.17}$$

where x^0, x^1, x^2, x^3 are at the point of intersection. Eq. (4.17) can be solved for

x^0, x^1, x^2, x^3 :

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sigma^+ + \sigma^- \\ \sigma^+ + \sigma^- - 2\sigma^1 \\ \sigma^+ + \sigma^- - 2\sigma^2 \\ \sigma^- - \sigma^+ \end{pmatrix} \tag{4.18}$$

As before, we define the light-cone coordinate

$$\Sigma = \frac{D}{E}.$$

Using this definition, we define position in analogy to Eq. (4.18):

$$\begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Sigma^+ + \Sigma^- \\ \Sigma^+ + \Sigma^- - 2\Sigma^1 \\ \Sigma^+ + \Sigma^- - 2\Sigma^2 \\ \Sigma^- - \Sigma^+ \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{D^+E^- + D^-E^+}{E^-E^+} \\ \frac{D^+E^-E^1 + D^-E^+E^1 - 2D^1E^+E^-}{E^1E^+E^-} \\ \frac{D^+E^-E^2 + D^-E^+E^2 - 2D^2E^+E^-}{E^2E^+E^-} \\ \frac{D^-E^+ - D^+E^-}{E^-E^+} \end{pmatrix} \tag{4.19}$$

Jaekel and Reynaud state the following definition of space-time which is a generalization of what we have derived explicitly in the (1,1) case and which should reduce to our expressions in M(1,2) and M(1,3) as well [7] [8]

$$X^\rho = \frac{P^\rho \cdot D - P_\sigma \cdot J^{\rho\sigma}}{P^2}. \tag{4.20}$$

4.3 Dirac theory in quantum relativity

Using our quantum algebraic methods we can derive a Dirac-like equation and a corresponding algebra. To begin with we introduce the Pauli-Lubanski vector, W_μ,

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma.$$

We now define a new mass operator M, given first in a Klein-Gordon-like equation

$$M^2 = P^\sigma P_\sigma - 2\gamma W_\nu P^\nu. \tag{4.21}$$

We can see that this expression yields the more familiar $P^\sigma P_\sigma = M^2$, since the term $P_\mu \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$ contains $P^\mu P_\sigma$ which is symmetric in μ and σ and $\epsilon^{\mu\nu\rho\sigma}$ which is antisymmetric in μ and σ. We now factorize,

$$M^2 = P^2 - 2\gamma W_\mu P^\mu = (P_\mu - 2\gamma W_\mu) P^\mu. \tag{4.22}$$

We now define

$$\gamma_\mu = \frac{P_\mu - 2\gamma W_\mu}{M}, \tag{4.23}$$

and substitute in Eq. (4.22) to obtain a Dirac-like equation

$$M = P_\mu \gamma^\mu. \tag{4.24}$$

Which is a Dirac-like equation. The gamma symbols Eq. (4.23) span the algebra $C\ell_{1,3}(\mathbb{C})$, the Dirac algebra, as is given by the relation [9]

$$\gamma_\mu \cdot \gamma^\nu = \eta_\mu^\nu$$

4.4 Zitterbewegung in quantum relativity

When discussing Zitterbewegung in quantum relativity we can use the Dirac equation Eq. (4.24)and essentially repeat the derivation given in Sec. 3.7 leading to Heisenberg operators $X^\mu = \tilde{X}^\mu + \Xi^\mu$. This shows that at the level of first quantization the issues related to Zitterbewegung are being reproduced in quantum relativity. A discussion of Zitterbewegung using quantum relativity and QFT(second quantization) is given by Jaekel and Reynaud in [9].

4.5 Conclusions

The purpose of this thesis consisted in contributing to unification in physics and in particular to the unification of quantum mechanics with relativity. More specifically, I have considered the issue of space-time localization in quantum theory. I have done this using the conformal group. This group has the attractive property of being sufficiently restricted that it leaves Maxwell's equations invariant but rich enough to allow the construction of space-time localization operators from its generators. I explored its invariance and algebraic properties. In the end I was able to give justification to the definition of space-time first proposed by Jaekel and Reynaud.

My work is one small piece in a much larger unification effort in theoretical physics. One advantage of quantum relativity is its ability to define physical quantities from the fundamental concepts of symmetry and observable phenomena reminiscent of Einstein's approach to special relativity. A challenge is the increasing complexity of the calculations involved. It remains to be seen how quantum relativity can be productively incorporated in the general unification effort.

Appendix A

Conventions

1. Unless explicitly stated otherwise we use natural units with $\hbar = 1 = c$.

2. Observables are denoted with capital letters, while variables are referred to by lower case letters.

3. The usual metric signature, $[+, -, -, -]$, is used unless stated otherwise.

4. Contravariant and covariant indices are denoted by superscript and subscript letters.

5. $$(A, B) = \frac{1}{i\hbar}[A, B]$$

   is the modified commutator.

6. $$A \cdot B = \frac{AB + BA}{2}$$

   is the symmetrized product.

7. Einstein summation notation is used, where $t^\mu s_\mu$ denotes the sum

   $$t^0 s_0 + t^1 s_1 + t^2 s_2 + t^3 s_3.$$

8. States transform under G with parameter α as

$$G : |\psi> \mapsto |\overline{\psi}> = e^{\alpha G}|\psi>$$

9. Dirac gamma matrices are expressed in the standard representation.
$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

10. The Pauli matrices are also represented in the standard representation.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

11. Greek indices run from 0 to 3 while Latin indices run from 1 to 3.

Appendix B

Group Theory

The purpose of this appendix is to define a group and give the basic properties of a group. It is a quick overview of group theory, and is not completely mathematically precise. The group is among the most essential algebraic structures used in physics [10].

Definition A *group* is a set X endowed with a product $\cdot$, which is written $(X, \cdot)$ with the following properties:

1. Elements of the set satisfy closure, $x \cdot y \in X, \quad \forall x, y \in X$.

2. Elements of the set satisfy associativity, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

3. The set contains a unique identity element, e with $x \cdot e = e \cdot x = x, \forall x \in X$.

4. Elements of the set have a unique inverse, $\forall x \in X, \ \exists x^{-1} \in X, \quad$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.

Remark The notation $a \cdot b$ for the product of elements a and b is unnecessary. From now on we will denote this by ab whenever it is clear what product we are using.

There is also a specific type of group that is useful in many contexts, an Abelian group:

Definition We say a group $(G, \cdot)$ is *Abelian* if the elements satisfy commutativity, ie $xy = yx$ for all elements $x, y \in G$.

Now, groups can come in many sizes. Let me write some tables of some very small groups. We will present these groups in what's known as a Cayley table. Row and column of the table correspond to the row's element product with the column's element. For example, the first table, with only two elements e, x says by reading the first row and the second column that $e \cdot x = x$, etc.

·	e	x
e	e	x
x	x	e

·	e	x	y
e	e	x	y
x	x	y	e
y	y	e	x

·	e	x	y	z
e	e	x	y	z
x	x	y	z	e
y	y	z	e	x
z	z	e	x	y

·	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

All of the groups shown above are Abelian, the smallest example of a non-Abelian group is the dihedral group of order 6, D_6

·	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	e	d	c	a
c	c	d	f	e	a	b
d	d	c	a	b	f	e
f	f	b	c	a	e	d

Definition The *order* of a group is the number of elements in the group. We say that a group is of infinite order if it has an infinite number of elements.

Now in physics, groups represent physical transformations. For example, we can talk of the group of rotations. It is clear, that the group of rotations cannot be written as a set of a finite number of rotations, because if that were so, we wouldn't have every rotation. In fact, groups such as the group of rotations, are a particular type of infinite groups.

Definition A *Lie group* is a group whose elements form a smooth n-dimensional surface (manifold), and where the product and inversion are smooth maps.

Examples of smooth manifolds include planes, spheres in n-dimensional space, etc. Smooth means that they are differentiable.

Lie groups can be difficult to manage, since they are infinite. The following notion makes it easier to attack Lie groups.

Definition A *Lie Algebra* L that generates a Lie group G is a set of all generators (operators) g such that any element in the algebra can be composed of exponentials of the operator with parameters. Thus $G = \{e^{ax} \mid x \in L,\ a \in \mathbb{C}\}$. A Lie Algebra also by definition satisfies some properties.

Let us examine this in some detail with the Lorentz group. The Lorentz group, $L \cong$O$(3,1)$ is the set of all origin fixing isometries. If $R \in L$ then $R(0,0,0,0) = (0,0,0,0)$, and $\|x^\mu - y^\mu\| = \|Rx^\mu - Ry^\mu\|$. The Lorentz group can be regarded as the set of all rotations in Minkowski space-time. Now what generates the Lorentz group? As can be shown the set $\mathcal{A} = \{c^{\mu\nu} J_{\mu\nu} | c^{\mu\nu} \in \mathbb{C}\}$ forms an algebra for the Lorentz group.

Appendix C

Derivations

C.1 A derivation of the Poincaré algebra

1. Determine the generators of the Lorentz group. A local Lorentz transformation is

$$x^\mu \mapsto e^{\delta\alpha_{\mu\nu}J^{\mu\nu}}x^\nu = (1+\delta\alpha_{\mu\nu}J^{\mu\nu})x^\nu = x^\mu + \alpha^\mu{}_\nu x^\nu.$$

This gives the solution

$$J^{\mu\nu} = \frac{1}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu).$$

2. Determine the generators of translations. A local translation is

$$x^\mu \mapsto e^{\delta\alpha^\mu P_\mu}x = (1+\delta\alpha^\mu P_\mu)x^\mu = x^\mu + \delta\alpha^\mu,$$

gives the condition

$$\delta\alpha^\mu P_\mu x^\mu = \delta\alpha^\mu,$$

yielding the solution $P_\mu = \partial_\mu$.

C.2 A derivation of the conformal algebra

An infinitesimal dilatation is of the form:

$$x^\mu \mapsto e^{\delta\alpha D} x^\nu = e^{\delta\alpha} x^\mu = (1+\delta\alpha)x^\mu = x^\mu + \delta\alpha D x^\nu.$$

Giving

$$D = x^\rho \partial_\rho.$$

An infinitesimal special conformal transformation admits the form:

$$x^\mu \mapsto \frac{x^\mu + \delta c^\mu x^2}{1 + 2\delta c^\rho x_\rho + (\delta c)^2 x^2} = x^\mu + \delta c^\mu x^2 - 2x^\mu \delta c^\rho x_r ho = (1 + \delta c^\lambda C_\lambda)x^\mu$$

which yields the solution

$$C^\mu = x^2 \partial_\nu - 2x_\mu x_\nu \partial^\nu.$$

C.3 Conformal transformations in cylindrical coordinates

Let us look at the example of cylindrical coordinates. Using the x and y plane I can show how r and θ transform. Let $c \equiv c^1$,$d \equiv c^2$. We know that

$$x \mapsto x' = \frac{x + c(x^2 + y^2)}{\sigma} \tag{C.1}$$

and we know the corresponding equation for y. We use this to obtain the expression for r' and θ'.

$$r\cos\theta \mapsto r'\cos\theta' = \frac{r\cos\theta + cr^2}{1 + 2(cr\cos\theta + dr\sin\theta + (c^2+d^2)r^2}$$

$$r\sin\theta \mapsto r'\sin\theta' = \frac{r\sin\theta + dr^2}{1 + 2(cr\cos\theta + dr\sin\theta + (c^2+d^2)r^2}$$

$$r \mapsto r' = \sqrt{(r'\cos\theta')^2 + (r'\sin\theta')^2} = \frac{r}{\sqrt{\sigma}} \tag{C.2}$$

$$\tan\theta' = \frac{\sin\theta + dr^2}{\cos\theta + cr}$$

.

$$\theta \mapsto \theta' = \arctan[\frac{\sin\theta + dr^2}{\cos\theta + cr}]. \tag{C.3}$$

Bibliography

[1] E. Cunningham, "The principle of relativity in electrodynamics and an extension thereof," Proceedings of the London Mathematical Society **8,** 77–98 (1909).

[2] H. Bateman, "The transformation of the electrodynamical equations," Proceedings of the London Mathematical Society **7,** 70–89 (1908).

[3] J. Townsend, *A modern approach to quantum mechanics* (University Science Books, 2000).

[4] A. Barut and R. Raczka, *Theory of group representations and applications* (Polish scientific publishers, 1977).

[5] E. Schrödinger, "Über die kräftefreie Bewegung in der relativistischen Quantenmechanik," Sitzungsberichte der Preussischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse pp. 418–428 (1930).

[6] E. Kirisis, *String Theory in a Nutshell* (Princeton University Press, 2007).

[7] J. . Reynaud, in *Frontier tests of qed and physics of the vacuum*, E. Zavattini, ed., (Heron Press, 1998), pp. 426–444.

[8] M. Jaekel and S. Reynaud, "Conformal symmetry and quantum relativity," Found.Phys pp. 439–456 (1998).

[9] J. . Reynaud, in *Frontier tests of qed and physics of the vacuum*, E. Zavattini, ed., (Heron Press, 1998), pp. 58–67.

[10] M. Hamermesh, *Group Theory and its Application to Physical Problems*, 3 ed. (Dover, Mineola, 1989).

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