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Representations for Understanding the Stern-Gerlach Effect

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REPRESENTATIONS FOR UNDERSTANDING THE

STERN-GERLACH EFFECT

by

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ABSTRACT

REPRESENTATIONS FOR UNDERSTANDING THE STERN-GERLACH

EFFECT

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The traditional explanation of the Stern-Gerlach effect carries with it several very subtle assumptions and approximations. We point out the degree to which this fact has affected the way we practice and interpret modern physics. In order to gain a more complete understanding of the Stern-Gerlach effect beyond the standard approximations and assumptions it typically carries, we introduce the inhomogeneous Stern-Gerlach effect in which the strong uniform field component is removed. This change allows us to easily identify precession as a critical concept. It also provides us with a means by which to study precession and analyze it critically. By applying and comparing several mathematical techniques to this problem we gain insight into the applicability of precession arguments and the role of standard approximations and assumptions in both analytic derivation and interpretation. This approach also allows for a more general discussion regarding the use of representations in physics and teaching.

ACKNOWLEDGMENTS

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Chapter 1

Introduction

The early twentieth century was a defining time in physics. In fact, by the mid-1900s a shift of ideas and methods had occurred on such a fundamental level that the discipline of the late 1800s only vaguely resembled the discipline that would close the next century. This shift was fueled by the wrestle between unexpected experimental results and man-made theoretical notions. Amid all this, in 1921 Otto Stern proposed one such experiment that not only validated the monumental shift but helped give it form.

In the Stern-Gerlach experiment a beam of silver atoms was passed through the poles of a magnet. Prior to this time the magnetic field may have been expected to blur the beam into one continuous image due to the magnetic properties of the atoms. However, a few years before the experiment was carried out the idea that the magnetic properties of atoms would only manifest themselves in discrete values upon measurement, not continuous ones, had been proposed. As a confirmation of this, the observed trace was indeed quantized. That is, the single beam of atoms did not blur continuously but split into two distinct parts. This was a verification of the emerging doctrine of quantization and, later, of the property of atomic spin.

Because of its simplicity and clarity the Stern-Gerlach experiment has now become exemplary of the axioms of modern physics. It is often discussed in textbooks introducing modern ideas and is taken as the clearest demonstration of the quantum measurement process. However, with this central role the Stern-Gerlach effect *directs* our study to such a large degree that we seldom make it the *object* of our study.

Because it so clearly demonstrates quantization, entanglement, measurement, and spin - all of which are new quantum concepts - the canonization of the usual description of the Stern-Gerlach effect (SGE) has also canonized their usual interpretation.

The usual interpretation of the SGE has gained clout however for good reason. It ties the classical and quantum systems of thought. It seems to explain a clearly quantum result in terms of almost purely classical concepts. For this reason we say it is clear. For this reason it is also approximate.

It was our struggle to understand the classical-sounding story of the SGE in *quantum* terms that led to this thesis. In the textbooks forces, trajectories, precessing vectors were all used to make the description clear while on another page we were forbidden to speak of such things in quantum descriptions (see [1] for example).

Most of our difficulties seemed to be connected to the phenomenon of precession so our questions began with comparing its classical and quantum justifications. This led to discussions of a more philosophical nature which in turn led to the interesting discovery that sometimes a problem is too complex to solve singlehandedly. We realized that a single problem could be presented, discussed, and solved in many different ways. By comparing and contrasting these results we found ourselves using a more experimental approach: we solved the same problem in various ways so as to have a sufficient "sample" while only tweeking particular parts and maintaining some "control" variables. After time, this lead to the formulation of our ideas on representations.

Although they appeared last chronologically, we discuss these ideas on the nature and value of representations in the next chapter. In our attempts to make this a general discussion we use non-technical examples as well as technical ones. The former we call Conceptual Representations and use them as examples of interpretational pictures in physics. The latter we call Mathematical Representations which exemplify several of the solution methods used in later chapters. We also relate representations to the the Kuhnian paradigms of [2] and discuss the consequences, both good and bad, of canonizing a given representation.

Chapters 3 and 4 use the SGE to show how one phenomenon can be represented in two very different ways. The account of Chapter 3 emphasizes the logical ordering of the concepts that are necessary to understanding the SGE from a quantum perspective. For this reason it is called a Thematic Account and is often the preferred method of textbooks (see [1] or [3]). In contrast, we give a Historical Account in Chapter 4. Such an account is characterized by its emphasis on the ordering of concepts and events chronologically. Because the events of history are not always logical this is not as widely found in teaching literature. However, in what follows we attempt to show that such accounts do give an accurate picture of *process of problem solving* as the attempts of researchers are not always logically ordered either.

In the comparison of the approaches of Chapters 3 and 4 it will be seen how taking either account as *absolute* can lead to problems. Chapter 5 discuss these problems first in terms of the complementary relationship between rightness and clarity. We then show how most problems with descriptions of the Stern-Gerlach effect, both technical and philosophical, seem to center on the phenomenon of precession and how the nature of precession depends on the choice of magnetic field.

Having identified the source of most technical and interpretive problems in Chapter 6 we outline a proposal to study the roles of both the magnetic field and precession in the standard description of the SGE. This gives rise to what we call the inhomogeneous SGE (ISGE). The remainder of this work is devoted to understanding it.

Because of the tension between having a clear account and a right account for the ISGE, in Chapter 7 we compare several different representations and techniques for the problem. Among them are matrix representations (section 7.1), differential calculus techniques (sections 7.2-7.6), Green's functions (section 7.8), and alternative representations for quantum mechanics such as the cliffor (section 7.7), and Dirac (section 7.9) pictures. In section 7.11 we suggest further approaches that could be used to arrive at a fuller understanding of the ISGE.

In order to place these numerous and varied methods in an appropriate and unifying context we begin with a discussion of the general role of representations.

Chapter 2

On Representations in Physics

Perhaps one of the most subtle aspects of learning to maneuver the physical sciences is gaining an appreciation of and familiarity with representations. They cannot be overly ostentatious or they would obscure the phenomenon of interest while on the other hand if they are too vague they fail to adequately communicate it. As these two extreme cases are somewhat at odds with each other effectively using various representations is tricky. This is a difficulty we must address due to our unavoidable use of representations.

2.1 The Necessity of Representations

In order to express a rational statement it must be presented in a *particular* way. That is, it must be given a representation so that it can be grasped by the mind in terms of a concrete language and based on a set of familiar concepts. For example, when we think of cars we do not have actual cars in our heads. We only represent the concept of cars to our minds as thoughts. So, we cannot just think but we must think particular *thoughts*. In other words, because cars are not equivalent to *thoughts* of cars a necessary process of translation takes place. We must therefore constantly represent abstract concepts, either to our mind or to others', in a particular way. The specific manner in which a statement is expressed for its communication or preservation in concrete form constitutes its representation. It is in this sense that representations cannot be avoided.[1]

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[1]The ideas on concepts assumed in this thesis were largely influenced by [4] in which a similar though not identical epistemology is developed in detail. [5] shows how this epistemological system fits with the more metaphysical questions of the physical sciences.

An analogous process occurs when translating between two languages. For example, one might say,

(a) *But look how the fish drink in the river.*

or

(b) *Pero mira cómo beben los peces en el río.*

Using these two statements as different representations of the same idea we can learn several things about representations in general.

(1) Representations are arbitrary *in principle.* Although (a) and (b) are very different expressions they express exactly the same phenomenon.

(2) Representations make assumptions. (a) assumes a knowledge of the English language while (b) assumes a knowledge of Spanish. Because assumptions can be either more or less general, representations can also occur simultaneously on different levels. Thus, representations can be "layered" with the more specific ones occurring on top of more general ones.

(3) Representations imply certain things. (a) implies that valid responses are expressible in English whereas (b) implies that Spanish should be used. For spoken languages this may be no surprise but in certain mathematical or physical situations there may be problems or answers that are not easily expressible in a particular way such as explaining the process of electron capture in an ancient African dialect, describing quantum phenomena with only classical concepts, or using a discretely indexed series to express a continuous process.

(4) Representations depend on cultural or philosophic values.[2] Thus, they are context dependent *interpretations.* To an English speaker (a) is a completely random and detached observation whereas to a Spaniard (b), although describing an identical phenomenon, brings to mind memories and scenes of a religious nature as it is a line from a well known Christmas carol. It is in this sense that the selection and use of a representation or interpretation is an unscientific but direction-giving part of physics.

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[2]Conversely, cultural and philosophic values often depend on commonly accepted representations and interpretations, e.g. the effects of Newtonian determinism and Darwinian evolution on religious discourse. For this reason, we must be careful about how we represent science in society.

(5) Representations change the communicated meaning of the phenomenon they describe. By utilizing attributes (1)-(4) above, representations can be selected to emphasize certain aspects, such as symmetries or biases, to our advantage. Unfortunately, they may also simultaneously and unintentionally obscure other aspects. The ability to make wise choices that properly balance this tradeoff is easily demonstrated but near impossible to teach. It seems to be the result of abstracting from experience and not from a concrete or deductive process.[3]

These are only some of the characteristics of representations as we use the term here. Others will be demonstrated shortly.

2.2 Two Types of Representations

Although there are many types of representations, perhaps as many as there are languages, there are two that are of particular interest in physics. They might be categorized as mathematical and conceptual representations. The former are typically thought of when representations are mentioned in scientific discourse while the latter also play a very important, though less emphasized, role. We will give a few examples to illustrate these two categories.

2.2.1 Mathematical Representations

One of the earliest examples of a mathematical representation - after students have mastered the ability to accept x as representing an unknown quantity - is the use of the 2-dimensional Cartesian grid. By drawing two intersecting lines we gain the ability to express quantitative relationships which we say are 1-dimensional. The choice however of *which* two lines is a choice of representation.

We often make the choice of using an "orthogonal" basis, that is, we choose two lines that not only intersect but that are perpendicular to each other (see fig. 2.1). The fact that they are perpendicular usually provides us with a great simplification over non-orthogonal axes. Also, note that if the information we were trying to

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[3]An example of this might be making a judicious choice of coordinates in a Lagrangian problem or arranging charges in an image problem. As general processes these are never really *explained* but only repeatedly *demonstrated* by those who have got the "hang of it."
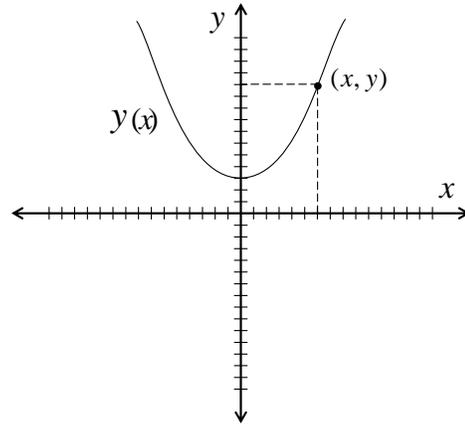
Figure 2.1: On the 2-dimensional Cartesian plane any point can be represented as a pair of numbers (x, y) specifying its relationship to a predetermined set of coordinate axes. Any *set* of points can be written as a relation $y(x)$.

represent were more complex we could choose a higher dimensional space, e.g. more perpendicular axes, by which to represent them. If we were bound to only graphical representations of coordinate systems such as in fig. 2.1 doing this for more than 2 or 3 dimensions would be impossible but because of the more abstract algebraic representations of Descartes' analytic geometry we can easily write functions in n-dimensional spaces as functions of n variables. We will however restrict ourselves to the simplest two-dimensional examples.[4]

Vectors and Rotations

Suppose we have a vector in a 2-dimensional Cartesian space. We often represent such an object as a directed line segment. The length of the line segment quantitatively encodes the magnitude and its direction the orientation of the physical quantity of interest.

Leaving the vector in this graphical form requires that any mathematical operation involving that vector be done in a graphical manner as well (see attribute (2) above). Thus, we speak of placing vectors, head-to-tail and forming parallelograms, etc. (see fig. 2.2(a)). This is sometimes referred to as a coordinate free *representation.*

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[4]Even when we discuss spin system we will restrict ourselves to 2-level systems for reasons discussed in section 3.2.1.

(a)

$$\boldsymbol{v} = (x_A, y_A) = x_A \hat{x}_A + y_A \hat{y}_A \qquad \boldsymbol{v} = (x_B, y_B) = x_B \hat{x}_B + y_B \hat{y}_B$$

A

(b)

y_A

x_A

A

B

(c)

y_B

φ

x_B
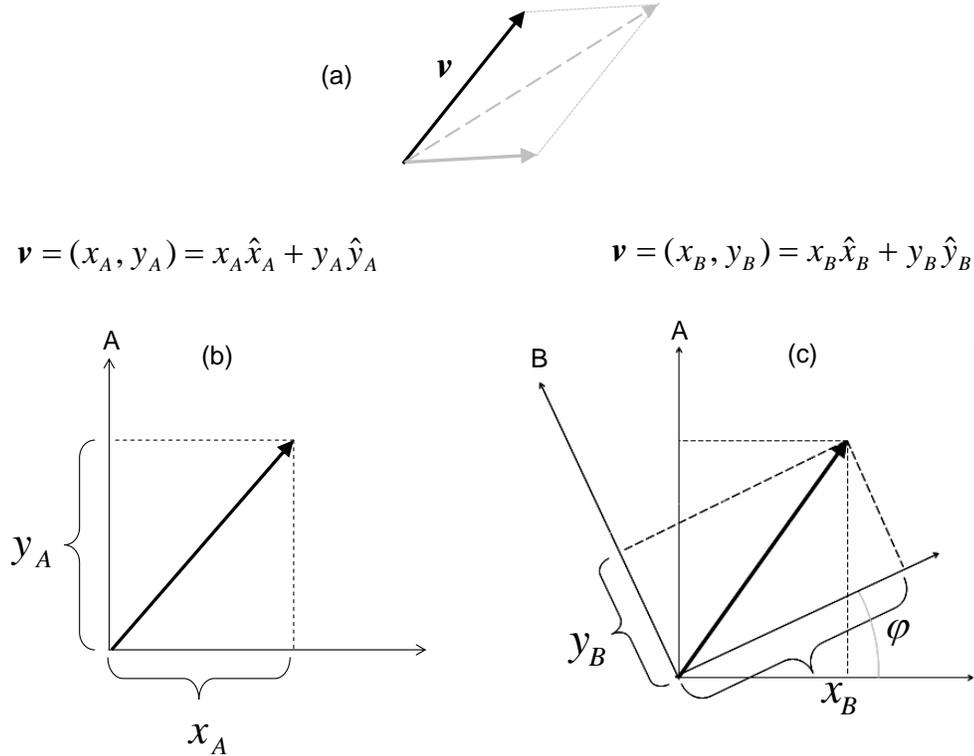
Figure 2.2: (a) Vector **v** can be represented graphically as a directed line segment. In a graphical manner we can add other vectors to **v** by forming parallelograms. (b) By imposing an orthogonal coordinate system A we can give our graphical representation a more compact algebraic form **v**= $x_A \hat{x}_A + y_A \hat{y}_A$. (c) If we rotate A to form a new coordinate system B the *representation* of **v** has changed to **v**= $x_B \hat{x}_B + y_B \hat{y}_B$ but the vector itself has not changed at all. Thus, some changes arise from the object itself while others arise only from the representation.

We can however use another layer of representation. If we create a set of perpendicular coordinate axes with which to represent the vector (see fig. 2.2(b)) it allows us to use a more compact and general algebraic method to sum or multiply vectors. We are therefore released from the limitations of graphical methods.

If there were a reason to, we could and often do, arbitrarily rotate the coordinate axes, perhaps to take advantage of a different symmetry (see fig. 2.2(c)). In doing so the length and orientation of the vector with respect to the physical quantity they express are unaffected while its specific mathematical representation in the given basis would change since the axes are now tilted. Although the components

have changed, the essential aspects of the vector have not and so any calculation of physical results are the same regardless of the choice of coordinate system.

If the vector represents a quantity that is horizontal, such as the displacement of a ball rolling to the right on a table, we often choose to orient the coordinate axes such that the displacement vector lies along the "x"-direction because this direction is typically associated with the horizontal. Such a choice may reduce the complexity of the problem to that of 1-dimension (see attribute (5) above).

Coordinate Systems

Another example of a mathematical representation can be given which is of a slightly different sort. Suppose we had two variables with a linear relationship. We could represent their relationship in a Cartesian grid by writing

$$y = mx + b, \tag{2.1}$$

which is the familiar equation of a line with slope m and y-intercept b. This form is simple and well known because it takes advantage of the linear symmetry of the given relationship.

Likewise, if we were asked to represent a circle instead of a line we might have chosen, based on the angular symmetry, a coordinate system that parameterized the angle about the origin. In standard polar coordinates the circle is written as

$$r = a \tag{2.2}$$

where a is the radius. Thus, taking advantage of the known symmetries allows us to express relationships - lines and circles - in very simple ways.

However, if we weren't as experienced with lines and circles or Cartesian and polar coordinates we might have chosen, perhaps based on some biased fancy,[5] to express the line in polar or the circle in Cartesian coordinates. Although this can be done it disguises the problem in a messy representation. Their algebraic representations become

$$a = \pm\sqrt{x^2 + y^2}, \tag{2.3}$$

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[5]Like a religious devotion to the circular form.
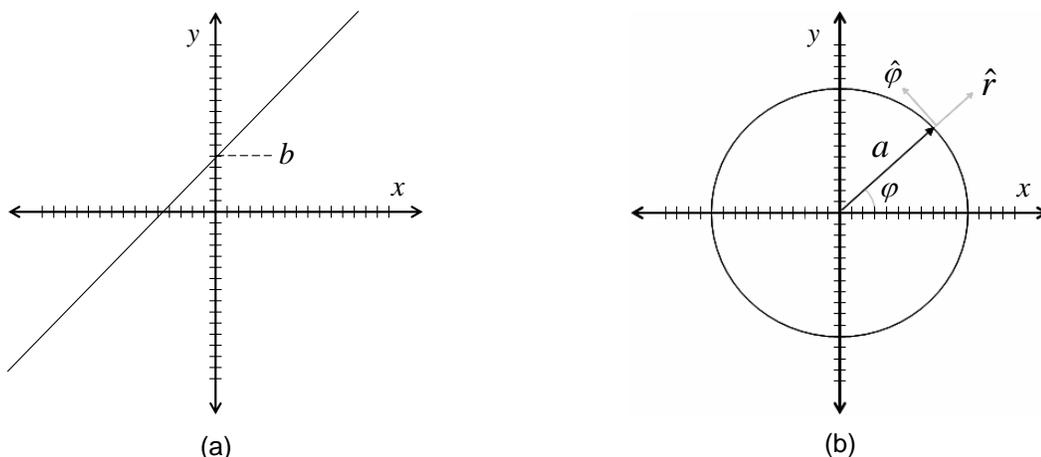
Figure 2.3: (a) Within a coordinate system described by the Cartesian coordinates (x, y) a line is easily described. However, when described by standard polar coordinates (r, ϕ) its simple representation is replaced by a non-linear equation in r and φ. (b) Conversely, when a circle of radius a is represented it is simple in polar coordinates whereas it is more complicated in Cartesian form. See Table 2.1 and 2.2.

which is now a piecewise function, for the circle in Cartesian coordinates and

$$r = \frac{b}{\sin \varphi - m \cos \varphi}, \tag{2.4}$$

which is now, ironically, a non-linear equation for a line in polar coordinates. Many more examples of basis sets,[6] vector spaces, algebras,[7] coordinates systems, and representations[8] could be given.

These examples not only demonstrate interesting behaviors that properly belong to the representation and not to the curves themselves, i.e. the piecewise nature of eq. (2.3), etc., but also the fact that representations can be layered, that is, you may have representations of representations. They also show that although the same phenomenon can be given in various ways we often choose among the possibilities in order to emphasize certain aspects.[9] However, if we have no intuitive guide as to

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[6]Basis sets can consist of many types of mathematical objects such as vectors, functions, matrices, etc. In fact, even real numbers can be thought of as a 1-dimensional space of which the basis set is the number 1. That is, all numbers can be written as a linear combination of 1's. Basis elements are also usually chosen to be mutually orthogonal and normalized though they need not be.

[7]A specific algebra, the Clifford algebra, will be mentioned in Chapter 7.

[8]Matrices, for example, require a particular representation.

[9]Even if nature were perfectly symmetric, we have no reason to disbelieve that our minds have a preferred direction that can break the stalemate of symmetry.

Relating Cartesian and Polar Coordinates	
Cartesian Coordinates $\Leftrightarrow$	Polar Coordinates
$x = r\cos\varphi$	$r = \pm\sqrt{x^2 + y^2}$
$y = r\sin\varphi$	$\varphi = \arctan\left(\frac{y}{x}\right)$
$-\infty \le x \le \infty$	$0 \le r < \infty$
$-\infty \le y \le \infty$	$0 \le \varphi < 2\pi$

Table 2.1: A table showing the transformation rules and other relationships between the Cartesian and Polar coordinate systems.

Some Representations in Cartesian and Polar Coordinates	
line: $\quad y = mx + b$	$r = \frac{b}{\sin\varphi - m\cos\varphi}$
circle: $\quad a = \pm\sqrt{x^2 + y^2}$	$r = a$

Table 2.2: A table showing how a line and a circle may be represented in both Cartesian and Polar coordinates.

what in the problem is worth emphasizing representations, even though they can still be given, can be unnecessarily messy.

2.2.2 Conceptual Representations

Just as mathematical representations are utilized only when speaking mathematical languages, conceptual representations must be used whenever concepts are used. Their difference is demonstrated in the fact that while mathematics presupposes certain concepts, e.g. numbers, concepts do not obviously presuppose a mathematical language. In this way, conceptual representations can be seen to be more fundamental than mathematical ones. If this is the case then it follows that conceptual representations are of extreme importance in physics, for math is. It also follows that selecting

a particular mathematical approach *does* carry conceptual and/or pedagogical consequences that may affect the approach and results of a problem.[10] Indeed, specific instances of conceptual representations are used when we employ such things as *models* and *interpretations*, without which science could not progress.[11]

A more suggestive term for all conceptual representations might be *paradigms*. This is meant to allude to [2] which discusses at length the shaping role of paradigms in science. We give shortly two specific examples of paradigm shifts.

Conceptual Systems and Orthogonal Concepts

Before giving some familiar examples of conceptual representations, or paradigms, it is interesting to establish a paradigm of our own to show the similarities in form with the mathematical representations given above. Just as in mathematics great utility is found in representing objects in a coordinate system defined by certain basis elements, conceptual pictures are formed in terms of their own basic set of components. In other words, when a given concept can be *explicitly* identified as a particular weighted combination of a few defining and elementary concepts, much as a vector can be described in terms of its components within a coordinate system, great efficiency and progress can be made. In this light, forming a paradigm is the qualitative equivalent to selecting an appropriate basis in which to describe phenomena. We will use this model in the examples that follow.

The Early Scientific Revolution

A canonical example of a scientific paradigm shift is the early scientific revolution that began with Copernicus and was consummated with Newton. We will not go into the historical details but we can see that the change that occurred during this time was primarily one of perspective. The gods had not altered their course, neither had the planets changed their motion, and yet physics was revolutionized

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[10]There is a very common opinion to the contrary.

[11]Although many downplay the importance of interpretations in physics today that little progress can be made without them is especially manifest in the concerted efforts that went into formulating a consistent interpretation of quantum mechanics in the first half of the last century [6].

and completely changed form. Simply put, before the revolution we operated in a conceptual "space" - a conceptual "coordinate" system - that set the concepts of simplicity, anthropocentrism, and the duality of the terrestrial and divine natures among others as a basis whereas after Copernicus, Kepler, Galileo, and Newton had done their work we saw different advantages and accordingly shifted our values so as to enshrine objectivity, mechanism, reductionism, determinism, and mathematical rigor. Our way of conceptualizing - of seeing problems - had been drastically altered. We had effectively rotated or redefined our conceptual system so as to take advantage of our new found values much as we did in section 2.2.1. As a result the way in which we interpreted and categorized our problems, methods, results, and even values changed as well.[12]

Wave-Particle Duality

There are more modern examples of paradigm shifts. One that has not resulted in the final selection of one system completely at the expense of another, as did the early scientific revolution, is found in quantum mechanics.[13] Here we have come to grips with the need to adopt at least two very different paradigms. We have learned to constantly shift our view between the two based on the nature of the questions asked. This is because, in the language of mathematical spaces, our concepts have been reduced to not just any set of concepts but to what we might call an *orthogonal* set of concepts. Bohr called these sorts of concepts "complementary" and developed many ideas based on his principle of complementarity [7]. Similar to the definition given in section 2.2.1 by orthogonal concepts we mean two concepts that are in no way expressible in terms of each other. If we take light, for example, as our object of description these two concepts are waves and particles.

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[12]For an account of this period and an idea of the concepts that formed the conceptual "space" in which these theories developed see [8] or [9].

[13]See [10] for an excellent discussion of the history and development of many quantum mechanical concepts including wave-particle duality and Bohr's complementarity which we mention here as well.

<div style="text-align:center">(a)   (b)</div>
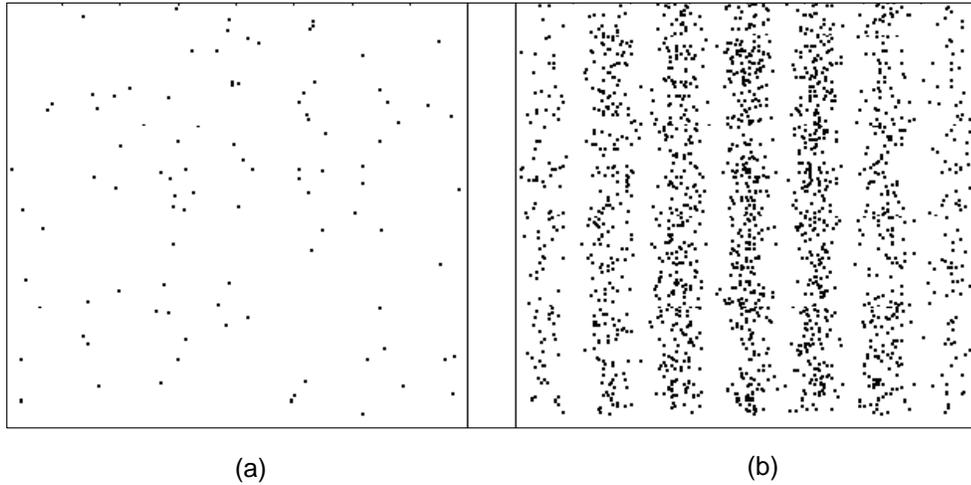
Figure 2.4: (a) If a very low intensity beam of light is sent through a set of narrowly spaced slits a pattern of dots accumulates one-by-one on the detecting screen. This is what we would expect if light were made up of particles. (b) However, if the beam is left for a long time or if the intensity is increased so that several particles strike the screen we can see that the particles (photons) are striking the screen in an orderly fashion, i.e. creating a series of bright and dark bands. Because this is exactly what we observe with all wave phenomena, such as with sound or water waves, this is what we would expect if light were a wave. Therefore, we see that while the path of light is wave-like (shown in (b)) the way in which it collides with detectors is particle-like (shown in (a)).
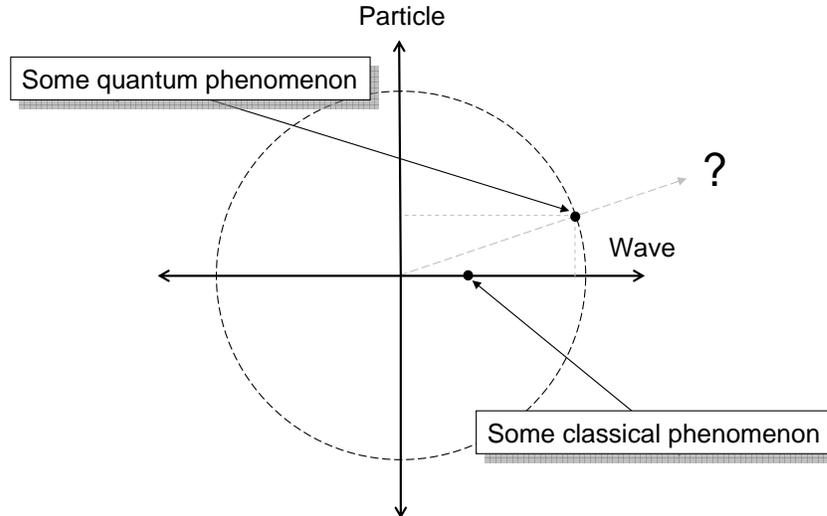
Figure 2.5: Because the duality demonstrated in the 2 slit experiment of fig. 2.4 is so common in fully describing quantum phenomena we have realized the necessity of using a 2-dimensional conceptual system. Each of the basis elements are classical concepts so only a 1-dimensional system is needed to describe a phenomenon classically. We will see in section 3.2.4 that any *representation* in terms of these dual, or complementary, concepts is constrained to something like a circle, i.e. as a system is more describable in terms of particles it is correspondingly less describable in terms of waves. The question remains as to whether the phenomena *themselves* can evolve *off* the circle.

<div style="text-align:center">15</div>

Particle Concepts	Wave Concepts
Mass	Frequency
Trajectory	Amplitude
Collision	Phase
Position	Wavefront
Force	Peak
Spin	Polarization
Volume	Phase Velocity
Speed	Interference

Table 2.3: A table of some concepts that we use to describe wave and particles. Particle concepts are not well defined when applied to waves and visa versa. The set of concepts listed in either column separately form a conceptual space or basis, much as the set $(\hat{x}, \hat{y}, \hat{z})$ does, within which the overall concept of "particle" or "wave" can be clearly used.

Some experiments show light to demonstrate the properties of waves while others demonstrate particulate nature. Because there are at least these two orthogonal ways of representing light there are at least two completely different sets of properties of which we can learn. As a particle, we can use the well-defined concepts of position, speed, trajectory, and force as applied to light while as a wave we may use wavefronts, interference, crests, troughs, frequency, and amplitude. It seems that depending on the precise experimental questions we ask we may use either the attributes and concepts of waves or particles in studying light.

It is interesting to note that we have effectively doubled the space in which we describe quantum mechanics by doubling the the number of concepts that can be applied to it. This is the reverse of the vector example in section 2.2.1 in which we chose a mathematical representation so as to reduce the dimensionality of the problem. Here we choose a conceptual representation that *increases* the conceptual dimensionality of the problem in order to give it a more complete representation; the problem can now be described as some combination of "waviness" and "particle-ness" just as a vector might be describable in its x and y orthogonal components. This is not only analogous to the introduction of spin into quantum mechanics into which the

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spin degree of freedom was added in order to account for observed phenomena but it is the gist of the entire approach of this work. We attempt to expand the conceptual space available to describing spin phenomena as manifest in the Stern-Gerlach effect. This demonstrates the practical effects a shift in conceptual representation may have.

2.3   Representations as Standards

When the utility of a particular representation is demonstrated with respect to a shared value system, whether mathematical or conceptual, they can become standard, or axiomatic. As such, new hypotheses and results are compared against it in order to gauge the validity and "truthfulness" of the hypothesis, *not* the standard. If there is an inconsistency it is the nature of the hypothesis or result that is usually questioned and not the standard.[14] This can have several undesirable consequences some of which are

(1) Their democratically-decided status as "standard" implies to many minds the absolute status of *unquestionable.*

(2) Once considered standard the necessity and prevalence of representations can easily be mistaken for complete objectivity.

(3) The characteristics of nature that representations assume are seen as naturally or mathematically imposed.

(4) The implications and emphases of the representation become confused with those of the original phenomena.

(5) The solution methods and interpretations natural to the representation are seen as required.

(6) The value system used to determine it and that arises from it is mistaken as absolute, etc.

In short, all unscientific aspects of the representation process can easily and mistakenly be given scientific status. When they thus become axiomatic representations are often taken for granted and much valuable insight is lost.

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[14]When enough discrepancies arise there may come a point when the standards themselves begin to be questioned. This occurs at the onset of scientific revolutions (see [2]).

On the other hand, standards and established norms - even to the hiding of irregularities and messiness - are crucial to the communication and development of any science. Therefore, sifting these subtleties out scientifically, not necessarily abandoning them, is necessary for further progress.

Chapter 3

A Thematic Account of the Stern-Gerlach Effect

Representations come in many forms. They may be qualitatively expressed in everyday language or quantitatively expressed in mathematical form. In any case, a particular representation is chosen in order to emphasize a particular aspect of the phenomenon being described. In this chapter we choose to represent the Stern-Gerlach effect (SGE) in a manner that will appear very similar to those given in various textbooks (see [1], [3], or [11]). We will seek to emphasize the logical ordering of the themes and concepts of the SGE in order to provide a clear and rational account. For this reason we call it a thematic account. In the next chapter the same story will be told but from a historical perspective.

3.1 A Classical Representation of the Stern-Gerlach Effect

We begin with a classical description of the SGE because it introduces the main concepts with which students are usually familiar when the SGE is first presented. It therefore provides an appropriate context for the quantum mechanical "textbook" description we subsequently give.

In the classical picture of the atom we treat the electron as tracing a definite orbit around the nucleus. This moving charge creates a current I that encloses a vector area $\mathbf{A}$, with direction $\mathbf{n}$ normal to the surface in a righthanded sense with the orbital motion. We may thus use the familiar equation for magnetic moments $\boldsymbol{\mu}$ from classical electromagnetic theory

$$\boldsymbol{\mu} = I\mathbf{A}. \tag{3.1}$$
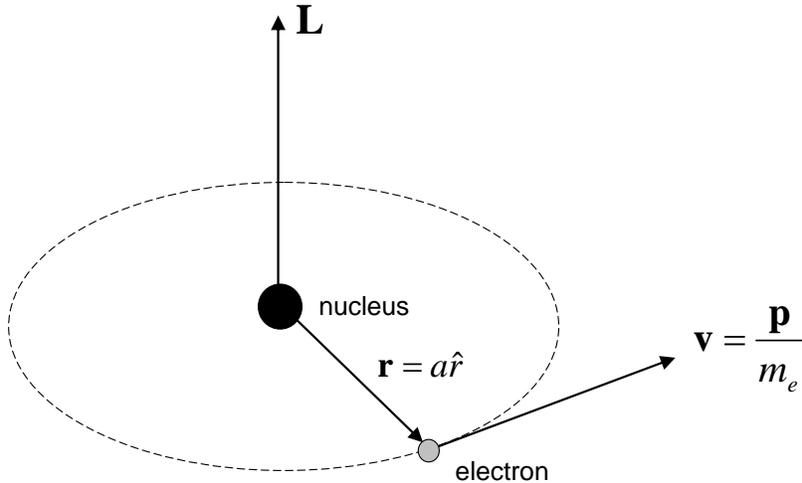
Figure 3.1: In the classical picture of the atom the electron orbits the nucleus much as a planet would orbit the sun. Because it has a continuous set of definite trajectories with velocity $\mathbf{v}$ and momentum $\mathbf{p} = m_e \mathbf{v}$ there is a definite and continuous orbital angular momentum $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$.

More specifically, if we assume that charge e orbits κ times[1] around a circular path of radius a with velocity $\mathbf{v}$ such that the period T of the particle is $T = 2\pi a/v$ we may write $I = \kappa e/T = \kappa e v/2\pi a$. Using $\mathbf{A} = \pi a^2 \mathbf{n}$ we have

$$\boldsymbol{\mu} = \frac{\kappa e a v \mathbf{n}}{2}. \tag{3.2}$$

Multiplying and dividing by the mass m of the particle allows us to recognize the total angular momentum $\mathbf{L} = amv\mathbf{n}$. We now have a general magnetic moment

$$\boldsymbol{\mu} = \frac{\kappa e}{2m} \mathbf{L} \tag{3.3}$$

for electrons. In the classical picture we assume that the particle carries the charge so, since in one period T the particle makes one complete revolution, so does the charge. Hence, classically we take $\kappa = 1$. The reason for introducing κ will become more apparent in section 3.2.1 (see footnotes 1, 5, and 6).

The energy of $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ is

$$U = -\boldsymbol{\mu} \cdot \mathbf{B} \tag{3.4}$$

---

[1]Usually $\boldsymbol{\mu}$ is derived classically without any mention of κ, since it is 1. Apart from demonstrating how we can make some common assumptions mathematically explicit, the use of it here is intended to emphasize two things. We leave this discussion however to section 3.2.1 and footnotes 5 and 6.

and the dynamics are defined by

$$\mathbf{F} = -\boldsymbol{\nabla}U = \boldsymbol{\nabla}(\boldsymbol{\mu} \cdot \mathbf{B}) \tag{3.5}$$

$$\text{and } \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \boldsymbol{\mu} \times \mathbf{B}. \tag{3.6}$$

Using eq. (3.3) inverted for $\mathbf{L}$ and defining a characteristic frequency $\boldsymbol{\omega} \equiv -\kappa e\mathbf{B}/2m$ the second of these equations becomes

$$\frac{d\boldsymbol{\mu}}{dt} = \boldsymbol{\omega} \times \boldsymbol{\mu}. \tag{3.7}$$

Thus, as can be seen in fig. 3.2, the particle's magnetic moment precesses about $\mathbf{B}$ at a frequency ω, which is proportional to the local magnitude of $\mathbf{B}$.
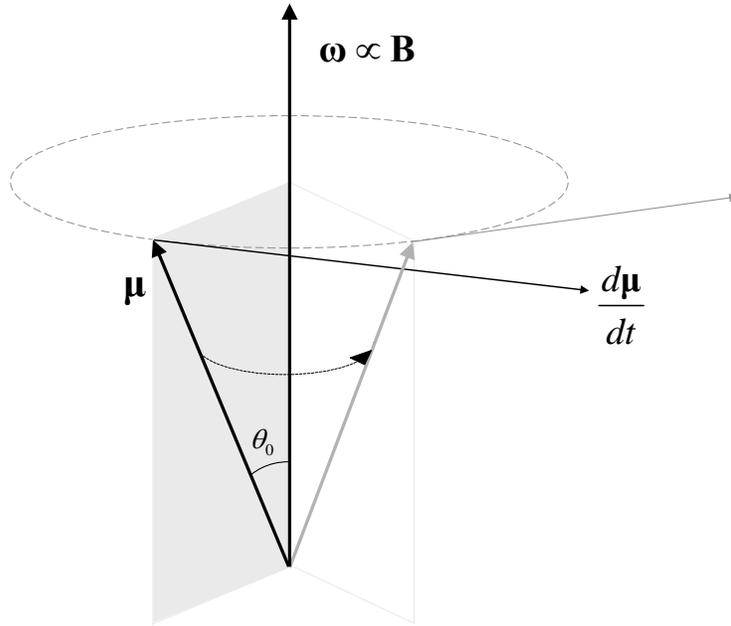
Figure 3.2: Because of the torque implied by $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$ the time rate of change of $\boldsymbol{\mu}$ is in the direction perpendicular to the plane of $\boldsymbol{\mu}$ and $\mathbf{B}$. This direction changes from point to point such that overall $\boldsymbol{\mu}$ rotates, or precesses, about the local field direction at a frequency ω proportional to μ, B, and the sine of the angle θ_0 between them. Without dissipation this continues forever. $\boldsymbol{\mu}$ would only align with $\mathbf{B}$ if the energy were somehow lost as in a compass by friction with the pivot point.

As this precession occurs the force equation also has an effect. Under the very common though very subtle assumption that $\boldsymbol{\mu}$ does not vary in space the gradient operator in eq. (3.5) moves past $\boldsymbol{\mu}$ giving the components, in index notation, $F_j = \mu_k \partial_j B_k$. Therefore, the force felt by the particle will be very subtly dependent upon both the magnetic moment $\boldsymbol{\mu}$, which is an inherent property of the particles, and the field gradient $\boldsymbol{\nabla}\mathbf{B}$. This means that in the SGE the particles will sift themselves out according to the magnitude and direction of their respective magnetic moments as they precess. In terms of experimentally controlled parameters, the *fact* of deflection
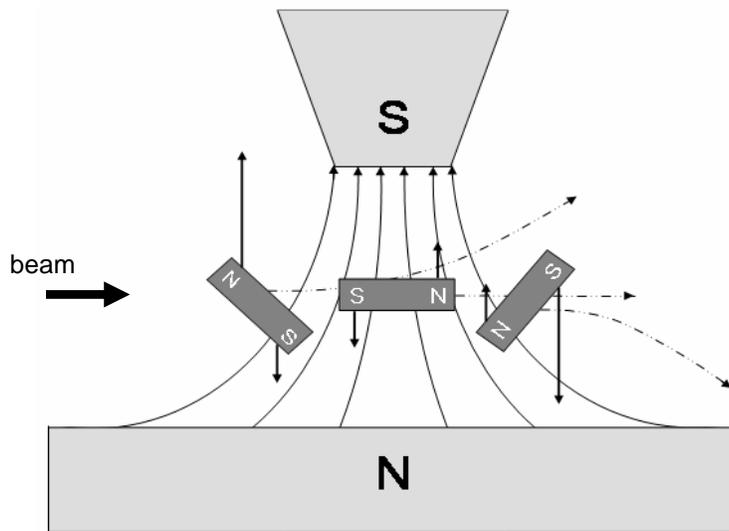
Figure 3.3:   A beam of atoms, whose magnetic properties can be represented as tiny bar magnets, enters an inhomogeneous magnetic field from the left. Depending upon the orientation of the bar magnetic in the field a net force will be exerted. This force causes a corresponding separation of the trajectories.

arises from the *presence* of a field gradient while the particular *direction* of deflection can be very subtly controlled by the chosen *form* of the field gradient according to $F_j = \mu_k \partial_j B_k$.[2]

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[2]Note that the direction of $\mathbf{F}$ is only *partly* selected by the form of the field used. It is also partly controlled by the inherent magnetic moments of the particles.

In order to classically derive the SGE we imagine passing a beam of particles each with a random magnetic moment $\boldsymbol{\mu}$ through a magnetic field that incorporates both characteristics of interest: a field gradient b that induces deflections and a clearly defined direction that directs them a certain way. The usual choice is

$$\mathbf{B} = (B_0 + bz)\hat{z}. \tag{3.8}$$

Note that in this particular form B_0 is not necessary. The field $bz\hat{z}$ has a clearly defined direction by itself. B_0 is usually included though so as to allow the independent adjustment of magnitude of the field, in this case $|B_0 + (bz)|$, and magnitude of its gradient b. This will become important later.

Applying this to eq. (3.5) yields

$$\mathbf{F} = \mu_z b\hat{z} = \frac{eb}{2m}\cos\theta_0\hat{z} \tag{3.9}$$

as the equation of motion. Thus, in the classical representation of the experiment we expect each particle to move under a force that is proportional to $\boldsymbol{\mu}$ and the field gradient b in the direction of $\hat{z}$. Inasmuch as the orientation of $\boldsymbol{\mu}$ is continuous and the force has a constant direction in all point of space a continuous, linear distribution of particles will register on a detecting plate placed some distance behind the magnet. This is the classical description of the SGE (see fig. 3.5(a)).

3.2   Quantum Mechanics

The quantum description of this experiment is much different. Before going into its derivation specifically a thematic approach requires that we introduce some concepts that will be needed but that are not part of the classical intuition we have built up. Together with the classical concepts, these will form a common conceptual context from which students seeing the SGE for the first time will draw.

The governing equation of quantum mechanics is Schrödinger's equation. In general form it is

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle \tag{3.10}$$

where $\hat{H}$ is the Hamiltonian operator and $|\Psi\rangle$ represents a general quantum state.[3]

In light of the previous chapter we note that by "general" we mean here only that it is free of *particular kinds* of representations not of *any* kind of representation. That is, $|\Psi\rangle$ is a general state in Hilbert space but has not yet been expressed in terms of a more narrow space, such as position or momentum space. Once we move to one such space this will be analogous to the example in section 2.2.1 when a vector's geometric, coordinate free representation as a directed line segment was made more concrete and useful through the introduction of a particular basis.

Because $\hat{H}$ is the Hamiltonian operator we may write it in a more suggestive form

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}, \tag{3.11}$$

where we now have the square of the momentum operator $\hat{\boldsymbol{p}}^2$ and the potential energy operator $\hat{V}$.

We can represent eq. (3.10) using eq. (3.11) in a familiar way by projecting the states and operators into a coordinate basis defined by the triplet (x, y, z). We will call this $\mathbf{x}$-space.[4] We also realize that $|\Psi\rangle$ may carry time-dependence so $|\Psi\rangle \rightarrow |\Psi(t)\rangle$. This gives us the Schrödinger representation which is most commonly dealt with in introductory treatments. With these choices[5] we follow the prescribed formalism of quantum mechanics for projecting into $\mathbf{x}$-space and multiply everything on the left with $\langle \mathbf{x}|$. We get,

$$\langle \mathbf{x}|\Psi(t)\rangle = \Psi(\mathbf{x}, t) \tag{3.12}$$

$$\langle \mathbf{x}|\hat{\mathbf{p}}^2\Psi(t)\rangle = -\hbar^2\boldsymbol{\nabla}^2\Psi(\mathbf{x}, t) \tag{3.13}$$

$$\langle \mathbf{x}|\hat{V}\Psi(t)\rangle = \hat{V}(\mathbf{x})\Psi(\mathbf{x}, t). \tag{3.14}$$

At this point we can see that it will be economic for us to suppress the explicit listing of functional dependencies when it is convenient to do so. They will still be included in the most general equations or when the meaning is unclear without them.

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[3]We use ˆ to differentiate operators whether matrix, vector, or scalar from other mathematical objects except in the case of $\boldsymbol{\nabla}$ for which it is obvious. Unfortunately the ˆ is also standard notation for unit vectors. When the context doesn't make this clear we'll make it more explicit.

[4]We will work in $\mathbf{p}$-space, which uses (p_x, p_y, p_z) as coordinates, in chapter 7.

[5]Later we will demonstrate the usefulness of other choices of representation.

Inserting these specified forms into eq. (3.10) we get

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\Psi + \hat{V}\Psi. \tag{3.15}$$

This is the time-dependent Schrödinger equation represented in **x**-space.

Because this is now represented as a familiar differential equation we can use familiar differential solution techniques to proceed. If $\hat{H}$ is time-independent, which implies that $\hat{V}$ is as well, we can use the separation of variables technique to separate the time behavior out. Assuming

$$\Psi(\mathbf{x}, t) = \psi(\mathbf{x})T(t) \tag{3.16}$$

and calling the constant of separation E yields the two equations

$$ET = i\hbar\frac{d}{dt}T \tag{3.17}$$

$$E\psi = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + \hat{V}\psi \tag{3.18}$$

for the time and space parts respectively. Note that the spatial equation is an eigenvalue equation of the form

$$\hat{H}\psi_n = E_n\psi_n \tag{3.19}$$

with eigenvalues E_n corresponding to eigenfunctions ψ_n and also that the partial time derivative in eq. (3.17) can become a total derivative because of the lack of spatial dependence in T.

Solving the time piece gives

$$T(t) = T(0)e^{-\frac{i}{\hbar}Et}. \tag{3.20}$$

A general solution of the full equation is then the linear combination

$$\Psi(\mathbf{x}, t) = \sum_{n=0}^{N} \psi_n(\mathbf{x})e^{-\frac{i}{\hbar}E_n t} \tag{3.21}$$

where N, the dimensionality of the quantum space, could be either finite or infinite. All constants have been absorbed into $\psi_n(\mathbf{x})$.

Had we represented our operators as N-dimensional matrices instead of algebraic operators we could have arrived at the same result. This would be a useful choice for systems of only a few dimensions such as many spin systems.

3.2.1 Spin

Spin is a necessary form of angular momentum in the quantum description of nature. It is analogous to the orbital angular momentum $\hat{\boldsymbol{L}}$ although it has not yet been consistently associated with a conceptual picture of any spinning or orbiting object. Orthodox interpretations of quantum mechanics take it to have no classical manifestation and to be intrinsic to quantum objects such as photons and electrons.

As a measure of quantum angular momentum, spin $\hat{\boldsymbol{S}}$ satisfies the same equations as $\hat{\boldsymbol{L}}$. In particular,

$$\hat{\boldsymbol{\mu}} = \frac{\kappa e}{2m}\hat{\mathbf{S}} \tag{3.22}$$

corresponding with eq. (3.3) but with an important difference. For quantum mechanical descriptions of electrons it has been found that there is a missing factor of 2. There are several ways to introduce this correction. One possibility is to assume that κ changes values in moving from the classical to the quantum description.[6] Our picture also changes then. Based on the interpretation we gave with eq. (3.3) with $\kappa = 2$ instead of 1 the charge apparently rotates *twice* for every *single* revolution of the particle. Although there is no mathematical difference, only interpretive changes, let us rename $\kappa \to g$ in order to connect with the literature at this point.[7] If this is the case then for one full revolution of the particle in its orbit, described by $\hat{\boldsymbol{L}}$, there

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[6]Another possibility, the usual choice, is to arbitrarily introduce a new factor $g = 2$ that didn't have an appreciable effect in the classical description and whose interpretation is consequently unclear. The method we use is completely non-standard.

[7]We did not call κ g until this point precisely because we did *not* want to "connect with the literature." If we had initially equated the two those familiar with the Landé g-factor would have immediately resisted associating it with our interpretation of κ because g is not usually given any classical motivation. However, if one had never used the symbol g in this context, as most students of a thematic classical picture haven't at this point, that wouldn't be possible. In order to create this unbiased effect we chose to call it κ. Thus, what authors do or don't do can lead the mind and prepare it for what they want the student to accept in the future or associate with in the past. The second reason for this is to demonstrate the arbitrariness of some interpretations. We could've interpreted κ as the relative area of the orbits of the charge and mass as in describing allowed orbital radii, the relative velocity of the charge and mass perhaps shedding light on the phase and group velocities, or the relative periods of the charge and particle dynamics. This latter case might have the same interpretation as our choice to associate κ with the number of rotations the charge makes in time T. Respectively, instead of κe we would have introduced κa, κv, or T/κ all of which would be be *mathematically equivalent*. Thus, while we have given *a* classical interpretation for κ, or in other words g, we have not given *the* interpretation. g is treated more rigorously in Dirac's relativistic formulation of quantum mechanics.

3-dimensional Physical Space   2-dimensional "Spin" Space
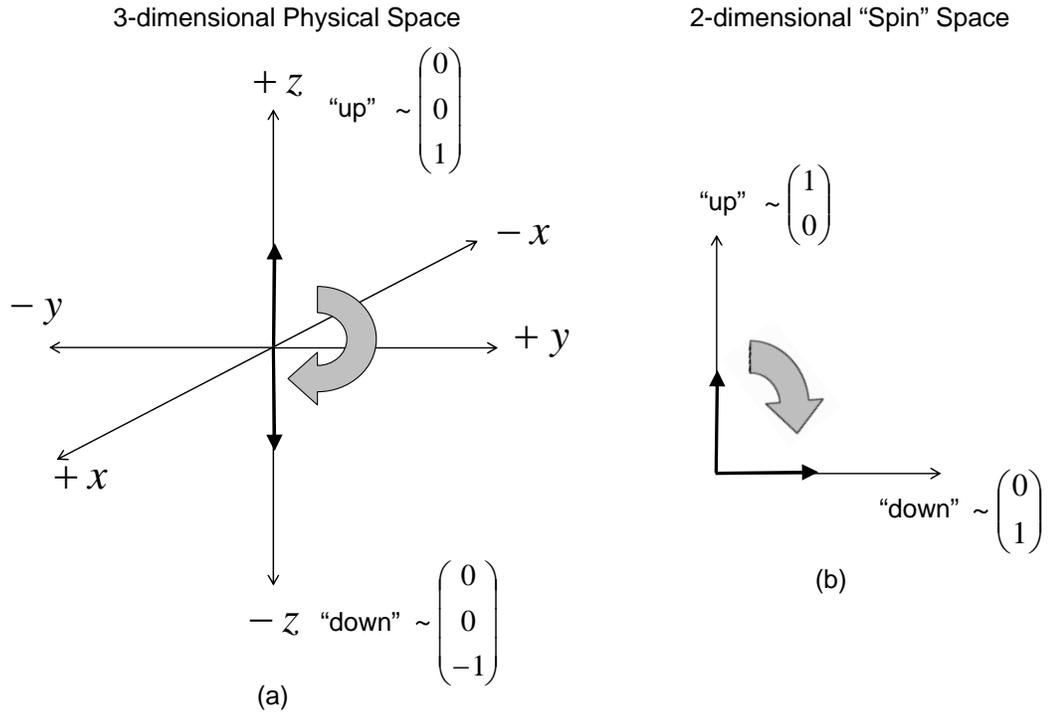
Figure 3.4:   (a) Rotations of 180° in the familiar 3-dimensional, physical space correspond to (b) rotations of only 90° in "spin" space. That is, the "up" and "down" directions in z are 180° apart in physical space but because they can completely describe all the possible outcomes of a spin-z measurement they completely span the space describing spin properties. Thus they can be thought of as an orthogonal basis of this space.

is only *half* a rotation of the charge that determines $\hat{\boldsymbol{\mu}}$ and corresponds to $\hat{\boldsymbol{S}}$. As it turns out, the fact that one full rotation in physical space corresponds to just half a rotation in spin space is exactly the result of other theoretical spin descriptions.

Also, in contrast with classical measures of angular momentum, $\hat{\boldsymbol{S}}$ can take on only $2s+1$ discrete values where s is the quantum number defining the spin characteristics of a system. For example, a measurement of the spin of the electron, for which $s = 1/2$, in a given direction can only have two results: spin "up" or spin "down"

in the given direction.[8] Due to their two-valuedness, single electron spin systems are two-level systems. These are extremely useful for modelling more complicated systems of more states because they are the simplest systems that incorporate both the properties of a single state with the phenomenon of transitions *between* states (see section 2.2.1, footnote 4). Consequently, whenever we refer to our system it is assumed to be a spin-1/2 system.

We can use these facts to define the operators corresponding to spin-1/2 systems. According to eq. (3.21) $s = 1/2$ means $N = 2$ so $\hat{\boldsymbol{S}}$ can be expressed with 2×2 matrices. Taking experimental results as constraints on the theory we often define the spin operator as

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}} \tag{3.23}$$

where

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \text{and } \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.24}$$

are the Pauli matrices in the standard representation that arbitrarily diagonalizes $\hat{\sigma}_z$.

We also represent $\hat{H}$ as a 2×2 matrix. This necessitates representing any spin state $|\psi\rangle$ as a two component vector, or spinor,

$$|\psi\rangle \rightarrow \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \psi_\uparrow \chi_\uparrow + \psi_\downarrow \chi_\downarrow \tag{3.25}$$

instead of as a scalar state as before. Also,

$$\chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.26}$$

are the eigenstates that diagonalize $\hat{\sigma}_z$. Thus the subscripts $\uparrow\downarrow$ respectively denote the spin as either "up" or "down" in the z-direction. Using eq. (3.21) we can write a general spin state with both time and space parts

$$\Psi = \psi_\uparrow \chi_\uparrow e^{-\frac{i}{\hbar}E_\uparrow t} + \psi_\downarrow \chi_\downarrow e^{-\frac{i}{\hbar}E_\downarrow t} \tag{3.27}$$

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[8]Perhaps this is easier to understand as spin "right" and spin "left" in the x or y-directions. Regardless of the direction we usually distinguish the two values as "up" and "down" (with quotation marks).

where $E_{\uparrow\downarrow}$ are the energies, i.e. the eigenvalues of $\hat{H}$, corresponding to the "up" and "down" states $\chi_{\uparrow\downarrow}$.

3.2.2 Expectation Values

Important in any theoretical treatment of quantum measurements is the concept of expectation values. For a given state $|\psi\rangle$ every operator $\hat{A}$ has an expectation value

$$\langle \hat{A} \rangle \equiv \langle \psi | \hat{A} | \psi \rangle. \tag{3.28}$$

Using the specific case that $\hat{A} = \hat{x}$ Griffiths [1] explains

> [This] emphatically does *not* mean that if you measure the position of one particle over and over again, $[\langle \hat{x} \rangle]$ is the average of the results...Rather, $\langle \hat{x} \rangle$ is the average of measurements performed on particles *all in the state* ψ, which means you must find some way of returning the particle to its original state after each measurement, or else you prepare a whole ensemble of particles, each in the same state ψ, and measure the positions of all of them: $\langle \hat{x} \rangle$ is the average of *these* results (p. 14).

Discussing this further Griffiths [1] continues, this time in terms of velocity $d\langle \hat{x} \rangle / dt$,

> Note that we're talking about the 'velocity' of the *expectation* value of x, which is not the same thing as the velocity of the *particle* (p. 15).

With these statements we see that expectation values are averages of repeated measurements on identical systems not of repeated measurements on a single system.

3.2.3 Measurement

As one can see, measurement plays a significant role in both our practice and interpretation of quantum mechanics. There is however much ambiguity and debate as to what exactly the process of measurement entails. In attempts by Bohr, Von

Neumann, Wigner, and others to clarify the ontological and/or epistemological nature of these issues some unfamiliar concepts such as the unpredictable "collapse" of the wave function or the placement of a "cut" between the quantum and classically described worlds have been introduced.[9] In what might be called the orthodox opinion "Observations not only *disturb* what is to be measured, they *produce* it...[When measuring position] we *compel* [the particle] to assume a definite position." (Jordan in [1] p. 3) However, because of the proliferation of the ambiguous and unfamiliar ideas of the "cut" and "collapse" there is only a quasi-standard conception of what measurement is.

Because the Stern-Gerlach experiment so clearly demonstrates the discrepancies between our classical and quantum expectations, with apparently simple and intuitive theoretical descriptions, it is considered a canonical, or defining, example of the quantum mechanical measurement process.

3.2.4   The Uncertainty Principle

Another characteristic of quantum mechanics that should be mentioned in the discussion of measurement and the SGE is the uncertainty principle. It simply states that for a set of non-commuting operators, say $\hat{A}$ and $\hat{B}$, whose commutator is

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = \hat{C} \tag{3.29}$$

the relation can be given

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{|\langle\hat{C}\rangle|}{2}. \tag{3.30}$$

where the square uncertainty of the particular measurement represented by any operator $\hat{A}$ is

$$\Delta\hat{A}^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2. \tag{3.31}$$

This is usually interpreted to mean that there is an irreducible uncertainty associated with the simultaneous measurements of $\hat{A}$ and $\hat{B}$. In other words, when applied to the non-commuting pair $\hat{x}$ and $\hat{p}_x$ the more confident we are of the result

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[9]For a standard discussion of measurement see [12]. For an expression of some concerns involved in these issues see [13] or [14].

of a measurement of the position x of a particle, i.e. smaller $\Delta\hat{x}$, the less certain we are of a simultaneous measurement of its momentum in that same direction $\hat{p}_x$, i.e. $\Delta\hat{p}_x$ increases. To what extent this is an ontological or epistemological principle is still a matter of interpretation.[10]

Of particular interest to our discussion here is the realization that the operators, denoting measurements of the three orthogonal spin directions $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$, do not mutually commute

$$[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k. \tag{3.32}$$

Their simultaneous existence as well defined mathematical quantities is therefore constrained by

$$\Delta\hat{S}_i\Delta\hat{S}_j \geq \epsilon_{ijk}\frac{|\langle\hat{S}_k\rangle|}{2}. \tag{3.33}$$

More specifically, in any conceivable observation of the SGE it is assumed that no two components of the spin will ever be specified with more certainty than eq. (3.33) allows.

3.2.5 Stern and Gerlach's Experiment

Now that we have built up concepts relevant to the quantum representation of the SGE we can outline what might be its typical quantum derivation (see [1] or [11]). Such a derivation usually proceeds with the intent of making as few changes as possible to the classical account.

There are some stark differences however. For example, since the quantum formalism is largely based on Lagrangian and Hamiltonian mechanics as opposed to Newton's, energy is given a more fundamental place than are forces. We therefore begin by constructing a Hamiltonian operator from the interaction energy in eq. (3.4). It is

$$\hat{H}_{interaction} = -\hat{\boldsymbol{\mu}} \cdot \mathbf{\hat{B}} \tag{3.34}$$

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[10]Again, inasmuch as [13] and [14] critique the measurement process they also raise several interesting issues involving the uncertainty principle.

where all objects are now operators. Using eqs. (3.22) with $\kappa = g = 2$ now, (3.23), and (3.24) with the field of eq. (3.8) we can represent this interaction with a matrix operator.

$$\hat{H}_{interaction} = -\frac{e}{m}\frac{\hbar}{2}\hat{\sigma}_j\hat{B}_j = -\frac{e}{m}\frac{\hbar}{2}(B_0 + bz)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.35}$$

Deriving the results of the SGE is easiest if we express the full Hamiltonian in the frame of the beam. This sets the kinetic energy terms to zero. For simplicity we also treat the field as an ideally impulsive field of duration T. In this case $\hat{H}_{interaction}$ becomes the full Hamiltonian for $0 \le t \le T$. The Hamiltonian for all times is

$$\hat{H}(t) = \begin{cases} 0 & \text{for } t < 0 \\ -\frac{e}{m}\frac{\hbar}{2}(B_0 + bz)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{for } 0 \le t \le T \,. \\ 0 & \text{for } t > T \end{cases} \tag{3.36}$$

If we restrict ourselves to times such that $0 \le t \le T$ then $\hat{H}$ is constant in time and we may solve for its eigenvalues which are the energies of the system. This can be done to get

$$E_{\uparrow\downarrow} = \mp\frac{e\hbar}{2m}(B_0 + bz). \tag{3.37}$$

There are two solutions labelled $\uparrow$ and $\downarrow$ because the system was describable by a 2×2 matrix. These are referred to as the spin "up" and the spin "down" states in the basis which diagonalizes $\hat{\sigma}_z$. They then mean "up" and "down" in z.

Using our previous results for describing a general state in terms of the eigenstates eq. (3.27) we may evaluate this state at $t = T$ and, after some rearranging, get

$$\Psi = \psi_\uparrow\chi_\uparrow e^{i\frac{e}{2m}B_0 T}e^{\frac{i}{\hbar}(\frac{ebT\hbar}{2m})z} + \psi_\downarrow\chi_\downarrow e^{-i\frac{e}{2m}B_0 T}e^{\frac{i}{\hbar}(-\frac{ebT\hbar}{2m})z}. \tag{3.38}$$

for the general state of the beam after emerging from the field at time T. This is valid then for all $t \ge T$.

If we compare this to the familiar form for an infinite plane wave travelling in the $\mathbf{k}=\mathbf{p}/\hbar$ direction with momentum $\mathbf{p}$

$$\Psi \sim e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-Et)} \tag{3.39}$$
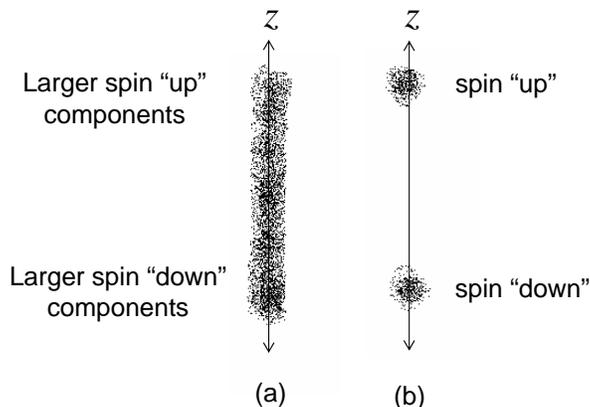
Figure 3.5: (a) The classical description of the SGE leads us to expect the particles with the largest spin *components* in the chosen direction to deflect the furthest up, the particles with the largest spin component *opposite* the field direction to deflect the furthest down, and a continuous range of deflections in between arising from the continuous range of possible spin projections on the z-axis which is preferred by the **B**-field. (b) Using a quantum mechanical representation the expected result suggests the experimental result. There is not a continuous range of possible spin deflection but only two distinct possibilities. The particles that have collapsed as if undergoing a upwards deflect likewise collapse to the spin "up" state while the particles that appear to be deflected downwards have collapsed into the spin "down" state relative to the chosen direction.

we see that there are two distinct momenta both proportional to the field gradient b and in the z-direction. Namely,

$$p_z = \pm \frac{ebT\hbar}{2m}. \tag{3.40}$$

By comparing this with eq. (3.38) we note that the $\pm$ momenta are associated, or *entangled*, with the spin "up" and "down" states respectively. Thus, the standard interpretation of the quantum formalism tells us that regardless of specific initial spin state of the beam particles in the SGE upon measurement the states will collapse to only one of two states: either they are travelling upwards along the chosen axis with a spin "up" in that direction or they are travelling downwards with spin "down." Inasmuch as the traditional SGE uses a field in which a single direction has been deliberately selected upon detection an image will appear that shows the particles to be collecting in two distinct and unconnected placed. This is in stark contrast to the classically expected result in which a single *continuous* trace appears.

Apart from clearly showing the quantized nature of spin and the divergence of quantum mechanical concepts from classical ones the SGE is taken as the canonical

example of the quantum process of measurement. This gives it an important place in the history and development of quantum mechanics.

Chapter 4

A Historical Account of the Stern-Gerlach Effect

In the previous chapter we built up a description of the SGE from fundamental concepts. This is not the only way to represent the story however. Just as we can represent a function in terms of different coordinates systems (see Table 2.2) we can also portray the development of the SGE in different ways. Each will emphasize a different aspect of the story. The historical account of the SGE, which is given here, though perhaps not found as universally in textbooks, is extremely valuable on its own.

The historical account given in this chapter does not present all the detail that could be given due to practical constraints.[1] We give here only enough detail to capture, in the end, the general nature of historical representations and, in particular, the divergence of this account of the SGE from the thematic one given previously.[2]

4.1 Atomic Models

Prior to 1900 classical mechanics was the prescribed methodology for progress in physics. It provided the most universally accepted and powerful paradigm in physics. It had begun in embryonic form with the revolution of Copernicus but was carried on by others such as Kepler, Galileo, Descartes, and set in full motion by

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[1]The careful reader will realize that this statement implies that the historical account here, and in fact historical accounts in general, are really specific types of thematic accounts. This is similar to our previous note that mathematical representations are a subclass of the more general conceptual ones. In other words, we select in chronological order only those events which we determine as appropriate to the historical themes we want to convey.

[2]For more detailed historical accounts of the original Stern-Gerlach experiment see [15] or [16]. For a historical development of quantum mechanics in general see [17].

Newton. To a large degree, since that time, physics has consisted of the working out of the implications of Newton's laws of motion. In the Kuhnian terminology of chapter 2 and [2], in Newton had culminated a shift of paradigm and most subsequent physics consisted of casting observed data in the paradigm-provided mold and not in creating the mold itself.

One phenomenon of interest during this period of "normal" science was the description of the atom. In the late 1800s a debate existed between those that thought nature was fundamentally continuous and those that considered it fundamentally discretized. These latter proponents were the atomists. But Einstein's work on Brownian motion in 1905 provided the groundwork for the first experimental demonstration of atomic behavior. Up to this point talk of atoms had been only theoretical and based on macroscopic secondary effects. Under the classical regime, atomic behavior, as every other phenomenon, was thought to strictly follow Newtonian laws. With the connection of Brownian motion to the discreteness of atomic particles, and other developments including but not confined to Einstein's other 1905 paper concerning the photoelectric effect and an earlier purely theoretical description of black body radiation by Max Planck in 1900 in which a completely *ad hoc* factor h was introduced the paradigm of discreteness, or quantization, gained widespread acceptance [18].

Accordingly, under the extant models had anyone proposed the SGE *at this time*, the theoretical description would have been similar to that which was given in section 3.1. It is important to realize however that the SGE was not even conceived of until much later, after other developments had occurred.

4.1.1 1913: The Bohr Model

From extensive spectroscopic measurements it was concluded that atoms of a particular type always seemed to emit the same definite and distinct amounts of energy. For example, when observing the light emitted from a tube of gas that had been excited with an electrical voltage the same spectrum of colors always appeared. Even more interesting was the fact that this spectrum was discrete.

In 1913 Niels Bohr developed a model of the atom that mathematically and conceptually captured this discrete behavior. It resembled, though not exactly, the familiar picture of the solar system with particles moving around the nucleus in various coplanar, circular orbits of *discrete* radii (see fig. 4.1(a)). These radii determined the energies of the atom as gravitational potential energy does in a solar model.

The discrete nature of the orbital radii also discretized the magnitude of the angular momentum **L** because of its dependence on the radius. More specifically, in order to agree with experimental findings Bohr asserted that

$$L = n\hbar, \text{ with } n = 1, 2, 3, ... \tag{4.1}$$

where $\hbar$ is related to Planck's recently introduced constant.

Because of its now outdated quantization rules and picture Bohr's scheme is known today as the "old quantum theory."

4.1.2   1916: The Sommerfeld Model

As with any model however the Bohr model of the atom did not fully describe the details of observed atomic phenomena. In 1916 Arnold Sommerfeld aided in extending the Bohr atomic model to other cases including relativistic effects and the quantization of all three components of **L** [19]. In doing so Bohr's conceptual representation was altered in a few ways.

The circular coplanar orbits that were visualized in the Bohr atom were replaced with orbits that could be distorted to elliptical shapes and could take on various orientations. They did not have to be coplanar (see fig. 4.1(b)). As a result Bohr's makeshift quantization of **L** in *magnitude* was extended to include the possibility of quantizing the *direction* of **L** as well.

4.2   1921-1922: Stern and Gerlach's Experiment

We stop here with the story of atomic models because this is the environment in which Otto Stern and Walther Gerlach found themselves in 1921. This was the model - or paradigm - with which they were working.
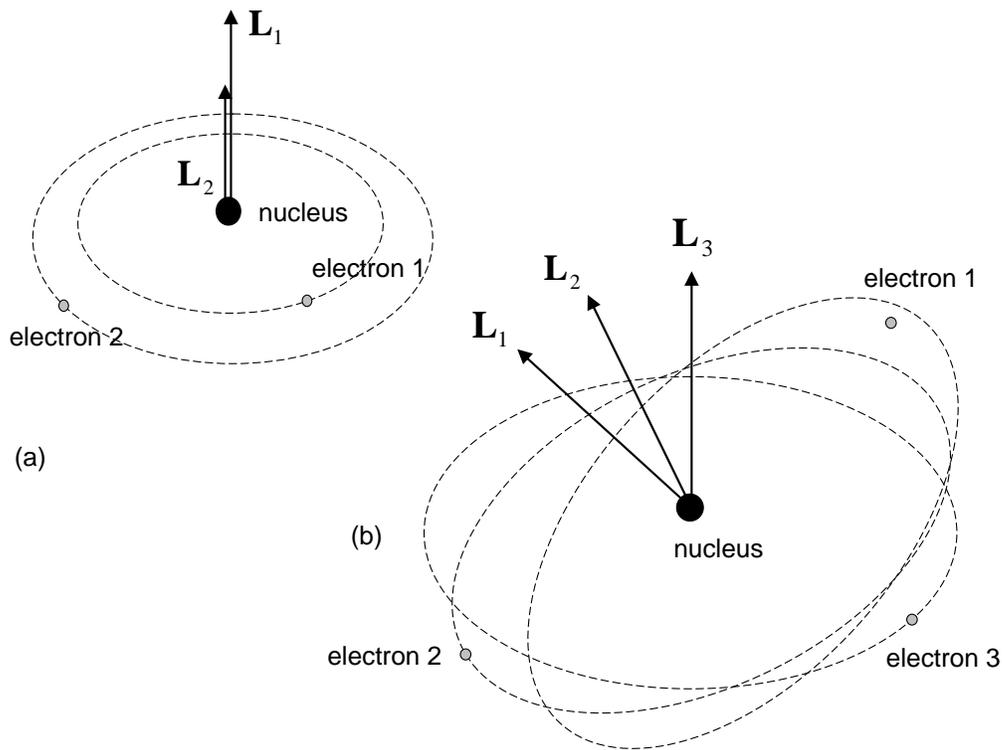
Figure 4.1: (a) In the Bohr atom the electron has a definite trajectory that can only occur at certain, discrete coplanar distances from the nucleus. Thus the magnitude L is likewise discrete or *quantized*. (b) In Sommerfeld's extension of Bohr's ideas the electron paths can occur in different planes though they were still constrained to quantized radii. Because this allowed for the additional quantization of $\mathbf{L}$ in its *spatial* orientation this is referred to as space quantization.

It was Stern and Gerlach's intent to either verify or discount the Bohr-Sommerfeld model of the atom by measuring the quantized states of $\mathbf{L}$.[3] As we have seen, based on their "old" quantum intuition Stern and Gerlach assumed that the atom possessed angular momentum made manifest in the orbit of the electron about the nucleus. This implied the presence of a magnetic moment $\boldsymbol{\mu}$ which could be manipulated via a magnetic field $\mathbf{B}$ as described in section 3.1. There were several considerations that

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[3]Ironically, although Stern is thought to have experimentally verified the Bohr model of the atom, he is reported to have said along with Max von Laue, "If this nonsense of Bohr should in the end prove right, we will quit physics!" [15]

would have been either explicitly confronted or unknowingly passed by. We list a few of these based on what seems appropriate for our purposes.[4]

4.2.1 Magnet Type

The first consideration may have been of the particular magnetic field configuration that would be used. As we saw in the classical picture of section 3.1 if Stern and Gerlach wanted to observe the magnitude of $\boldsymbol{\mu}$ *in a particular direction* there were two essential components to the field. They needed (1) a non-uniform component to $\mathbf{B}$ so as to cause the force differential needed to sift the particles by an observable amount. And (2) Stern and Gerlach needed a preferred direction to $\mathbf{B}$ in order to define the component of $\boldsymbol{\mu}$ being measured. In section 3.1 this was done with the introduction of B_0.

Spatial variations of the field had to be considered as well. Once generated with an appropriate momentum towards the detector, the particles had to enter and exit the field. Considered in the frame of the particles this would introduce the same effects as a time-dependent field. Thus maintaining the desired uniformity along the beam as well as avoiding unwanted dynamic effects would have to be considered in order to make the results clear.

4.2.2 Particle Choice

One way of avoiding several issues with the fringe field effects was to chose a very specific type of particle. Following Maxwell's equations this changing $\mathbf{B}$-field would create an electric field that could exert Lorentz forces on particles carrying charge. These forces would easily blur the beam in unintended directions disguising the outcome and interpretation of the experiment.

Stern had worked with beams of silver atoms before so this was the natural choice [16]. They are electrically neutral but still possess a magnetic moment. That is, in their neutral state they carry as many protons as electrons but only have one

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[4]The following may or may not have been exactly how these issues played out in the minds of Stern and Gerlach. It only *represents* how those issues may have been resolved in the given historical context.
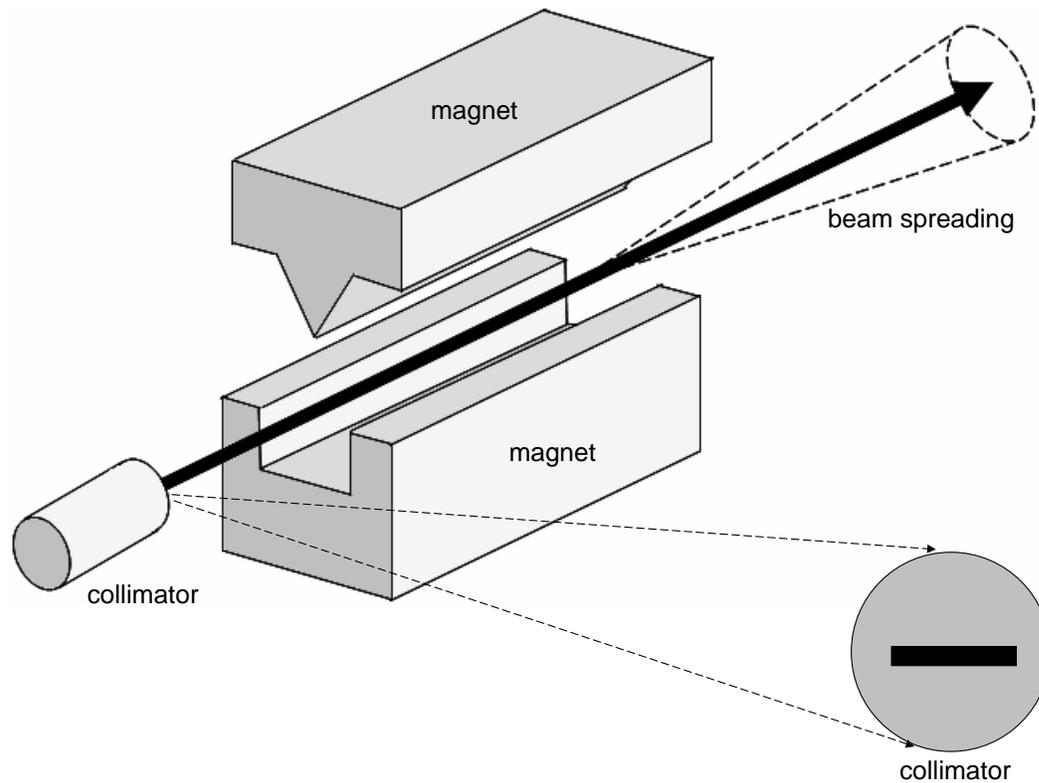
Figure 4.2: After being collimated into a long, narrow cross section, the Stern-Gerlach beam, made up of silver atoms, enters the evacuated space between two vertically oriented magnet poles, one with a notch cut into it and the other with a point. The resulting non-uniform magnetic field was 3.5cm long and .1T in strength with a 10T/cm field gradient. This caused the beam to separate in different directions due what is now thought of as spin. The beam was then detected on a cold plate of glass [16].

valence electron so that, while all charge cancels out, there is an "extra" electron with a magnetic moment orbiting the nucleus overall. Thus, a beam of silver atoms was used.

4.2.3 Beam Width

The atoms of the beam were accelerated from a vapor towards the magnets for measurement. This process imparts a random distribution of both momenta and magnetic moment to the particles of the beam. The magnetic moments were considered random only in direction. This is exactly what the experimenters wanted
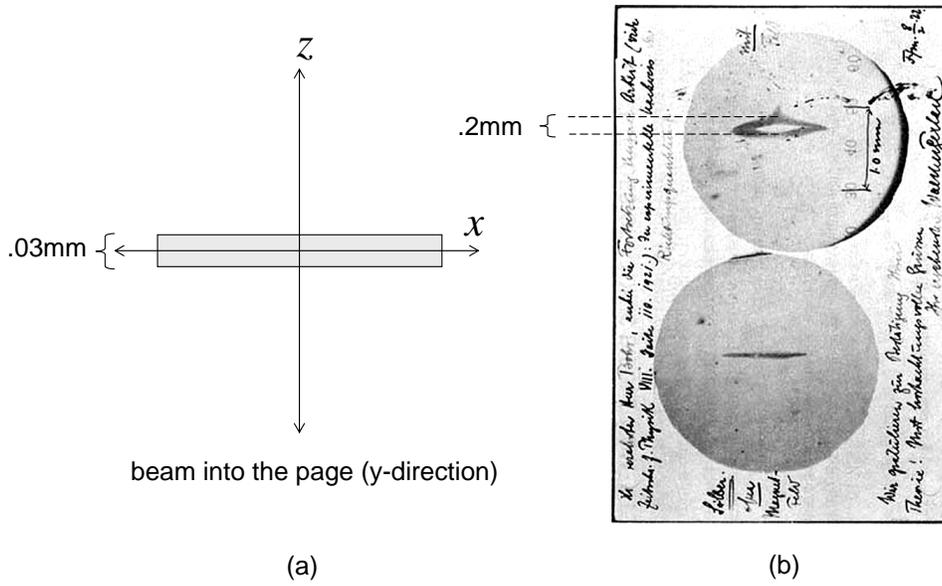
Figure 4.3:   (a) The cross section of the Stern-Gerlach beam in the y-direction was long and narrow in the xz-plane. (b) The actual images taken from the post card Gerlach sent to Bohr announcing his results. He writes, "Attached [is] the experimental proof of directional quantization. We congratulate [you] on the confirmation of your theory." Taken from [20].

in order to measure the nature of the distribution of directions. However, the atoms had to be selected according to their momenta in order to carefully direct them at the appropriate location in the measuring apparatus. This was done by collimation.

Stern and Gerlach chose to collimate the beam using a long narrow slot oriented perpendicularly to the axis along which the measurement was to be taken (the z-axis in fig. 4.3(a)) and the beam axis. This gave them a very narrow beam in the direction of interest but yielded a more diffuse beam in the perpendicular direction. As they were only concerned with one direction - the direction selected by the magnetic field - this was sufficient.

4.2.4   Results

After sending the beam through the poles of the chosen magnet it proceeded to a glass plate for detection some distance away where it left a deposition. After several attempts at fine tuning the apparatus, particularly the vacuum system, and the intervention of a cheap cigar [15] a trace was recovered that gave Stern and

Gerlach *exactly what they had predicted from theory* - a definite separation into two distinct traces oriented along the direction of the uniform field (see fig. 4.3(b)).

4.3   1925-1926: Quantum Mechanics

To this point the SGE had been spoken of in terms of mostly classical concepts and equations. *At most* it participated only in substantiating the *old* quantum theory of the Bohr-Sommerfeld atom. The fundamental equations of modern quantum mechanics had not even been developed yet. It wasn't until 1925 that Heisenberg developed his matrix formulation for the fundamental characteristics of quantum mechanics with his equation

$$i\hbar\frac{d}{dt}\hat{A} = [\hat{A}, \hat{H}] + i\hbar\frac{\partial}{\partial t}\hat{A} \tag{4.2}$$

in which the arbitrary operator $\hat{A}$ and the Hamiltonian $\hat{H}$ are the relevant objects. The next year Schrödinger proposed his more widely recognizable wave mechanics formulation using the then still mysterious wave function Ψ.[5] His famous equation

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\Psi + \hat{V}\Psi \tag{4.3}$$

is the time-dependent Schrödinger equation. It is typically found in textbook discussions of quantum mechanics and was also used in the previous chapter.

4.4   1925: Spin

The development of these two formalistic systems, the Heisenberg and Schrödinger pictures, signaled our entrance into a new era of physical science. We were undergoing a shift of paradigm. With each new unexpected result - and there were many - we were having to define and redefine the conceptual basis upon which we could build our theories and against which we would push off when we once again felt equipped to return to "normal" science.

One such result that required the introduction of a new concept for its proper placing in the framework arose from atomic physics. The spectral lines of a given

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[5]Max Born later gave Ψ its present interpretation as a probability amplitude.

element could be observed to bifurcate into two closely spaced identical lines when the element was placed in an extremely strong external magnetic field. This was the Anomalous Zeeman effect. The word that expressed the concept that was needed to connect this behavior to others in the quantum framework was coined by two graduate students, Samuel Goudsmit and George Uhlenbeck, in 1925. It was spin.

It appears that the quantum mechanical concept of spin wasn't associated with the SGE of 1922 until 1927 [21].

4.5   1935: Measurement

During the years under discussion here not only was the scientific community working out the technical implications of the new quantum regime but they were simultaneously attempting to define its conceptual foundations. As we have pointed out, one of the difficult phenomena to translate was that of measurement.

The exact timing of the designation of the SGE as the clearest experimental demonstration of the measurement problem is not well defined. From a historical perspective the phenomenon of measurement didn't force itself into the forefront of scientific philosophical discussion until 1935 when Einstein, Podolsky, and Rosen published a paper attempting to salvage some familiar, classical notions from the broadening quantum conceptual revolution.[6] Ironically, their proposal instead gave rise to the EPR paradox which, along with some later theoretical and experimental verification, was a clear and fundamental repudiation of some of the very ideas they attempted to save. It placed the quantum concepts of entanglement and measurement in the forefront of our framework. Perhaps because these two concepts are easily and clearly represented in a discussion of the SGE, as we have shown in the previous chapter, simultaneously with a clearly distinct classical analogue, the SGE has consequently been given important status as well. For this reason it is important to address the problems in our descriptions of it.

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[6]Although the EPR paper was published in 1935 debates and controversy regarding other aspects of the proper interpretation of quantum mechanics had carried on for as many as 10 years. The discussions of the Solvay conference of 1927 are particularly interesting [6].

Chapter 5

Problems With Accounts of the Stern-Gerlach Effect

We have given both a thematic, or textbook, account of the SGE and a historical account of the same. The former is based on the logical ordering necessary to systematically construct appropriate concepts and the latter by the *chrono*logical ordering of human experience. As representations both accounts emphasize different aspects. They also obscure other characteristics either inadvertently or because those characteristics are explicitly deemed less valuable. In the following we discuss the assumptions and inconsistencies that are hidden in the two previous accounts but that are rashly dismissed either because of tradition, practicality, or misunderstanding. Before considering the limitations of these two accounts however we will consider some limitations that apply more generally to quantum mechanics, and even science, as a whole.

5.1 Problems from Quantum Mechanics

In section 2.2.2 we introduced wave-particle duality as an example of orthogonal concepts. We called such concepts "orthogonal" because, like orthogonal vectors or functions, the concepts implied by one cannot in any way be represented in terms of the concepts implied by the other. In the language of logic the two are mutually exclusive. In this sense orthogonal concepts are also describable in terms of Bohr's principle of complementarity in which two disparate but complementary concepts are used in order to fully describe a single phenomenon.

5.1.1 Rightness and Clarity

Among the many sets of concepts that can serve as an effective conceptual basis for describing any scientific representation of a phenomenon are the concepts of rightness and clarity.[1] They are an effective basis because they are "orthogonal" or complementary. As applied to representations we can see that the more effort that

Some Orthogonal Concepts	
Wave	Particle
Position	Momentum
Theoretical	Experimental
Quantitative	Qualitative
Historical	Thematic
Abstract	Concrete
Form	Function
Creativity	Conformity
Constancy	Flexibility
Humility	Confidence
Patience	Ambition
Justice	Mercy

Table 5.1: A listing of some orthogonal concepts. Bohr applied his principle of complementarity to ideas such as these [7]. Perhaps Kuhn's description of "normal" and "revolutionary" science could be added as well. Just as with coordinate axes it is precisely because of their orthogonality that these sets of concepts are useful in describing other, more complicated, ideas.

is put into making a presentation simple, clear, and accessible the more idealization, approximation, and artificiality are in it. Whereas if a strenuous effort is made to represent all the facts in their proper context so much complication and technicality are introduced that a useful understanding of them becomes near impossible. In short, the more clear a representation is made to be the less right it is whereas the

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[1]Bohr is sometimes attributed with the recognition of this complementary relationship between "rightness" and "clarity." However, the most relevant discussion that we can find by Bohr is in [7] in which he speaks of the complementary relationship between the "use" and "meaning" of words. See also [22] for some of his related writings.

more right it is the less clear it can be. Thus there is a fundamental tension between the rightness and clarity of a statement.

5.1.2   Representations as a Map

This relationship between the rightness and clarity of a statement can be demonstrated by considering a map. Maps are representations of regions of space. The more detailed the map the bigger and bulkier must be its pages with expanded scaling and legends whereas the more compact and simple the map the less detail can be described by it. To maximize its correspondence to reality at the expense of user-friendliness would make the map no different than the terrain it describes while increasing its immediate and efficient use would surely omit some detail that could become important.

5.1.3   Communication

The simultaneous maximization of both the clarity and rightness of a statement is the aim of effective representation and communication. However, in quantum mechanics this discrepancy between what *is* and what we *understand* is manifest in even stronger terms than in other fields. It was a common opinion among the architects of quantum theory that although quantum phenomena were incompatible with classical concepts, due to the classical nature of the equipment, i.e. its compatibility with humans on the macro-scale, only these concepts could be used to describe them. For example, Bohr [22] has said that

> In this context, we must recognize above all that, even when the phenomena transcend the scope of classical physical theories, the account of the experimental arrangement and the recording of observations must be given in plain language. (p. 72)

and Heisenberg [23] has written that

Any experiment in physics, whether it refers to the phenomena of daily life or atomic events, is to be described in the terms of classical physics. (p. 44)

In other words, the common opinion is that although nature fundamentally behaves quantum mechanically humans can only understand it in terms of classical concepts because we can only interact with this level.[2] Unfortunately for us, classical concepts are also inadequate.

Thus, it is a frustrating axiom of the modern paradigm that *clear* communication is exactly opposed to *correct* communication. As one aspect is refined or improved the other is helplessly compromised. This may be suggestively expressed in the schematic form

$$\Delta\text{Clear}\Delta\text{Right} \geq \text{constant}, \tag{5.1}$$

reminiscent of eq. (3.30).

When students find this out for themselves it is not unlike entering hell in the *Divine Comedy*.[3]

5.2   Problems with Historical Accounts

In historical accounts some details might be missed - either deliberately or ignorantly. However, of more concern for us here is their lack of clarity. Consequently we will not spend time here to discuss the problems of historical accounts as regards to facts but we will only emphasize their pedagogical limitations.

Kuhn [2] tells us of some of the concerns common to these sorts of accounts.

More historical detail, whether of science's present or of its past, or more responsibility to the historical details that are presented, could only give artificial status to human idiosyncrasy, error, and confusion. Why dignify what science's best and most persistent efforts have made possible to discard? (p. 138)

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[2]For a related discussion see also [24] (p. 127-130).
[3]In Dante's classic *The Divine Comedy* the inscription above the doorway leading to hell read "Abandon hope, all ye who enter here."

Indeed, including all the facts associated with each "dead end" pursued by each researcher is impractical and surely confusing to students at an introductory level. At that point students need a clear, linear progression that builds the concepts in a logical way so as to make them as graspable as possible. Unfortunately this is not how history typically unfolds. The development of science is often more messy than it is linear. We usually run into several problems and backtrack several times before making a small but successful advance. Sometimes paths are vigorously pursued only to show they don't yield the anticipated result.[4] Thus, in the attempt to make an explanation of the SGE correspond to the reality of the events, which for some purposes is a worthy goal,[5] for instructional purposes clarity is lost.

5.3 Problems with Thematic Accounts

On the other hand, the strength of thematic accounts of scientific events is precisely their clarity. However, according to the dual relationship hinted at in eq. (5.1) this implies only a crude correspondence to fact or reality. As the bulk of this work will assess the accuracy of thematic accounts with a special emphasis on their technical aspects (see the next chapter, in particular) we will spend the remainder of this chapter developing these ideas.

5.3.1 An Artificial History

First, in a general sense opposite to historical accounts discussed in section 5.2 the thematic accounts of textbook or classroom discussions are very linear and cumulative in their presentation and use of concepts. That is, before a particular concept is needed in the description of an event that concept is either motivated or derived from previous concepts. This of course removes concepts out of their historical context and therefore assigns them an artificial and unscientifically acquired meaning. This also necessitates the eventual acceptance of certain axioms which ultimately have

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[4]While such a path may be fruitless as regarding the *desired* result, inasmuch as an understanding of what definitely will not work is gained this is a vital step *forward* in producing progress.

[5]Since section 5.3.1 outlines problems with artificial histories it also indirectly discusses the benefits of historically accurate accounts.

no justification but are deemed useful. However, convenience is too often confused with correctness. These considerations lead us to the conclusion that there is little that necessarily corresponds to reality in these accounts. Their final justification is only that they are clear and that they posses an accurate predictive power. This says little, however, about the accuracy of the conceptual structure that had to be built up.

From contrasting the thematic account in Chapter 3 with the historical one of the Chapter 4 we can surmise that there are several misconceptions that students could have after having been taught the SGE.

(1) The original SGE was observed several years *prior* to the development of the quantum theory and spin. Hence, although it is often used as a confirmation or demonstration of modern quantum concepts such as spin and measurement it was in no way motivated by or intended for this purpose.

(2) The observation of the SGE actually *confirmed* the historically prevalent theory of the time - the Bohr-Sommerfeld model of the atom - which is now considered false. It did not surprise the experimenters for this is exactly what Stern had predicted previously [25]. It did not signal the opening of an as yet unknown door in physics or philosophy. Those doors had either already been open, such as with the Bohr model's quantization, or would not be open for years, such as the incorporation of spin into the quantum framework. Instead it reinforced the relatively unexciting continuance of the prevalent paradigm.

(3) It is interesting to note that the SGE clearly demonstrates how the right results[6] - even experimental results - can be used to justify an erroneous account of the phenomena. They then can also be given canonical and defining status in a completely new conceptual system [26]. This demonstrates well the subjectivity of representations in science well.[7]

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[6]This is clumsy language. Results can never be wrong especially in experimental physics; only our questions or our interpretations of the results can be difficult.

[7]This is similar to the researcher who can always seem to get the answer he's looking for precisely because he's looking so hard for it.

Because it is the primary purpose of textbooks and all thematic accounts to communicate a paradigm rather than to communicate facts they necessarily disguise the process of paradigm formation and selection, which is the process of scientific revolution.[8] Kuhn [2] writes

> The result [of the thematic approach of textbooks] is a persistent tendency to make the history of science look linear or cumulative... The textbook tendency to make the development of science linear hides a process that lies at the heart of the most significant episodes of scientific development. (p. 137-140)

Hiding the "significant episodes" in science is a general deficiency of thematic accounts but the main strength of historical ones.

5.3.2 Misunderstanding the Practical Aspects of the Stern-Gerlach Effect

Related to misconceptions as to the historical ordering and significance of events are the practical experimental considerations which we included in sections 4.2.1-4.2.3. They concern (1) the species of particle in the beam, (2) selection of beam cross section, and (3) the type of magnetic field used.

The usual de-emphasis of these three points in thematic accounts of the SGE has some interesting consequences.

(1) (See section 4.2.2) Because all the talk of the SGE is couched in terms of spin-1/2 particles the idea, if not the word explicitly, of the electron is used. It is considered the canonical spin-1/2 particle. However, this practice hides a whole field of very interesting research. Students don't realize that there is a time-dependence to the magnetic field due to its approaching speed relative to the particle which exerts a transverse Lorentz force on it. Because this phenomenon is dependent on the presence of charge the SGE has never been observed with electrons. In fact, Bohr and Pauli, among others were of the opinion that the electron SGE could *never* be observed [27].

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[8]Kuhn asserts that this is shown by the fact that textbooks must be rewritten at the completion of each paradigm shift, or revolution.

It is awkward to use electrons as a canonical example of spin-1/2 particles when they cannot display the SGE as do other spin-1/2 particles like silver.

Despite this, there is much research going on to overcome these blurring effects with fruitful results (see [28]). Because this whole field opens more questions than it answers, which compromises clarity, it is often neglected in thematic accounts.[9]

(2) (See section 4.2.3) The astute textbook reader might find an inconsistency between what is said and what is shown. Typically when the idealized SGE is discussed in textbooks it is spoken of in terms of either a point-like or infinite plane wave beam, if it is spoken of at all. However, when accompanied by an image of the original trace of the experiment (see fig. 4.3(b)) it is confusing. If both the discussion and the trace are taken as accurate then it can appear that there was a continuous blurring in the horizontal direction as well as the discrete separation in the field direction. If this is not explicitly dealt with it is inconsistent with the communicated interpretation whereas if it is pointed out it can incorporate many unnecessary experimental details making the account bulky and difficult.[10]

The previous point is especially important when considering the SGE as a demonstration of quantum measurement because it is accepted that the *entire* experimental context defines the phenomenon. In other words, if this is not made clear then it is unclear as to what we are really measuring. Students may be left wondering if a point-like SGE has ever been attempted and if not, why not? This is difficult to determine from thematic accounts because so little is ever said about the specifics of the collimation.

(3) (See section 4.2.1) As a discussion of the misconceptions regarding the specifics of the magnetic field introduces the more technical aspects of thematic accounts and has lead to numerous questions it is treated in its own section.

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[9]This statement carries the implication that inquiry-based teaching is a tricky practice.

[10]Regardless of the beam cross sectional shape there is a "broadening" of the beam caused by "the thermal distribution of velocities in the beam" mentioned in [16] (p. 176). This thermal blurring would have occurred in both the parallel and transverse directions of the field but it is not due to the SGE and spin. Careful students may confuse the broad transverse collimation (x-axis in fig. 4.3(a)) with thermal and spin blurring neither of which are deemed to have a significant effect in the traditional SGE.

5.4 The Choice of Magnetic Field

As we saw in section 3.1 the magnetic field

$$\mathbf{B} = (B_0 + bz)\hat{z} \tag{5.2}$$

is the field typically used to theoretically represent the question that Stern and Gerlach asked. It is usually given in this form in many discussions of the SGE. It is rarely mentioned however, especially in explicit terms, that there are other physical constraints on the field.

Maxwell's equations for all electromagnetic fields in vacuum specify that $\mathbf{B}$ must satisfy

$$\nabla \cdot \mathbf{B} = 0 \tag{5.3}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{5.4}$$

where $\mathbf{E}$ is the electric field and c the speed of light. Because of eq. (5.3) $\mathbf{B}$ must be inhomogeneous in *at least two* directions

$$\nabla \cdot \mathbf{B} = \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z = 0. \tag{5.5}$$

So a better choice for the field would be

$$\mathbf{B} = -bx\hat{x} + (B_0 + bz)\hat{z} \tag{5.6}$$

because

$$\frac{\partial}{\partial x}B_x = -\frac{\partial}{\partial z}B_z. \tag{5.7}$$

As a fortunate consequence $\nabla \times \mathbf{B} = 0$ so in accordance with eq. (5.4) there are no dynamically changing fields. This is the *simplest* magnetic field that satisfies the conditions of Stern and Gerlach's question - it has an inhomogeneous part and a *clearly defined direction* via $B_0\hat{z}$ - as well as Maxwell's physical constraints. For reasons of clarity the field is often not treated this way. This is however the difference between a description of the SGE that is consistent with the fundamental equations of electricity and magnetism. and one that is unphysical.

It should also be pointed out that there is another constraint on the field. It may prove important later to require the field to remain finite at all distances $r = \sqrt{x^2 + z^2}$. As it is now, the field linearly blows up with increasing r away from the origin. That is,

$$\lim_{r \to \infty} \mathbf{B} = \infty. \tag{5.8}$$

To get around this we can either append the field with an exponential factor that enforces the required asymptotic fall off or merely truncate our range of interest to a finite region. The second of these is more artificial than the first. These considerations will be taken up in more detail in sections 7.5.7 and 7.8-7.9.

5.5 Precession Arguments

Notwithstanding the inaccuracies in the field of eq. (5.2), as we have shown, it can still yield correct results. This indicates that there must be some physical justification for neglecting it. When the necessary complication of the full field eq. (5.6) is used it is the phenomenon of precession that is said to justify the neglect of the transverse inhomogeneity. We will consider how this is done in both the classical and quantum regimes.

5.5.1 Classical Precession

We have shown in section 3.1 and it has been more rigourously demonstrated in [29] and [30] that the presence of the homogeneous field component $\mathbf{B}_0$ controlls precession of the classical vector $\boldsymbol{\mu}$. From the torque equation

$$\frac{d}{dt}\boldsymbol{\mu} = \boldsymbol{\omega} \times \boldsymbol{\mu} \tag{5.9}$$

$\boldsymbol{\mu}$ can be found and substituted into the force equation eq. (3.5). [29] and [30] show how when the time average of the force is calculated only the force in the z-direction turns out to be non-zero. From this fact it is argued that the inclusion of precession *in the right way* justifies the neglect of all x-oriented dynamics.

It is interesting to consider what the "right way" consists of. Even in this classical picture the approximations are crude. As [30] explicitly points out the required relation is not just that there is a homogeneous component to the field but that

$$|\mathbf{B}_{homogeneous}| \gg |\mathbf{B}_{inhomogeneous}| \tag{5.10}$$

or, in the case of eq. (5.6)

$$|B_0| \gg |br|. \tag{5.11}$$

What is more subtle is that [30] then applies this particular criterion with the approximation that

$$\mathbf{B} \approx B_0 \hat{z}. \tag{5.12}$$

In other words, the inequality eq. (5.11) is only useful if it justifies the complete neglect of the inhomogeneity. So, in the classical treatment of [29] and [30], the field necessary to induce adequate precession, i.e. a strong uniform field, is exactly the field that prohibits the separation of spins by a net force, i.e. an inhomogeneous part. This is crude at best and inconsistent at worst.

There is one further condition on the experimental set up that is required for practitioners of classical physics to invoke the precession argument. Conditions must be chosen such that the time of interaction, or the time it takes the particle to pass through the region of space in which the field is significant, must be much greater than the period of precession. This was assumed in performing the time averaging integrals mentioned above. This condition allows the x-directed forces to adequately "wash out." Without this the average x-force would tend to favor one side or the other (see fig. 5.1).

5.5.2   Quantum Precession

In thematic accounts in which the quantum version of the precession argument is used to justify the use of a non-Maxwellian field configuration we realize that it is the only the *expectation value* of the spin that precesses. Based on the discussion in section 3.2.2 the interpretation of this statement is very subtle.
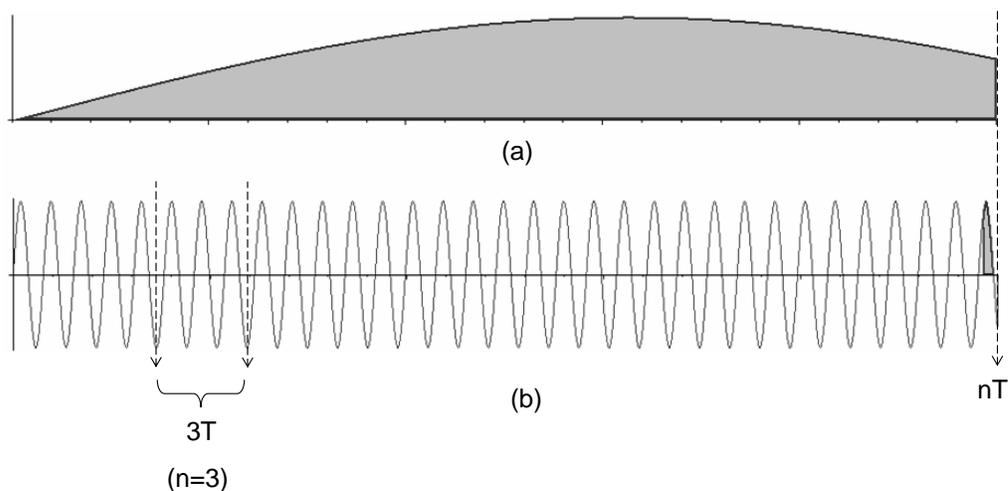
Figure 5.1: (a) If the time of interaction nT is roughly equivalent to or less than the period of the precession frequency T, that is $n \leq 1$ where n is the number of cycles is a given time, then the net area between the precessing curve and the axis, i.e. the shaded region, is large. (b) If $n \gg 1$ such that several precession cycles occur during the time of interaction nT then the portion of the shaded region that does not cancel out is much smaller. In other words, for the same total time of interaction the net shaded region for a slowly precessing spin is much larger than that for a rapidly precessing spin. Inasmuch as the shaded region corresponds to the accumulated effect of the transverse force in time, i.e. its time-average (see eq. (5.24) in the next section), the transverse deflection of the beam only washes out in (b). For (a) we would still expect a significant transverse deflection.

Note that in the present context we use "spin" $\hat{\boldsymbol{S}}$ instead of "magnetic moment" $\hat{\boldsymbol{\mu}}$ according to their relation

$$\hat{\boldsymbol{\mu}} = \frac{ge}{2m}\hat{\mathbf{S}}. \tag{5.13}$$

Recall also that

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}} \tag{5.14}$$

where

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \text{and} \ \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.15}$$

as before.

Precessing Expectation Values

In order to show this we consider a spin-1/2 particle in a uniform **B**-field

$$\mathbf{B} = B_0 \hat{z}. \tag{5.16}$$

Using this in the Hamiltonian

$$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} \tag{5.17}$$

we can find the energies

$$E_{\uparrow\downarrow} = \mp \frac{eB_0\hbar}{2m} \tag{5.18}$$

and general states

$$\Psi = \psi_\uparrow \chi_\uparrow e^{-\frac{i}{\hbar}E_\uparrow t} + \psi_\downarrow \chi_\downarrow e^{-\frac{i}{\hbar}E_\downarrow t} \tag{5.19}$$

as we did in section 3.2.1

With these states, the expectation value of $\hat{\boldsymbol{S}}$, or more specifically $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$, and $\langle \hat{S}_z \rangle$, can be evaluated. For $\hat{S}_x$ we have

$$\langle \hat{S}_x \rangle = \langle \Psi | \hat{S}_x | \Psi \rangle = \frac{\hbar}{2} \left(\psi_\uparrow^* \psi_\downarrow e^{\frac{i}{\hbar}(E_\downarrow - E_\uparrow)t} + \psi_\downarrow^* \psi_\uparrow e^{\frac{i}{\hbar}(E_\uparrow - E_\downarrow)t} \right). \tag{5.20}$$

Note that all the time-dependence is carried by factors that depend on the difference of the energy of the two states, more specifically $\pm(E_\uparrow - E_\downarrow)$.

Applying this procedure to $\hat{S}_y$ and $\hat{S}_z$ as well we get

$$\langle \hat{S}_x \rangle = \frac{\hbar}{2} \sin\theta_0 \cos\left(\frac{eB_0}{m}t\right) \tag{5.21}$$

$$\langle \hat{S}_y \rangle = -\frac{\hbar}{2} \sin\theta_0 \cos\left(\frac{eB_0}{m}t\right) \tag{5.22}$$

$$\langle \hat{S}_z \rangle = \frac{\hbar}{2} \cos\theta_0 \tag{5.23}$$

where θ_0 is the constant angle the spin vector makes with the field direction (see fig. 3.2). Thus, as time progresses $\langle \hat{S}_z \rangle$ is constant and $\langle \hat{S} \rangle_{transverse} = \langle \hat{S}_x \rangle \hat{x} + \langle \hat{S}_y \rangle \hat{y}$ rotates, or precesses, in the xy-plane.

To clearly see the effect of this precession on quantum expectation values we take the time average. In general if we wanted to find the time average of some

oscillating function $f(t)$ where the frequency of oscillation is $\omega = 2\pi/T$ over n periods T. We would evaluate

$$f(t)_{avg} = \frac{1}{nT} \int_0^{nT} f(t)dt \qquad (5.24)$$

Applying this averaging formula to the expectation values above we see that

$$\langle S_x \rangle_{avg} = 0 \qquad (5.25)$$
$$\langle S_y \rangle_{avg} = 0 \qquad (5.26)$$
$$\langle S_z \rangle_{avg} = \frac{\hbar}{2}. \qquad (5.27)$$

Thus, over significant time intervals the expectation values $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ "wash out," or go to zero, whereas $\langle \hat{S}_z \rangle$ is constant.

Problems with Precession

In the standard description of the SGE this "washing out" effect of the expectation values is referred to in order to justify the neglect of the transverse inhomogeneity of eq. (5.6) and the use of the divergenceless field eq. (5.2). Some merely cite the classical case of the phenomenon as justification (see [3]) while others refer to the fully quantum demonstration given above (see [1] and [11]). Even in the latter case however, when the demonstration is explicit, it is inconsistent for at least three reasons.

(1) The averaging procedure used above was only valid "over significant time intervals" as compared to the precession period T (see discussion of fig. 5.1). This was critical in justifying the precession arguments we used above. However, inasmuch as the period depends on the various energy states $E_{\uparrow\downarrow}$ as

$$T = \frac{2\pi}{\omega} = \frac{2\pi\hbar}{\mp(E_\uparrow - E_\downarrow)} \qquad (5.28)$$

this can only be rigorously justified *after* having an understanding of the energies of the system. Because we use the precession argument to enable us to find the energies we cannot also use the energies to justify the argument. This is circular reasoning. Although intuitively easy, which is valuable, it is logically invalid and probably glosses over several interesting questions.

(2) The precessing solutions found in the homogeneous field of eq. (5.16) were rigourously obtained but then subjectively applied to a completely different problem - one involving the field of eq. (5.6). This assumes that the interaction of these phenomena - the uniform and non-uniform parts of the field - is linear and can be naively superposed. However, it will be seen in Chapter 7 that solving only the non-uniform part is not at all trivial. This suggests that more than just a linear interaction is occurring.

(3) Finally, just because the expectation value time-averages to zero doesn't mean the *measured* value *is* zero or even that it is close to zero! According to the interpretation discussed in section 3.2.2 what we have shown here only implies that the *average* of several measurements all performed on identical systems will be zero but any one could be arbitrarily large. This says nothing about one measurement in particular.[11]

From all these considerations it is obvious that at best the precession argument that is traditionally invoked in thematic accounts of the SGE disguises several interesting questions and at worst is invalid and inaccurate.

5.5.3   Precession and the Uncertainty Principle

If there is a possibility that the precession argument is misapplied in the standard interpretation of the SGE then there is also a possibility of other misinterpretations. One of these has to do with the uncertainty principle.

We saw in section 3.2.4 that for any two non-commuting operators there is a corresponding uncertainty relationship which is typically interpreted as a constraint on the physical process of measurement. That is, as a quantity is measured to a given degree of precision the quantum state of the system being measured is altered in such a way as to limit the precision with which another conjugate quantity can be simultaneously measured.

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[11]Exactly how much it says about the distribution of a collection of many measurements is dictated by eq. (3.31).

Because the three components of the spin operator $\hat{\boldsymbol{S}}$ do not mutually commute they satisfy an uncertainty relationship as well

$$\Delta \hat{S}_i \Delta \hat{S}_j \geq \epsilon_{ijk} \frac{|\langle \hat{S}_k \rangle|}{2}. \tag{5.29}$$

We interpret this to mean that we cannot simultaneously measure two components of the spin to an arbitrary degree of accuracy. It is precisely the phenomenon of precession that allows this interpretation.

If for some reason precession could not be invoked as a valid occurrence then in the SGE our classical intuition would lead us to believe that the particle would arrive at the detection screen purely due to spin forces. If the particle were found at a 45° angle from the location of the localized y-directed beam in the field then we would assume the particle felt an equal force in both the x and z-directions. The formalism tells us that these forces, and thus deflections, arise in proportion to the spin component in the corresponding direction. Therefore, we would classically interpret such a result as a simultaneous measurement of both the x and z-components of the magnetic moment.

In a quantum context the same arguments apply but with more at stake. It is the relative strength of one direction to the other that justifies the presence of precession and it is this precession that gives the "washing out," or ambiguity, of the transverse spin components necessary to the standard interpretation of the SGE in terms of the uncertainty principle. From this it seems that if the preferred direction were removed for the Stern-Gerlach measurement of a *single particle*, precession would also be removed and there would be nothing to rescue us from simultaneously measuring, or assigning definite values to, two orthogonal spin components of a single particle. Thus, it is possible that our present understanding of the uncertainty principle only follows from our choice of field and not from the nature of the particles themselves.

We accordingly will use the problems outlined here as a motivation and guide to the work and questions of the next chapter. In addition to the broad problems outlined so far there are many other smaller and more specific problems that arise

in discussions of the SGE. These will be brought up in the appropriate places in the next chapter as we investigate the nature of the SGE in several contexts and in more depth. This offers us several interesting possibilities for not only gaining a deeper understanding of the SGE but, more generally, of our interpretations of physics.

Chapter 6

The Proposal

The SGE is widely thought to have a description that is not only conceptually very clean with relatively few mathematical technicalities but that clearly demonstrates some fundamental characteristics of quantum behavior. Yet despite its clear break from classical expectations as we saw in the previous chapter our justifications for the SGE description have yet to break free of classical traditions. Consequently, there are few, if any, fully quantum descriptions that do not make reference to outdated notions.[1]

As an example of this, we see that in order to motivate the use of a magnetic field which clearly violates Maxwell's equations many authors cite and/or derive the *classical* phenomenon of precession. When particular experimental conditions are met, i.e. $n \gg 1$, this causes an averaging away of spin components transverse to the magnetic field so that the transverse behavior can be ignored from the beginning. However, as we saw in section 5.5 this is an *ad hoc* assumption and has not been shown to easily follow from rigorous solutions. We are therefore unaware of its implications. In addition, while this approach may be sufficient classically when applied to quantum mechanical descriptions of nature only *expectation values* can be expected to precess. Invoking this argument then in reference to the SGE blurs the distinction between classical and quantum conceptual systems which, because of their stark differences and the necessity of clear and consistent conceptual representations for rational communication (see section 2.1), leads to further confusion. If

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[1][31] and [32] are interesting papers that make reference and attempt to correct this dearth of quantum treatments of the SGE so work in this area is mounting.

the quantum formalism is complete it should provide us with an appropriate basis for describing the SGE.[2] Not only will this demonstrate consistency but it will allow us to uncover questions that have previously remained unasked merely because of self-enforced conceptual boundaries.

6.1 Motivation: Positivist and Realist Representations

The formulation of a specific question that attempts to reveal the true nature and applicability of the SGE with its precession arguments can be motivated by contrasting two opposing conceptual, or representational, systems. We will call these the positivist and realist views (see [1] and [10]).

The positivist position, which most closely resembles the orthodox or mainstream position, has become known as the Copenhagen interpretation. It states that prior to the measurement of a particular quantum property that property *ontologically*[3] did not have a well defined value. It is the act of measurement that *compels* the particle to assume a definite value. In terms of a spin measurement via a Stern-Gerlach apparatus, spin is not considered definite, i.e. it is not considered to take on *one* orientation or *one* magnitude, until the particle strikes the detection plate.[4] Prior to this, the most that can be said of the particle's "spin" is that it was in an ontological superposition of spin "up" and "down" states. In terms of position the particle was in a superposition of deflecting "up" and deflecting "down".

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[2]Because we have spoken of the SGE as an *axiom* of modern physics there could be an interesting discussion here on the applicability of Gödel's Incompleteness Theorem.

[3]This is the most controversial word in the debate between interpretive schemes in physics. As we will see shortly realists would replace it with "epistemologically." Although we will use these concepts here it should be realized that the *specific* identification of the content of the realist and positivist positions is difficult because there is so little consensus as to particulars. There are intermediary positions in which existence is still independent of measurement but completely unapproachable due to its effects. Philosophically this might correspond to variants of Kant's discussion of noumena and phenomena. We choose to use these basis concepts in their *orthogonal* form so as to increase the clarity of the discussion.

[4]This statement emphasizes clarity at the expense of rightness. Exactly *when* measurement occurs is an open question contributing to the quantum problem of measurement summarized in section 3.2.3. Does it occur upon collision of the particle with the plate, with the gaze of a conscious observer, or somewhere in between (see [13] and [14])? In an ironic twist of quantum fate because we cannot measure measurement it seems we will never quite have a *definite* answer.

In the realist opinion, these properties exist independent of the measurement process but are affected in complicated ways by it. In other words, the *existence* of these properties is independent but their specific *value* is not.[5] As applied to spin, in this view there is always a spin property to a particle, perhaps with value zero, though its particular orientation and magnitude are changed in ways that are heavily dependent upon the entire experimental arrangement, i.e. the distance to the screen, the orientation of the field, etc. [14].[6]

Because in the positivist view the spin is not said to be an element of reality until measurement occurs it seems inconsistent to invoke the precession argument to justify the neglect of part of the field. Its neglect must be attributed to something other than the property of spin. The realist view has no such problem. In this sense, the SGE could help direct our interpretations of physics along either more realist and positivist lines.

6.2 The Necessity of Precession and the Magnetic Field

Because developing a deeper understanding of the SGE depends critically on the precession argument we must ascertain the true effect of precession, its validity, and its range of applicability. This can be done by considering its interaction with the spin or magnetic moment and its dependence on the homogeneous field component.

Is there any reason to apply this homogeneous field component? Its only purpose classically seems to be to induce precession about the direction of interest so that only components in that direction will be clearly observed. As mentioned in section 5.5.3 the question becomes more interesting when we consider its implications in the quantum picture.

Although it seems it was introduced historically only to label the direction of interest it seems it has been preserved through the quantum revolution only because our conceptual picture, the prevailing positivist view, requires it. Put differently, whereas in the old paradigm it was *justified* by our desire to measure only a single

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[5][5] outlines both the realist metaphysics and epistemology that has most influenced this thesis.
[6][33] is a more technical development of the ideas and interpretation in [14].

component of the magnetic moment, it has now become the *justification* for our belief that measuring only one component is possible, via the uncertainty principle.

This demonstrates the little noticed effect of paradigm on experimental practice. Kuhn [2] explains that

> consciously or not, the decision to employ a particular piece of apparatus and to use it in a particular way [as with the homogeneous field component] carries an assumption that only certain sorts of circumstances will arise. (p. 59)

In addition to the usual theoretical expectations these instrumental expectations "have often played a decisive role in scientific development."

6.3   Proposal: The Inhomogeneous Stern-Gerlach Effect

The most direct way that we propose to discover the true nature and effect of the precession argument on not only the measurement of spin but on our theoretical description and general interpretation of measurement as well is to theoretically remove the homogeneous field component B_0 which selects a universally preferred direction and which is largely the precession inducing agent. More explicitly, we propose to solve the SGE with a beam travelling in the y-direction by replacing the typically studied field configuration

$$\mathbf{B} = (B_0 + bz)\hat{z} \tag{6.1}$$

with the its more natural choice

$$\mathbf{B} = -bx\hat{x} + (B_0 + bz)\hat{z}. \tag{6.2}$$

Here B_0 is only included so that by it the phenomenon of precession can be *explicitly shown* as opposed to its usual *imposition*. We can then either set $B_0 = 0$ in order to solve the inhomogeneous SGE (ISGE) or leave it as a large non-zero constant to test the known Stern-Gerlach limit.

Note that either choice is physically consistent with the requirement that

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{6.3}$$

The field does however blow up linearly far away from the origin.

6.3.1 The Experimental Arrangement

If we consider the physical realization of this field we can perhaps avoid this difficulty. If we set $B_0 = 0$ the field eq. (6.2) may be considered an approximation to a configuration of four parallel wires carrying current I located at the four corners of a rectangle of sides $2x_1$ and $2z_1$ (see fig. 6.1(a)). With the origin in the xz-plane equidistant from each wire the field may be described by the column vector

$$\mathbf{B} = \begin{pmatrix} B_x \\ B_z \end{pmatrix} = \tag{6.4}$$

$$= \frac{\mu_0 I}{2\pi} \begin{pmatrix} \frac{z_1-z}{(x_1+x)^2+(z_1-z)^2} - \frac{z_1-z}{(x_1-x)^2+(z_1-z)^2} + \frac{z_1+z}{(x_1+x)^2+(z_1+z)^2} - \frac{z_1+z}{(x_1-x)^2+(z_1+z)^2} \\ \frac{x_1+x}{(x_1+x)^2+(z_1-z)^2} + \frac{x_1-x}{(x_1-x)^2+(z_1-z)^2} - \frac{x_1+x}{(x_1+x)^2+(z_1+z)^2} - \frac{x_1-x}{(x_1-x)^2+(z_1+z)^2} \end{pmatrix}$$

where the coordinate origin is naturally chosen at the field center.

At $(0,0)$ there is absolutely no field only a gradient b which can be arbitrarily large. For us it would be ideal to send a point beam along the y-direction. This is however impossible. Practically, the beam must have some non-zero width. However,

(a)

(b)
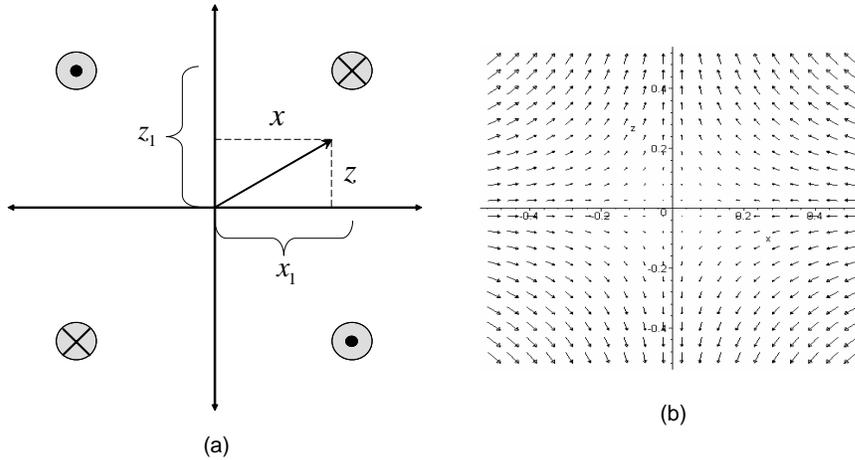
Figure 6.1: (a) An arrangement of 4 long, parallel wires lying in the y-direction. The dots represent current flowing out of the page and the crosses represent current flowing into the page. (b) A plot of the vector field produced by the four wire set up in (a) near the origin.

if we merely restrict ourselves to behavior *near* the origin we can drop all orders of x and z that are second order or greater. For simplicity we will also take $x_1 = z_1$, i.e. the wires arranged in a square. Doing this arrives at precisely the field of eq. (6.2) with $B_0 = 0$. In our search for solutions we must remember that any solutions we find are only valid for small x and z. Sending a beam of neutral spin-1/2 particles, i.e. neutral silver atoms, along the y-axis in such a field will exhibit the ISGE.

(a) (b) (c) (d)
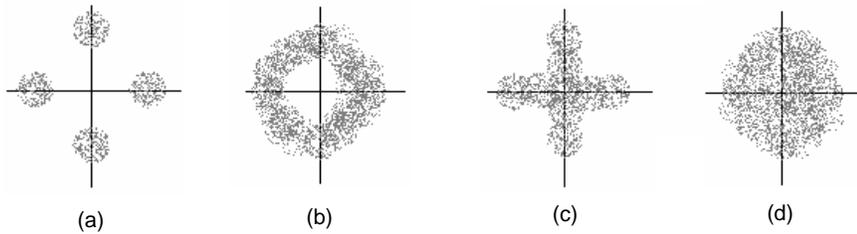
Figure 6.2: There are at least four possible outcomes of the ISGE. They are: (a) The beam splits in both x and z-directions, (b) the beam splits in the radial direction only, (c) the beam splits into two angular directions only, and (d) the beam does not split but blurs in all directions. Some blurring will arise from collimation so in this last case it would be important to note whether the observed blurring was attributable to spin separations.

6.3.2 Possible Outcomes

In the case that $B_0 = 0$ what beam trace can we expect? Fig. 6.2 shows four possibilities. If we choose the experimental conditions, i.e. the time of interaction nT and the strength of the field gradient b, so as to further invalidate precession arguments any result would be instructive. Moreover, because of the absence of precession, if the experiment is performed with only a *single* particle its deflection and detection *anywhere* on the plate would naively define 2-components of the "force" or, consequently, 2-components of the spin. For reasons discussed in section 5.5.3 this is of extreme interest.

Chapter 7

A More Complete Study of the Stern-Gerlach Effect

In accordance with the proposal of the previous chapter we seek here to find solutions and insights into the Stern-Gerlach Effect (SGE) via the Inhomogeneous Stern-Gerlach Effect (ISGE). Because of the difficulty and nature of the task we will represent this problem using several solution methods, formalisms, and pictures in an attempt to fully understand this effect in a quantum context along with its theoretical and philosophical implications.

7.1   A Matrix Representation in an Inertial Frame

In order to gain an initial familiarity with the full SGE, and in particular the ISGE, we can follow the method used in Chapter 3 only with the slightly modified field eq. (6.2). We begin with a Hamiltonian operator describing the ideally impulsive interaction of duration T as

$$\hat{H}(t) = \begin{cases} 0 & \text{for } t < 0 \\ -\frac{e}{m}\frac{\hbar}{2}\begin{pmatrix} (B_0 + bz) & -bx \\ -bx & -(B_0 + bz) \end{pmatrix} & \text{for } 0 \le t \le T \\ 0 & \text{for } t > T \end{cases} \qquad (7.1)$$

7.1.1   The Assumptions

This is a special representation of the Stern-Gerlach Hamiltonian. As was stated, but not discussed, in Chapter 3 it treats the dynamics from the frame of the particles in the beam which we take to be inertial. As this assigns the definite value

of zero to the operators $\hat{p}_x$ and $\hat{p}_z$ it correspondingly limits the possible definition of the position according to eq. (3.30). The incoming beam is therefore a plane wave travelling purely in the y-direction with infinite extent in the x and z-directions. As it was shown in Chapter 3 the y-behavior can be separated off without any assumption on $\hat{p}_y$.

For completeness we should also consider the effects of the idealization involving the impulsive field. In reality no field can be turned on infinitely quick or confined perfectly to a given region of space without some variation. In our case, in the rest frame of the beam particles, this would be manifest as a time-varying **B**-field, which via Maxwell's equation eq. (5.4) sources electric fields. This could cause some complicated effects as the particle enters and exits the field. However, a numerical treatment of the ISGE involving charged particles and using a field with a non-zero "turn-on" and "turn-off" time showed no significant differences from the idealized case [28]. Because of this and because we have chosen to work only with neutral particles we think that the idealization of an impulsive field is adequate.

7.1.2 The Eigenstates

The energies can be found in the same manner as in section 3.2.5 with a similar result. We get

$$E_{\uparrow\downarrow} = \mp \frac{e\hbar}{2m}\sqrt{(bx)^2 + (bz + B_0)^2}. \tag{7.2}$$

where the $\uparrow\downarrow$ refer to two possibly resultant states as represented in the z-basis. We should also point out the apparent spatial dependence of the energies. This was also present in the treatment in Chapter 3. In order to make sense out of these spatially dependent eigenvalues the factors of x and z in eq. (7.2) should be thought of as parameters and not as spatial coordinates. If the beam were localized enough they might refer to the coordinates of the peak of the beam packet in the xz-plane. But for a beam that is extended compared to the region of interest as we have, (x, z) could be thought of as the initial position of the particles within the beam as in a realist, or pilot-wave-type visualization (see section 7.9.1 for further discussion). Either way, they should be thought of as parameters and not variables.

Although we will not need them we point out that the eigenvalues eq. (7.2) correspond respectively to the non-normalized eigenstates

$$|\psi_{\uparrow\downarrow}\rangle = \begin{pmatrix} 1 \\ -\frac{p_{\uparrow\downarrow}^2}{e\hbar bx} + \frac{B_0}{bx} + \frac{z}{x} \end{pmatrix} \tag{7.3}$$

with $p_{\uparrow\downarrow}^2 = 2mE_{\uparrow\downarrow}$.

For $t < 0$ the field is off so we have a free particle at rest since $\hat{H} = 0$ ($E = 0$). At $t = 0$ the field is switched on and as we found earlier the states are describable as

$$\Psi = \psi_\uparrow \chi_\uparrow e^{i\frac{e}{2m}\sqrt{(bx)^2+(bz+B_0)^2}t} + \psi_\downarrow \chi_\downarrow e^{-i\frac{e}{2m}\sqrt{(bx)^2+(bz+B_0)^2}t}. \tag{7.4}$$

This persists until $t = T$ after which the particle is again unaffected so the final states can be found by merely evaluating eq. (7.4) at $t = T$.

Before moving on we'd like to explicitly point out three very subtle assumptions that are made in the approach used here and in section 3.2.5.

(1) Before entering the field region we assume the particle is at rest, or we consider the system from the rest frame of the particles. However, after emerging from the field we have *supposedly* shown how the particle has picked up a momentum that cause separation of the spin components. In general we cannot neglect the kinetic energy any longer and should append a propagating factor $e^{-i\boldsymbol{p}^2t/2\hbar m}$. Therefore, in this we have assumed that somehow the acquired energy is negligible.

(2) We said our results in Chapter 3 corroborated the experimentally observed result because it clearly showed two distinct momenta entangled with the spin properties of the particles. It was actually two infinite plane waves moving with equal and opposite momenta. Thus, based on our treatment in Chapter 3 we would expect nothing but an infinite blur upon measurement because of the infinite uncertainty in the position of the particles in the beam. Two distinct traces will not occur.

(3) Finally, in Chapter 3 but even more so here the interpretation of the ↑↓ is unclear. As far as we can tell it references the eigenstates of the Hamiltonian operator. In the present context that is eq. (7.3).

7.1.3   Demonstrating Precession

If we take the limit that ϵ is small where $\epsilon = b/B_0$ such that $|B_0| >> |br|$ for all relevant values of $r = \sqrt{x^2 + z^2}$ we can demonstrate the *effect* of the precession argument but without reference to the classical or quantum concept of precession.

If we expand the squares in the phase ϑ of eq. (7.4) acquired after time T we can write

$$\vartheta = \pm i \frac{e}{2m} \sqrt{B_0^2 + 2B_0 bz + (br)^2} T. \tag{7.5}$$

Since we take B_0 to be large let us factor it out of the radical

$$\vartheta = \pm i \frac{eB_0}{2m} \sqrt{1 + \frac{2bz}{B_0} + \left(\frac{br}{B_0}\right)^2} T \tag{7.6}$$

Expanding the radical for small ϵ we get

$$\sqrt{1 + 2\epsilon z + \epsilon^2 r^2} = 1 + \epsilon z + \frac{\epsilon^2}{2} x^2 + ... \tag{7.7}$$

So to first order the phase is

$$\vartheta = \pm i \frac{e}{2m} (B_0 + bz) T \tag{7.8}$$

which is exactly the phase of the solutions we found in Chapter 3 (see eq. (3.38)). This is a purely mathematical demonstration of what is happening in the standard derivations of the SGE. It shows that despite the logical and mathematical heuristics that often go into making a thematic discussion of the SGE intuitive there is a more mathematical and *clearly demonstrable* reason the physical picture works. Whether precession accounts for this is a matter of interpretation.

7.1.4   The Inhomogeneous Stern-Gerlach Effect

Now that we can clearly see how B_0 justifies the precession picture and the neglect of the x-directed field we can let $B_0 \to 0$ to solve the ISGE. This should clarify the role of precession even further.

Returning to eq. (7.4), but with $B_0 = 0$, we can write the final inhomogeneous Stern-Gerlach states

$$\Psi = \psi_\uparrow \chi_\uparrow e^{\frac{i}{\hbar}\left(\frac{ebT\hbar}{2m}\right)r} + \psi_\downarrow \chi_\downarrow e^{-\frac{i}{\hbar}\left(\frac{ebT\hbar}{2m}\right)r}. \tag{7.9}$$

(a)

(b)
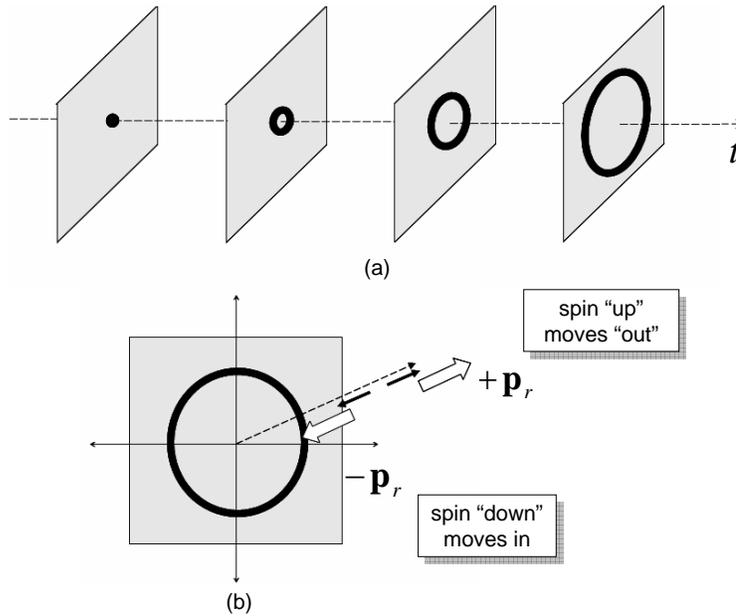
Figure 7.1: As time progresses (to the right) the spins separate radially "up" or "down" depending on their spin orientation relative to the radial direction.

Using the same arguments as in section 3.2.5 we may interpret this as two circular waves travelling either radially outward or inward with a momentum

$$p_r = \pm \frac{ebT\hbar}{2m} \tag{7.10}$$

depending on its spin's projection along the r-axis. Relative to this axis the spin "ups" move radially outward while the spin "downs" move towards and through the center of the field.

7.2 Position and Momentum Representations

With an idea of what the solutions might behave like we can now attempt a more rigorous solution that involves fewer assumptions and compare. We begin this process with the time-dependent Schrödinger equation with operators in **x**-space

$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2|\Psi\rangle + \hat{V}|\Psi\rangle, \tag{7.11}$$

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where

$$|\Psi\rangle = \Psi_\uparrow \chi_\uparrow + \Psi_\downarrow \chi_\downarrow = \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} \qquad (7.12)$$

is a spinor. In our case the potential energy arises from the interaction of the magnetic moment $\hat{\boldsymbol{\mu}}$, or spin $\hat{\boldsymbol{S}}$, with the field $\hat{\boldsymbol{B}}$. That is,

$$\hat{V} = -\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{B}}. \qquad (7.13)$$

If we use the previous definitions of $\hat{\boldsymbol{\mu}}$ in terms of the Pauli spin matrices $\hat{\sigma}_j$ we have

$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2|\Psi\rangle - \frac{e}{m}\frac{\hbar}{2}\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{B}}|\Psi\rangle. \qquad (7.14)$$

We can easily separate off both the time and y-dependence using the same procedure as in section 3.2. Assuming $|\Psi\rangle = T(t)Y(y)|\psi(x,z)\rangle$ we get the three equations

$$T(t) = T(0)e^{-iEt/\hbar} \qquad (7.15)$$

$$Y(y) = Y(0)e^{ik_y y} \qquad (7.16)$$

$$k^2|\psi\rangle = -\boldsymbol{\nabla}^2|\psi\rangle - \frac{e}{\hbar}\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{B}}|\psi\rangle, \qquad (7.17)$$

where E and k_y are separation constants and $k^2 = 2mE/\hbar^2 - k_y^2$. Using the full field of eq. (6.2) and standard representation for the Pauli matrices $\hat{\sigma}_j$ we can write the matrix equation for $|\psi\rangle$ as two, coupled differential equations for the spin "up" and "down" components[1]

$$k^2\psi_{\uparrow\downarrow} = -\boldsymbol{\nabla}^2\psi_{\uparrow\downarrow} + \frac{eb}{\hbar}\left[x\psi_{\downarrow\uparrow} \mp z\psi_{\uparrow\downarrow}\right] \mp \frac{eB_0}{\hbar}\psi_{\uparrow\downarrow}. \qquad (7.18)$$

Here and throughout the remainder of this work this compact notation is used in which the top (bottom) signs in $\mp$ correspond to the first (second) subscript of $\psi_{\uparrow\downarrow}$.

In order to arrive at analytic solutions of this equation the "up" and "down" components must be decoupled. To do this would require applying the laplacian operator, $\boldsymbol{\nabla}^2$, to both equations. Although this decouples the "up" and "down"

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[1]Recall that "up" and "down" are in quotes only because they are terms relative to the basis by which we represented their spin operators. In our case, "up" and "down" in z.

behavior it also yields two fourth order partial differential equations (PDEs). This effectively prohibits us from using familiar differential equation solution techniques because they are typically only formulated for second order equations.

We conclude that although $\mathbf{x}$-space is an intuitive space to work in this context it requires techniques that we do not have. We can then transform eq. (7.18) into an "orthogonal" representation in hopes that the tradeoff between intuition and technical detail will be in our favor.

As it turns out, this is precisely what happens when mapping these equations to $\mathbf{p}$-space. When making this change from $\mathbf{x}$ to $\mathbf{p}$-space, via a Fourier transformation, we make the following changes

$$x_j \;\; \rightarrow \;\; i\hbar\frac{\partial}{\partial p_j} \tag{7.19}$$

$$-i\hbar\frac{\partial}{\partial x_j} \;\; \rightarrow \;\; p_j, \tag{7.20}$$

that is, the coordinates become operators and the operators become coordinates. We also realize that

$$\psi_{\uparrow\downarrow}(x,z) \rightarrow \phi_{\uparrow\downarrow}(p_x,p_z). \tag{7.21}$$

Making this change eq. (7.18) becomes

$$\hbar^2 k^2 \phi_{\uparrow\downarrow} = (p_x^2 + p_z^2)\phi_{\uparrow\downarrow} + ieb\hbar^2 \left[\frac{\partial}{\partial p_x}\phi_{\downarrow\uparrow} \mp \frac{\partial}{\partial p_z}\phi_{\uparrow\downarrow}\right] \mp eB_0\hbar\phi_{\uparrow\downarrow}, \tag{7.22}$$

in the momentum representation.

For convenience we write this in dimensionless form

$$\frac{\partial}{\partial p_x}\phi_{\downarrow\uparrow} \mp \frac{\partial}{\partial p_z}\phi_{\uparrow\downarrow} = -i\left[\xi - \alpha(p_x^2 + p_z^2) \pm \beta\right]\phi_{\uparrow\downarrow} \tag{7.23}$$

where $\xi \equiv k^2\hbar/e\Delta b$ is a unitless energy, $\alpha \equiv \hbar/e\Delta^3 b$, and $\beta \equiv B_0/\Delta b$ is the ratio of the homogeneous to the inhomogeneous field. Δ is a characteristic length scale of the system. Note also that the p_j are now dimensionless momentum variables.

The utility of this particular representation is that the derivative properties of $\mathbf{x}$-space are replaced with algebraic properties in $\mathbf{p}$-space. Thus, the *fourth* order PDEs with *quadratic* terms in x and z that would arise from decoupling in $\mathbf{x}$-space
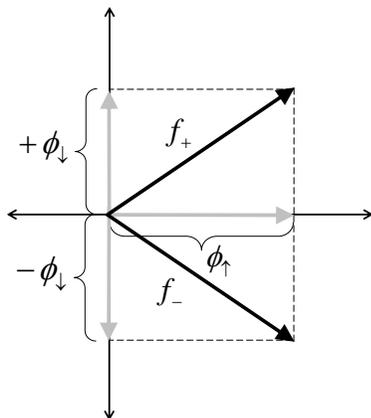
Figure 7.2: $f_\pm$ can be thought of as a set of vectors in the imaginary plane with components $\pm\phi_{\uparrow\downarrow}$. It is easier to decouple eq. (7.23) when it is rewritten in terms of $f_\pm$.

now yields a simpler result. We get *second* order PDEs that are now *quartic* in the coordinates p_x and p_z. Applying well known solution methods for PDEs therefore becomes much more straightforward.

7.3 Rotated and Unrotated Representations: Decoupling

As it turns out, in the present form the decoupling process is quite messy. We can however rotate the basis in which they are represented in the complex plane and simplify the process.

It is useful to note that this is the same approach that we used in section 2.2.1 in which we rotated the coordinate system with which we described an object in order to simplify it algebraically except that now our objects are complex. Thus, this rotation can take place in the complex plane.

For simplicity, instead of dealing with the functions $\phi_{\uparrow\downarrow}$ we now choose to introduce the functions $f_\pm$ where

$$f_\pm = \phi_\uparrow \pm i\phi_\downarrow. \tag{7.24}$$

Because the function $f_\pm$ is a linear combination of the $\phi_{\uparrow\downarrow}$ it can be thought of as a simple 2-dimensional complex vector represented in the 2-dimensional ϕ-basis (see fig. 7.2).[2]

Rewriting eq. (7.23) we get

$$\hat{\mathcal{L}}_\pm f_\mp = \mp\left[\xi - \alpha(p_x^2 + p_z^2)\right]f_\pm \tag{7.25}$$

where $\hat{\mathcal{L}}_\pm$ is an operator of the form

$$\hat{\mathcal{L}}_\pm \equiv \frac{\partial}{\partial p_x} \pm i\frac{\partial}{\partial p_z} \pm \beta \tag{7.26}$$

Keep in mind that these equations are actually a set of two equations written in a compact form.

With the introduction of $f_\pm$ the decoupling of these equations is simpler. By applying $\hat{\mathcal{L}}_\mp$ to eq. (7.25) we can decouple to get

$$A^2\hat{\mathcal{L}}_\pm\hat{\mathcal{L}}_\mp f_\pm + A\hat{\mathcal{L}}_\pm A\hat{\mathcal{L}}_\mp f_\pm + f_\pm = 0 \tag{7.27}$$

where A is the function

$$A(p_x, p_z) \equiv \frac{1}{\xi - \alpha(p_x^2 + p_z^2)}. \tag{7.28}$$

Eq. (7.27) is two decoupled second order PDEs.

7.4   Cartesian and Polar Representations: Separation

If the p_x-dependence can be separated from the p_z-dependence then this equation can be treated as two ODEs instead of two PDEs which is a great simplification. Unfortunately in the Cartesian representation such separation is not possible.

However we have noted that by transforming our coordinate system from a Cartesian to a polar form we can simplify the function A. Instead of being a function of 2 variables, with the polar substitutions

$$\rho^2 = p_x^2 + p_z^2 \tag{7.29}$$

$$\varphi = \arctan\left(\frac{p_z}{p_x}\right) \tag{7.30}$$

---

[2]It is also interesting to note the similarity of $f_\pm$ to right and left circularly polarized light. What this exactly means in the context of spin, for example, what sort of apparatus would filter these spin polarizations, is an interesting, and open, question.

in momentum coordinates the function A becomes a function of only one variable. Due to this symmetry the equations simplify. In fact, in the limit that $\beta \to 0$, which is the ISGE case, they become separable. In particular, the equations for $f_\pm$ become

$$0 = (\xi - \alpha\rho^2)\left(\rho^2 f_{\pm\rho\rho} + f_{\pm\varphi\varphi}\right) + \rho(\xi + \alpha\rho^2)f_{\pm\rho}$$
$$\mp 2i\alpha\rho^2 f_{\pm\varphi} + \left(\alpha^3\rho^8 - 3\alpha^2\xi\rho^6 + 3\alpha\xi^2\rho^4 - \xi^3\rho^2\right)f_\pm, \quad (7.31)$$

where each occurrence of the ρ or φ in the subscripts denote a partial derivative with respect to the corresponding variable.

For purposes of greater economy we let

$$\rho \to +\sqrt{\frac{\xi}{\alpha}}\sqrt{\rho} \quad (7.32)$$

giving eq. (7.31) the form

$$(1 - \rho)\left(4\rho^2 f_{\pm\rho\rho} + f_{\pm\varphi\varphi}\right) + 4\rho f_{\pm\rho} \mp 2i\rho f_{\pm\varphi} + \frac{\xi^3}{\alpha}\rho\left(\rho^3 - 3\rho^2 + 3\rho - 1\right)f_\pm = 0. \quad (7.33)$$

We can now see that

$$f_\pm(\rho, \varphi) = R_\pm(\rho)P_\pm(\varphi) \quad (7.34)$$

is a suitable separation *ansatz* if we choose

$$P_\pm(\varphi) = e^{in_\pm\varphi} \quad (7.35)$$

as the solution to the angular part. By the single-valuedness requirement we know $n_\pm$ can take on only integer values, $n_\pm = ..., -2, -1, 0, 1, 2, ...$ With $\zeta = \alpha/\xi^3$ the radial momentum equation then becomes

$$R_\pm'' + \frac{1}{\rho(1 - \rho)}R_\pm' - \frac{\left[\rho^4 - 3\rho^3 + 3\rho^2 - \left(1 + \zeta n_\pm(n_\pm \pm 2)\right)\rho + \zeta n_\pm^2\right]}{4\zeta\rho^2(1 - \rho)}R_\pm = 0, \quad (7.36)$$

an ODE in standard form with primes representing total derivatives. It is interesting to note that $R_\pm$ is real valued and that $P_+(\varphi)$ is unchanged from $P_-(\varphi)$ except for the particular integer n_+ or n_-. Also, the appropriate free particle solutions can be recovered in the limit as the entire field is turned off, i.e. $\zeta \to 0$.

7.5 Series Solution Representation

One of the most familiar solution techniques for a second order ODE such as we have is the Frobenius method. We first consult Fuchs' theorem for the applicability of this method.

7.5.1 Singularity Structure and Fuchs' Theorem

Fuchs' theorem states that

> ...we can always obtain at least one power-series solution, provided we are expanding about a point that is an ordinary point or at worst a regular singular point [34]. (p. 527)

Expansion about any other type of point may yield a solution. This theorem dictates only when obtaining a solution is guaranteed.

Following the methods of [34] (p. 516-517) it can be shown that eq. (7.36) has regular singluar points at $\rho = 0, 1$ and an irregular singular point at $\rho = \infty$. Choosing to do a Frobenius expansion about $\rho = 0$ is then the most straightforward choice.

7.5.2 The Indicial Equation

Using a series form[3] for $R_\pm$

$$R_\pm(\rho) = \sum_{j=0}^{\infty} (a_\pm)_j \rho^{\lambda_\pm + j} \tag{7.37}$$

we can arrive at the indicial relation for $\lambda_\pm$

$$\lambda_\pm = \pm \frac{n_\pm}{2}. \tag{7.38}$$

Note that eq. (7.38) actually expresses four equations: the $\lambda_+ = \pm n_+/2$ correspond to R_+ and the $\lambda_- = \pm n_-/2$ correspond to R_-. In both cases we will take $\lambda_\pm$ to be the larger of the two roots, i.e. the + solutions. Therefore, $\lambda_\pm = +n_\pm/2$, which are half-integers.

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[3]Parentheses have been placed around the coefficients $(a_\pm)_j$ to avoid associating the $\pm$ with the index j. They refer respectively to $R_\pm$.

7.5.3 Recurrence Relation

With the roots of the indicial equation we obtain a 5-term recurrence relation indexed by j

$$(a_\pm)_j = \frac{(a_\pm)_{j-4} - 3(a_\pm)_{j-3} + 3(a_\pm)_{j-2} - \left[1 + 4\zeta\left[n_\pm(\epsilon_\pm - j) - j(j-3) - 2\right]\right](a_\pm)_{j-1}}{4\zeta j(j + n_\pm)} \tag{7.39}$$

where

$$\epsilon_\pm = \begin{cases} 2 & \text{for } +, \text{ corresponding to } R_+ \\ 1 & \text{for } -, \text{ corresponding to } R_- \end{cases} \tag{7.40}$$

If we allow $n_\pm$ to range over negative values the coefficients blow up. This may give us a physical constraint on $n_\pm$.

Following a promising comment by [35] (p. 532) we make every attempt to reduce the number of terms in this recurrence relation from 5 to 3, perhaps even 2.

7.5.4 Extracting Asymptotic Behavior

Often the structure of the recurrence relation can be simplified if by some informed guess the asymptotic behavior can be extracted. This method is often employed when solving the quantum simple harmonic oscillator as in [1] and [36].

Unfortunately employing the same method here in various ways yielded no simplified result. In fact, in many cases by extracting the behavior *complicated* the structure of the recurrence relation giving us 8 or 9 terms.

This suggests that we have hit some "critical mass" or "local minimum" of mathematical sophistication in eq. (7.36) beyond which the problem increases in complexity. This is typical of non-linear systems as it seems that a simple combination of more basic parts can't be trivially assembled to describe the full behavior. Nevertheless, this is how many standard discussions proceed. The precession or quantization behavior is superposed with the classical description and concepts in order to achieve some understanding. There may still exist however some more obscure choice in the extraction method by which the recurrence relation is simplified. This is suggested by [35].

7.5.5 Series Truncation

Another common method also used in [1] and [36] to solve the quantum simple harmonic oscillator involves the truncation of the series. If a set of conditions can be found so as to force an appropriate set of consecutive coefficients in the series to go to zero then, by the recurrence relation, all succeeding coefficients will be zero as well. This truncates the infinite series into a finite polynomial. In the case of the simple harmonic oscillator this choice also provides the quantization condition on the energies.

However, because eq. (7.39) is a 5-term recurrence relation there are 4 independent conditions that must be made to go to zero in order to ensure *all* succeeding terms vanish as well. Four consistent conditions could not be found.

Although this is not a definitive proof it is evidence that in studying the ISGE we should not expect a quantization similar to that of the quantum simple harmonic oscillator. This realization can serve as a guide as to which systems we can model our solution methods and intuition after. For example, scattering and unbound systems might be preferred.

7.5.6 Radius of Convergence

Despite the comments of [35] there remains no obvious way of reducing our recurrence relation to two or three terms. In order to rigorously test the convergence of the series a more compact form for the recurrence relation is needed. However, from numerical testing of convergence and from the location of the regular singularity at $\rho = 1$ we conclude that the radius of convergence for the solutions to eq. (7.36) is $\rho = 1$.

This tells us that the solutions we seek, if we could find them, are not normalizable for $-\infty < p_x < \infty$ and $-\infty < p_z < \infty$. By Parseval's theorem this implies that the solutions are also not normalizable for $-\infty < x < \infty$ and $-\infty < z < \infty$ in the **x**-representation. This suggests that whatever solutions we are looking for are either unphysical or require some subtle normalization procedure as in the delta-function normalization of plane waves. This may either be a result of the nature of the ISGE

or of our crude and approximate field eq. (6.2) which we noted is unphysical since it blows up as $r \to \pm\infty$.[4]

7.5.7 Near-Origin Approximation

In replacing the four wire field of eq. (6.4) with the approximate field of eq. (6.2) with $B_0 = 0$ we limited ourselves only to a study of the system's behavior near the *spatial* origin. The solutions we have generated are therefore only valid in this region. It is reasonable to assume then that if in eq. (7.36) we keep terms only up to first order in x and z, so as to focus on solutions for small values of x and z, i.e. near the origin, that we are actually only excluding insignificant behavior.

There is one complication however that should be considered. While this approximation is straightforward in $\mathbf{x}$-space we are now operating in $\mathbf{p}$-space which introduces other subtleties. For example, in the transformation from the $\mathbf{x}$-representation to the $\mathbf{p}$-representation we know that

$$x_j^n \to (i\hbar)^n \frac{\partial}{\partial p_j^n} \tag{7.41}$$

which can be seen by a Fourier transformation. So higher powers of x_j become higher order derivatives with respect to p_j. In other words, the behavior of the solutions near the origin in $\mathbf{x}$-space is encoded in the *curvature* of the solutions *everywhere* in $\mathbf{p}$-space. Thus, in the process of keeping only near-origin behavior we will drop all second order derivative terms from eq. (7.36). Although this seems to make the appropriate restriction the neglect of derivatives in an ODE typically changes the nature of the equation completely. Thus, the question remains open as to how crude this is as an approximation.

Perhaps the neglect of second order derivatives is no more crude than it was in the $\mathbf{x}$-space method used in Chapter 3. There we assumed we could derive a solution in the rest frame of the particles. Since a reasonable result was obtained there it didn't seem to be too restrictive although it involved dropping the second order derivative operator $\boldsymbol{\nabla}^2$. This effectively defined the particle momentum in x and z as exactly

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[4]It is unknown why both Fuchs' theorem and the comment in [35] have not been substantiated.

zero, which by the uncertainty relation eq. (3.30), completely blurs its position in the corresponding plane. Thus, it was equivalent to describing the beam as an infinite plane wave.

In the context we use it here the approximation is not so clearly understood. It is however mathematically equivalent so we have good reason to believe that it has a similar although much less familiar interpretation.

Having said this we make the approximation and our second order ODE of eq. (7.36) reduces to first order. This can be easily integrated. Combining again with the angular solutions with $n_+ = n$ and $n_- = m$, we get

$$\phi_{nm\uparrow\downarrow} = \frac{\epsilon_{\uparrow\downarrow}}{2} e^{\frac{\rho}{16\zeta}\left(\rho^3 - 4\rho^2 + 6\rho - 4\right)} \left[A_{nm}\rho^{n^2/4} e^{-\rho\frac{n}{2}\left(\frac{n}{2}+1\right)} e^{in\varphi} \pm B_{nm}\rho^{m^2/4} e^{-\rho\frac{m}{2}\left(\frac{m}{2}-1\right)} e^{im\varphi} \right] \tag{7.42}$$

where

$$\epsilon_{\uparrow\downarrow} = \begin{cases} 1, \text{ for } \uparrow \\ -i, \text{ for } \downarrow \end{cases} \tag{7.43}$$

These are the stationary state wave functions for the "up" and "down" components near the origin in **p**-space for the ISGE. A general solution would be a linear combination of these with the appropriate p_y and t behaviors appended since they have been separated off.

The probability density (PD) of either the "up" or "down" components for a particular choice of n and m is found using

$$\text{PD} = \phi_{\uparrow\downarrow}^* \phi_{\uparrow\downarrow}. \tag{7.44}$$

Applying this to our solutions for a particular choice of n and m we get

$$\phi_{\uparrow\downarrow}^* \phi_{\uparrow\downarrow} = \pm \frac{1}{4} e^{\frac{\rho}{8\zeta}\left(\rho^3 - 4\rho^2 + 6\rho - 4\right)} \left[|A_{nm}|^2 \rho^{n^2/2} e^{-\rho n\left(\frac{n}{2}+1\right)} + |B_{nm}|^2 \rho^{m^2/2} e^{-\rho m\left(\frac{m}{2}-1\right)} \right.$$

$$\left. \pm \rho^{(n^2+m^2)/4} e^{-\rho\left(\frac{n}{2}\left(\frac{n}{2}+1\right)+\frac{m}{2}\left(\frac{m}{2}-1\right)\right)} \left\{ A_{nm}B_{nm}^* e^{i(n-m)\varphi} + A_{nm}^* B_{nm} e^{-i(n-m)\varphi} \right\} \right] \tag{7.45}$$

where A_{nm} and B_{nm} are constants of integration.

Realize that A_{nm} and B_{nm} relate to $R_{\pm}(0)$. This in turn relates to $f_{\pm}(0)$ which relates to *both* $\phi_{\uparrow\downarrow}(0)$. Furthermore, eq. (7.45) is a probability density for a given spin in **p**-space. Thus, it is not trivial to correlate A_{nm} and B_{nm} to the initial spin states or give eq. (7.42) or eq. (7.45) a clear physical interpretation.

7.6   The Confluent Heun Equation

We could also take advantage of the fact that eq. (7.36) has a similar singularity structure as the Heun equation (see [37], [38])

$$y''(x) + \left[\frac{a_1}{x} + \frac{a_2}{x-1} + \frac{a_1 + a_2}{x - x_0} + \frac{a_3}{x^2} + \frac{a_4}{(x-1)^2} + \frac{a_5}{(x-x_0)^2} \right] y(x) = 0. \quad (7.46)$$

This is in normal form. It has one regular singularity at each of four points $x = 0, 1, x_0, \infty$.

By conflating the singularities $x = x_0$ and $x = \infty$ we get the confluent Heun equation (CHE)

$$y''(x) + \left[\tilde{a}_0 + \frac{\tilde{a}_1}{x} + \frac{\tilde{a}_2}{x-1} + \frac{\tilde{a}_3}{x^2} + \frac{\tilde{a}_4}{(x-1)^2} \right] y(x) = 0 \quad (7.47)$$

which has the same singularity structure as eq. (7.36) at least in terms of number and location.

As it turns out this is not enough. Although the CHE has an irregular singularity at $x = \infty$, as does eq. (7.36), it is not of the same rank. In other words, the singularity at infinity for the CHE does not blow up as fast as the corresponding singularity in eq. (7.36). We have tried to extract at least a portion of the asymptotic behavior as discussed in [37] and [38] in an attempt to rectify these two singularities but we have not been successful. Thus, we have learned that a discrepancy in the *rank* of the singularities is enough to prohibit our use of the CHE in describing the ISGE.

7.7   A Cliffor Representation

There are many mathematical representations we could use to facilitate the solution process of the ISGE. As each one is designed with a certain end in mind each

one has its particular strengths and limitations. We have already seen the problem of the ISGE treated using both matrices and differential calculus and encountered their special challenges as well as their advantages. Another mathematical space within which the quantum phenomena of spin is particularly interesting is the Clifford algebra Cl_3, also known as the Pauli algebra.

In this algebra the Pauli matrices $\hat{\sigma}_j$ are not given a particular matrix representation but are treated in a representation-"free" manner. More specifically, we define a space with a product that preserves the algebraic properties of the $\hat{\sigma}_j$ but that does not require a specific matrix representation in order to manipulate them. In this space the *matrices* $\hat{\sigma}_j$ become *vectors* $\mathbf{e}_j$ and can be taken as an orthonormal basis that spans the space. In this way the spin properties, now represented in terms of the algebraic properties of the vectors $\mathbf{e}_j$, are naturally associated with the 3 orthogonal directions of physical space (see [39]). Thus, by avoiding matrix representations we can also avoid many complications that arise from those representations, focusing on the phenomenon of interest, and provide a more intuitive framework in which to conduct a study of spin.

If we express eq. (7.14) in a frame moving along with the particles in the beam, as we did in section 7.1 we can neglect the kinetic energy terms. We then have

$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = -\frac{e}{m}\frac{\hbar}{2}\hat{\sigma}_j B_j|\Psi\rangle \tag{7.48}$$

where there is an implied sum over j and B_j is a scalar. Using the simple mappings

$$|\psi\rangle \quad \rightarrow \quad \psi = \psi^0 + \mathbf{i}\boldsymbol{\psi} \tag{7.49}$$

$$\sigma_j|\psi\rangle \quad \rightarrow \quad \mathbf{e}_j\psi\mathbf{e}_3 \tag{7.50}$$

$$i|\psi\rangle \quad \rightarrow \quad \mathbf{i}\psi\mathbf{e}_3 \tag{7.51}$$

$$i\sigma_j|\psi\rangle \quad \rightarrow \quad \mathbf{i}\mathbf{e}_j\psi \tag{7.52}$$

that can be found in [39] we can express eq. (7.48) in Cl_3

$$\frac{d\psi}{dt} = \frac{e}{2m}\mathbf{i}\mathbf{B}\psi. \tag{7.53}$$

where ψ is a Clifford representation, or a cliffor, of a spinor in Cl_3 and $\mathbf{i}$ is a cliffor with the same algebraic properties as the usual unit imaginary i but can additionally

be interpreted as an oriented unit volume in Cl_3. $\mathbf{iB}=iB_j\mathbf{e}_j$ is the magnetic bivector corresponding to a plane normal to $\mathbf{e}_j$. Because we have neglected translational terms in eq. (7.48) the spatial dependence of ψ is negligible and so the partial derivative has been replaced with a total derivative. This makes the solutions easier but surely hides important information.

The cliffor equation eq. (7.53) can be easily integrated to yield

$$\psi(t) = e^{\frac{e}{2m}\mathbf{iB}t}\psi(0). \tag{7.54}$$

The particular form of $\mathbf{B}$ has not be given yet. It has only been assumed that $\mathbf{B}$ is constant in time. Substituting eq. (6.2) with $B_0 = 0$, ψ becomes

$$\psi(t) = e^{\frac{eb}{2m}(-x\mathbf{ie}_1+z\mathbf{ie}_3)t}\psi(0). \tag{7.55}$$

If we define a space dependent unit bivector A

$$A(x,z) \equiv \frac{-x\mathbf{ie}_1 + z\mathbf{ie}_3}{|-x\mathbf{ie}_1 + z\mathbf{ie}_3|} \tag{7.56}$$

such that

$$\mathbf{iB} = |br|A \tag{7.57}$$

then ψ clearly takes the form

$$\psi(t) = e^{(\omega t)A}\psi(0) \tag{7.58}$$

where $\omega = e|br|/2m$. In this form eq. (7.58) is recognizable as the clifford representation of rotations in the plane defined by the bivector $A(x,z)$ (see [39]). Therefore, this picture of the ISGE emphasizes the local precession of the spin state ψ at a location dependent characteristic frequency ω. However, the identification of two distinct momenta entangled with spin is not as clear.

If we would have included the kinetic energy terms from the beginning in eq. (7.48) we could have used other geometric techniques of Cl_3 for this more general solution. The solution method above does demonstrate however that the removal of the matrix level of representation in Cl_3 causes *systems of coupled equations* like eq. (7.18) to be replaced by a *single* equation in which *separation* of the variables

becomes the issue. This could be a very desirable result depending on the familiarity with and availability of either decoupling and separation techniques.

Finally, there is at least one other insight we gain from this approach which is extremely interesting. In the derivation leading up to eq. (7.53) all factors of $\hbar$, which are characteristic of quantum behavior, cancelled out. Thus, in the classical limit, which is typically formalized by the limiting process $\hbar \to 0$, eq. (7.53) remains *unchanged.* This fact is further emphasized when we recall that we have also described this state using a formalism that is entirely expressible in physical 3-dimensional space. In other words, the cliffor approach used here suggests that the ISGE is completely independent of the quantum regime. That is, it can be equally considered a *classical* phenomenon. This underscores the peculiar nature of the SGE, namely that it closely ties our quantum and classical descriptions of nature. Further study would have to be pursued in order to determine to what degree this classicality is associated with the spin, the ISGE system, the neglect of the kinetic energy, or the formalism itself (for further discussion see [40], [41], or [42]).

7.8 Green's Function Representation

Sections 7.2-7.6 exhausted several possibilities in order to solve a second order PDE. In the end the only thing we could do to avoid diverging solutions was to drop the second derivative terms the **p**-space equation. This made the series soluble but may have also inadvertently excluded other interesting behavior. By applying other methods we may be able to avoid this.

In section 7.5.5 we concluded that the Stern-Gerlach system is similar to scattering systems. If fact, we can consider it a special case of a scattering problem in which an incoming beam of particles with spin undergoes a magnetic interaction via a magnetic potential. Represented in this way the tools and methods of canonical quantum scattering theory become tools and methods easily adaptable to understanding the ISGE. For example, Green's functions, propagators, and the Born approximation may be applied to solve the ISGE. These methods are familiar in **x**-space.

We can think of the time-independent Schrödinger equations for spin "up" and "down" as two Helmholtz equations sourced by the functions $g_{\uparrow\downarrow}(x,z)$

$$\left(\boldsymbol{\nabla}^2 + k^2\right)\psi_{\uparrow\downarrow} = g_{\uparrow\downarrow} \tag{7.59}$$

where

$$g_{\uparrow\downarrow} = \frac{eb}{\hbar}\left(x\psi_{\downarrow\uparrow} \mp z\psi_{\uparrow\downarrow}\right). \tag{7.60}$$

Notice that $g_{\uparrow\downarrow}$ couples the equations.

In general, the sources $g_{\uparrow\downarrow}$ are extended functions over some region of space. The Green's functions $G_{\uparrow\downarrow}(x,z;x',z')$ are defined as the spin "up" or "down" field component at point (x,z) produced by a unit point source located at (x',z'). $G_{\uparrow\downarrow}$ therefore satisfies

$$\left(\boldsymbol{\nabla}^2 + k^2\right)G_{\uparrow\downarrow} = -\delta(x-x')\delta(z-z'). \tag{7.61}$$

[34] gives the solution to these equations, the 2-dimensional Helmholtz Green's functions for unbounded space, as

$$G_{\uparrow\downarrow} = \frac{-1}{4k_x k_z}e^{ik_x(x-x')}e^{ik_z(z-z')} \tag{7.62}$$

identical for both spin "up" and "down" cases with $k^2 = 2mE/\hbar^2 - k_y^2 = k_x^2 + k_z^2$.

According to the formal theory of Green's functions we can construct the solutions to the whole field $\psi_{\uparrow\downarrow}$ in eq. (7.59) by treating the source $g_{\uparrow\downarrow}$ as a collection of point sources and summing over their individual field contributions $G_{\uparrow\downarrow}$. That is,

$$\psi_{\uparrow\downarrow}(x,z) = \int_{allspace} G_{\uparrow\downarrow}(x,z;x',z')g_{\uparrow\downarrow}(x',z')dx'dz'. \tag{7.63}$$

7.8.1 A Magnetic Field with Gaussian Fall Off

The advantage of approaching the problem from the perspective of these integral equations as opposed to the differential equations of previous sections is that it is much easier to include a magnetic field that has a realistic asymptotic behavior. We can make a slight modification to our field here that would not have been practical in our earlier approaches.

If we pick **B** such that

$$\mathbf{B} = b(-x\hat{x} + z\hat{z})e^{-(x^2+z^2)/a}, \ a > 0 \tag{7.64}$$

where a is some characteristic length scale, it will capture the inhomogeneous behavior of the four wire field near the origin but will also provide appropriate fall off at large distances $r \gg a$. This gaussian factor would have greatly complicated the ODE approach of sections 7.2-7.6, especially in **p**-space, but can be more easily used with integrals. Doing this will hopefully eliminate much of the problems with the ODE approach of the previous section which seemed to result from the unrealistic field configuration.

7.8.2   The Born-Approximation

In order to get around the fact that $g_{\uparrow\downarrow}$ in eq. (7.63) couples the two solutions together we must either decouple the equations, which will lead to two fourth-order operators, or we can use Born's iterative approximation method with a well known Helmholtz Green's function. Sacrificing rightness for clarity we choose the latter approach. In principle, it can easily be employed to obtain solutions up to any desired accuracy. Because the field now decays at large distances there is hope that only a few iterations will capture the essential behavior of the ISGE.

In this method the wave function $\psi_{\uparrow\downarrow}$ is seen as a sum of successively smaller corrections

$$\psi_{\uparrow\downarrow} = \psi_{\uparrow\downarrow}^{(0)} + \psi_{\uparrow\downarrow}^{(1)} + \psi_{\uparrow\downarrow}^{(2)} + \psi_{\uparrow\downarrow}^{(3)}... \tag{7.65}$$

Note here that the *larger* orders (numbers in parentheses) label successively *smaller* corrections. Each successive order is found by using eq. (7.63) with $g_{\uparrow\downarrow}$ approximated from the preceding order.[5] So,

$$\psi_{\uparrow\downarrow}^{(n)}(x, z) = \int_{allspace} G_{\uparrow\downarrow}(x, z; x^{'}, z^{'})g_{\uparrow\downarrow}^{(n-1)}(x^{'}, z^{'})dx^{'}dz^{'} \tag{7.66}$$

---

[5]It is interesting to note that we also applied this iterative technique to the differential equation without the Gaussian fall off to the field. We solved a homogeneous differential equation then substituted that solution back into the equation now with an approximated source. When this was iteratively done the solutions were found to blow up, likely for the same reasons other derivative methods did as well although when checked this approach worked for simpler cases.

where $\psi_{\uparrow\downarrow}^{(n-1)} = \psi_{\uparrow\downarrow}^{(0)} + \psi_{\uparrow\downarrow}^{(1)} + ... + \psi_{\uparrow\downarrow}^{(n-2)}$ and

$$\left(\boldsymbol{\nabla}^2 + k^2\right)\psi_{\uparrow\downarrow}^{(0)} = 0. \tag{7.67}$$

In this way, beginning with the homogeneous (zeroeth order) solutions to the Helmholtz equation - the free particle solutions

$$\psi_{\uparrow\downarrow}^{(0)}(x, z) = e^{ik_x x}e^{ik_z z} \tag{7.68}$$

- and proceeding through iteration, the solutions to the ISGE can be found to any desired degree of accuracy. While this can be done we will not pursue it here. We will instead comment later on a similar propagator approach in section 7.11.1.

7.9 Schrödinger and Heisenberg Representations

The final representation that we will discuss in technical detail has less to do with the spatial behavior of the solutions and focuses more on the dynamics. Early on, before we even moved into either $\mathbf{x}$ or $\mathbf{p}$-space we assumed the states $|\Psi\rangle$ carried the time characteristics of the evolution of the system. This is the Schrödinger representation of quantum mechanics which is expressed in his equation for $|\Psi\rangle$

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle \tag{7.69}$$

given in section section 3.2. We could have associated the time development with the *operators* involved instead. Because states are only revealed to us through measurement - the action of an operator on a state - this is really an indistinguishable and arbitrary choice. Such a choice constitutes the Heisenberg representation. This follows Heisenberg's equation

$$i\hbar\frac{d}{dt}\hat{A} = [\hat{A}, \hat{H}] + i\hbar\frac{\partial}{\partial t}\hat{A} \tag{7.70}$$

for an operator $\hat{A}$ as given in section 4.3.

There is another possibility. Instead of an all-or-nothing treatment we could choose to associate *some* of the time-dependence with the state and some with the operators. This is known as the Intermediate, or Dirac, picture (see [36]). This gives us much more freedom as we can choose from many different options exactly how to divide up the dynamics of the system. We will discuss two particular ways here.

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7.9.1 A Mixed Picture

If our Hamiltonian for the ISGE is

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + \hat{V} \tag{7.71}$$

then we can choose to divide up the time-dependence in many ways. For example, we can treat our operators $\hat{A}$, including $\hat{H}_0 = \hat{\boldsymbol{p}}^2/2m$, in the Heisenberg picture as varying in time while treating $\hat{V}$ using the Schrödinger picture with the states carrying the time-dependence. Thus we may separate our equation into two parts

$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = \tilde{V}|\Psi\rangle \tag{7.72}$$

$$i\hbar \frac{d}{dt}\hat{A} = [\hat{A}, \hat{H}] + i\hbar \frac{\partial}{\partial t}\hat{A} \tag{7.73}$$

where

$$\tilde{V} = \hat{U}\hat{V}\hat{U}^\dagger = e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{V}e^{-\frac{i}{\hbar}\hat{H}_0 t}. \tag{7.74}$$

In the case of the ISGE

$$\hat{V} = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} \tag{7.75}$$

and there may be several interesting choices for $\hat{A}$

$$\hat{A} = \hat{\mathbf{p}}, \hat{\boldsymbol{\sigma}}, \hat{\mathbf{x}}, \text{ etc.} \tag{7.76}$$

Since we assume that our operators have no *explicit* time-dependence we can drop the partial derivative terms and can evaluate the commutator in eq. (7.73) for the operators listed in eq. (7.76) to yield

$$\frac{d}{dt}\hat{\mu}_j = 0 \tag{7.77}$$

$$\frac{d}{dt}\hat{x}_j = \frac{\hat{p}_j}{m} = \hat{v}_j \tag{7.78}$$

$$\frac{d}{dt}\hat{p}_j = 0. \tag{7.79}$$

So $\hat{\mu}_j$ and $\hat{p}_j$ are constant in time and $\hat{x}_j = \hat{v}_j t + \hat{x}_{0j}$. The index notation denotes the separate components of the vectors, $j = 1, 2, 3$.

Before we can solve eq. (7.72) we must express $\tilde{V}$ in a more usable form. We can apply the identity

$$e^{\eta\hat{N}}\hat{M}e^{-\eta\hat{N}} = \hat{M} + \frac{\eta}{1!}[\hat{N},\hat{M}] + \frac{\eta^2}{2!}[\hat{N},[\hat{N},\hat{M}]] + \frac{\eta^3}{3!}[\hat{N},[\hat{N},[\hat{N},\hat{M}]]] + ... \qquad (7.80)$$

to eq. (7.74) with $\eta = it/2m\hbar$, $\hat{N} = \hat{\boldsymbol{p}}^2$, and $\hat{M} = -\hat{\boldsymbol{\mu}}\cdot\boldsymbol{B} = -e\hbar(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{B})/2m$.

In index notation with all indices ranging from 1 to 3 our magnetic field is $\hat{B}_j = -b(x\delta_{j1} - z\delta_{j3})$ so it can be shown that

$$[\hat{p}_j^2, \hat{\sigma}_i\hat{B}_i] = 2i\hbar b(\hat{\sigma}_x\hat{p}_x\delta_{j1} - \hat{\sigma}_z\hat{p}_z\delta_{j3}). \qquad (7.81)$$

Fortunately $\hat{p}_k^2$ commutes with all operators in eq. 7.81 so the expansion eq. (7.80) has only two terms. Thus,

$$\begin{aligned}
\tilde{V} &= -\hat{\boldsymbol{\mu}}\cdot\boldsymbol{B} - \frac{eb\hbar t}{2m^2}(\hat{\sigma}_x\hat{p}_x - \hat{\sigma}_z\hat{p}_z) & (7.82)\\
&= \frac{eb\hbar}{2m}\left[(\hat{\sigma}_x\hat{x} - \hat{\sigma}_z\hat{z}) - \frac{t}{m}(\hat{\sigma}_x\hat{p}_x - \hat{\sigma}_z\hat{p}_z)\right]. & (7.83)
\end{aligned}$$

We can then write eq. (7.72) as

$$\frac{d}{dt}\Psi_{\uparrow\downarrow} = -i\frac{eb}{2m}\left[(x\Psi_{\downarrow\uparrow} \mp z\Psi_{\uparrow\downarrow}) - \frac{t}{m}(\hat{p}_x\Psi_{\downarrow\uparrow} \mp \hat{p}_z\Psi_{\uparrow\downarrow})\right]. \qquad (7.84)$$

So far this approach is similar to the differential equation approach of section 7.2 but because of our particular choice of representation there are some significant differences. They are:

(1) We did not separate out the y and t behavior. Instead we have separated the behavior in a different way.

(2) The equations are first order in both time and space.

(3) These equations are at least as general as those of section 7.2 and at least as simple because some behavior has still been treated separately.

(4) We did not assume, but have *explicitly shown*, that the $\hat{p}_j$ are constant in time. This allows a us a clearer picture of how we could apply a slightly more practical choice for $\hat{p}_j$ rather than merely setting it to zero.

(5) Just as we saw in the cliffor treatment of section 7.7, here all factors of $\hbar$ have cancelled out. However, unlike that previous result this cancellation took place

while still including translational behavior. In this approach we also did not get rid of the matrix representation for the Pauli matrices.

In accordance with (4) if we do take the limit that $\hat{p}_j \rightarrow 0$ as we did before we can recover the same results (see section 7.1). In this case eq. (7.84) would become much simpler, namely

$$\frac{d}{dt}\psi_{\uparrow\downarrow} = -i\frac{eb}{2m}\left(x\psi_{\downarrow\uparrow} \mp z\psi_{\uparrow\downarrow}\right). \tag{7.85}$$

These equations can be decoupled to yield the two second order ODEs

$$x\frac{d^2}{dt^2}\psi_{\uparrow\downarrow} - v_x\frac{d}{dt}\psi_{\uparrow\downarrow} + \left[\left(\frac{eb}{2m}\right)^2\left(x^3 + xz^2\right) \mp \frac{eb}{2m}(v_x z - xv_z)\right]\psi_{\uparrow\downarrow} = 0 \tag{7.86}$$

where x and z are functions of time in general (see eq. (7.78)).

More specifically, for an infinite plane wave travelling in the y-direction v_x and v_z both have definite zero values. Making this assignment eq. (7.86) becomes

$$\frac{d^2}{dt^2}\psi_{\uparrow\downarrow} + \left(\frac{eb}{2m}\right)^2\left(x_0^2 + z_0^2\right)\psi_{\uparrow\downarrow} = 0. \tag{7.87}$$

The x and z reduce to x_0 and z_0 respectively because $v_x = v_z = 0$. Notice now that there is no difference between the $\uparrow\downarrow$ cases. When compared to the approach of section 7.1 this may tell us something about the meaning of the $\uparrow\downarrow$.

The solutions to this equation are

$$\psi_{\uparrow\downarrow} = C_1 e^{\frac{i}{\hbar}\left(\frac{eb\hbar t}{2m}\right)r_0} + C_2 e^{-\frac{i}{\hbar}\left(\frac{eb\hbar t}{2m}\right)r_0}. \tag{7.88}$$

If the interaction lasts for time T then this can be interpreted in the same manner as in section 3.2.5 or 7.1. In fact, the momenta

$$p_r = \pm\frac{eb\hbar T}{2m} \tag{7.89}$$

is identical. However, in this approach we see *as a consequence of the derivation* and not as a matter of interpretation as before that the spatial dependence in the exponent is actually a parameter describing the *initial* position of the beam or particle in the field.

From this approach we can also more naturally attribute the constants C_1 and C_2 to the initial spin conditions in the z-basis. C_1 is the fraction of the initial

spins that were in the spin "up" direction and C_2 is the fraction initially in the spin "down" direction as defined in the z-basis. For example, if the initial beam is perfectly polarized beam in $+z$, i.e. spin "up", then $C_1 = 1$ and $C_2 = 0$

$$\psi_{\uparrow\downarrow} = e^{\frac{i}{\hbar}(\frac{eb\hbar t}{2m})r_0} \tag{7.90}$$

and we have a plane wave travelling radially outward the radial direction being defined as pointing from the origin to r_0.

If the spins were initially polarized in the "down" direction then the plane wave would be travelling radially inward continuing on through the center.

We can also discuss orthogonal spins. Say the initial beam had initially passed through a Stern-Gerlach magnet such that the resulting polarization was in the x-direction. In this case $C_1 = 1/\sqrt{2}$ and $C_2 = 1/\sqrt{2}$. Thus

$$\psi_{\uparrow\downarrow} = \frac{1}{\sqrt{2}}e^{\frac{i}{\hbar}(\frac{eb\hbar t}{2m})r_0} + \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}(\frac{eb\hbar t}{2m})r_0} \tag{7.91}$$

and we have half the beam separating in the $+r$-direction and half in the $-r$-direction just as we might expect based on the results of the traditional SGE.

Notice that both this and other similar approaches either tell us nothings at all about the angular behavior of the solutions or they tell us that they are angularly symmetric in constrast to eq. (7.35) of section 7.4.

If to any of the foregoing derivations for the ISGE a large field component $B_0\hat{z}$ were added it is presumed that the radial behavior would partially average leaving only the behavior of the selected direction thus recovering the traditional SGE limit.

7.9.2   The Heisenberg Picture

There are other choices within the Intermediate picture we could make in dividing up the time-dependence of the system. [31] chooses to assign all temporal behavior to the operators thus adopting a purely Heisenberg approach.

[31] also uses a field similar to eq. (6.2) but truncates it to a finite region with a step function in order to avoid unwanted asymptotic behavior. Derivation of both the standard SGE as well as the ISGE is then possible. Because the method is very

similar to ours above we only cite their results. They find for the expectation values of x and z

$$\langle \hat{x}(t) \rangle \;=\; \langle \hat{x}_0 \rangle + \frac{t}{m}\langle \hat{p}_{x_0} \rangle + \frac{eb\hbar t}{2m^2 v}\langle \hat{\sigma}_{x_0} \rangle \tag{7.92}$$

$$\langle \hat{z}(t) \rangle \;=\; \langle \hat{z}_0 \rangle + \frac{t}{m}\langle \hat{p}_{z_0} \rangle - \frac{eb\hbar t}{2m^2 v}\langle \hat{\sigma}_{z_0} \rangle \tag{7.93}$$

with the assumptions that the factor $eb\hbar/2m^2 v$ is small enough to neglect it to second order and that the velocity v of the beam in the laboratory frame is constant.

It is concluded that for eigenstates of $\hat{\sigma}_z$

...in fact only the *average* x deflection vanishes. A spin-up particle will be found to undergo an x displacement but with equal probabilities in the $+x$ and $-x$ directions [31]. (p. 580)

It is claimed that this can be more clearly seen from the rest of [31] which we will discuss later in section 7.11.1.

7.10 A Comment on Precession in the Inhomogeneous Stern-Gerlach Effect

We have found some evidence that in the ISGE the particles will undergo a deflection in the positive radial direction for one sense of spin orientation and in the negative radial direction for the other (see section 7.1.4). This radial direction is defined by the line connecting the particle's parameterically described initial position in the beam r_0 and the origin (see sections 7.1.2 and 7.9.1). We have also seen in section 7.7 that in the ISGE rotations, or precession, could occur in the plane perpendicular to **r**.

7.10.1 The Inhomogeneous Stern-Gerlach Effect as a Local Stern-Gerlach Experiment

To give us a unified model of what is going on at the field point r_0 let's consider our purely inhomogeneous magnetic field

$$\mathbf{B} = b(-x\hat{x} + z\hat{z}) \tag{7.94}$$

which was valid in a small region equidistant from the four parallel wires running in the y-direction (see fig. 6.1). Let us suppose that in a coordinate system centered on the point $\mathbf{B} = 0$ that our particle is initially at the point

$$\mathbf{r} = r_0\hat{r} = x_0\hat{x} + z_0\hat{z}. \tag{7.95}$$

The field at this point is

$$\mathbf{B}(x_0, z_0) = b(-x_0\hat{x} + z_0\hat{z}). \tag{7.96}$$

If we add and subtract this field (eq. (7.96)) to the field everywhere (eq. (7.94)) then although there is no net change we can write the general field as

$$\mathbf{B} = b\left[(-x + x_0)\hat{x} + (z + z_0)\hat{z}\right] - \mathbf{B}(x_0, z_0). \tag{7.97}$$

Now doing a coordinate transformation such that

$$\tilde{x} = x - x_0 \text{ and } \tilde{z} = z + z_0 \tag{7.98}$$

we have

$$\mathbf{B} = b(-\tilde{x}\hat{x} + \tilde{z}\hat{z}) + \mathbf{A} \tag{7.99}$$

where

$$\mathbf{A} = -\mathbf{B}(x_0, z_0) = -b(x_0\hat{x} + z_0\hat{z}) = br_0\hat{r} \tag{7.100}$$

is a *local* homogeneity along the direction pointing from $(0,0)$ to (x_0, z_0) in the (x, z) coordinates.

Writing the field in this suggestive form and with the indication of section 7.7 that precession is occurring about the vector[6] $\mathbf{A}$ we can then interpret the ISGE as a *local* SGE in which there is a local uniform field component about which precession can be thought to occur as well as a non-uniform component which causes the particles *in that region* to separate according to their spin orientations. However, at any given point

$$|\mathbf{B}_{homogeneous}| = br_0 = b\sqrt{x_0^2 + z_0^2} \tag{7.101}$$

---

[6]A (not bolded) was a *bivector* in section 7.7.

and

$$|\mathbf{B}_{inhomogeneous}| = b\sqrt{x_0^2 + z_0^2 + x^2 + z^2 - 2xx_0 + 2zz_0} \qquad (7.102)$$

so, for most particles, there is no clear reason to believe that the validating condition eq. (5.11) for the precession argument is met. Only near the point $(x_0, -z_0)$ will the condition be satisfied.

7.10.2   Making Precession Insignificant: A Semi-Classical Argument

As we have said, one can trivialize the occurrence of precession in another way. If the experiment is constructed in such a way that the apparent duration of the field is short compared to the precession period then the averaging procedure exemplified in eq. (5.24) does not go to zero (see section 5.5). The precession argument becomes invalid.

As we have seen, in the ISGE the incident beam seems to spread in the radial direction with momentum

$$\mathbf{p} = \pm \frac{eb\hbar T}{2m}\hat{r}. \qquad (7.103)$$

So as the field gradient b or the time of interaction T increase, the "up" and "down" components of the incident wave are able to spread further apart.

Let us assume that, as is the case with silver atoms, the particles are massive enough that it is consistent to discuss them in terms of trajectories on their way to the detecting plate. This justifies a semi-classical treatment of the ISGE in which the particles are thought to have classical-like trajectories but with a "spread" factor appended. In our case

$$p_r \rightarrow p_r + \Delta p_r \qquad (7.104)$$

where Δp_r follows the uncertainty relation eq. (3.30).

Following this we can say that the particles have distinguished their trajectories according to their spin "up" and spin "down" components in the direction of the radial vector $\mathbf{r}_0$ by assuming different $\mathbf{p}$'s or propagation directions.

Δr_0	The maximum radius within which the particle is likely to be initially found
r_0	The initial position of the particle within the beam spot
p_{r_0}	The initial radial momentum of the particle
r	The position of the particle after interaction
R	The radius of the spreading due to spin interactions with the field
R'	The radius of the spreading due to collimation, i.e. an initial p_r distribution
T	The time which the particle is interacting with the field.
τ	The time of flight between leaving the field and hitting the detecting plate.

Table 7.1:   A table showing the physical definitions of some important variables.

If the radial distance travelled is R during the time of flight after leaving the field τ at a constant speed v then we may write

$$p_r = \frac{eb\hbar T}{2m} = mv = m\left(\frac{R}{\tau}\right) \tag{7.105}$$

so

$$R = \frac{eb\hbar T\tau}{2m^2}. \tag{7.106}$$

Notice the similarities between this and the small factor in eq. (7.92) and eq. (7.93).

However, even if we neglect the splitting that occurs from the magnetic field some blurring of the beam will occur due to diffraction effects of the beam collimation, i.e. effects from the Δp_r in eq. (7.104). In a quantum context this exemplifies the uncertainty principle. If we confine the beam to a given region of space through collimation we necessarily broaden the spread of momenta in that direction. So let us confine all particles in the beam to a circular beam region of radius Δr_0. The spread of radial momentum obeys

$$\Delta r_0 \Delta p'_{r_0} \geq \frac{\hbar}{2}, \tag{7.107}$$

the uncertainty principle. Note that the primes refer to dynamics that result from *collimation* whereas the unprimed variables refer to the dynamics caused by the interaction of the particle with the *field*.
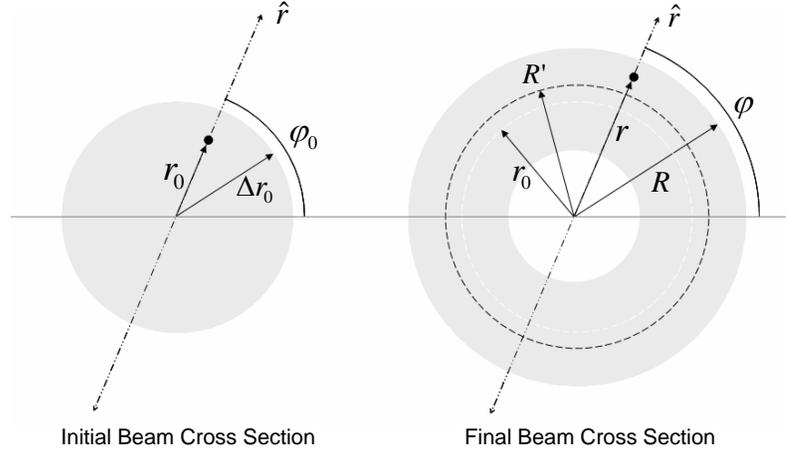
Figure 7.3: This shows graphically what some of the parameters are in finding a constraint on the precessionless ISGE in the semi-classical approach. The light dotted circle is the original beam radius and the dark dotted circle is the radius to which the beam would have spread due to collimation effects alone. It is still unclear whether or not $\varphi_0 = \varphi$. See also Table 7.1

In the semi-classical picture we can describe this as a particle located at r_0 following a classical trajectory with a momentum $p_{r_0} \pm \Delta p_{r_0}$ somewhere within the region of Δr_0. This is perhaps reminiscent of the pilot-wave picture in which a particle is said to dynamically evolve in unknown ways inside some region of likelihood.

Ignoring the fields then for a moment we can find the amount of radial displacement R' a particle will undergo due to diffraction of the collimator in the same way that we found the dynamical displacement R. Assuming we can add the *maximal* amount of uncertainty to an initial radial momentum of $p_{r_0} = 0$, i.e. initially no spreading, we have

$$p_{r_0} + \Delta p_{r_0} = \Delta p_{r_0} = mv' = m\left(\frac{R'}{\tau}\right) \tag{7.108}$$

but with a maximum uncertainty state we also have

$$\Delta p_{r_0} \approx \frac{\hbar}{2\Delta r_0} \tag{7.109}$$

so

$$R' \approx \frac{\hbar\tau}{2m\Delta r_0}. \tag{7.110}$$

We interpret R' to be the maximum amount of spreading that will occur due only to the collimation of a given beam of width Δr_0. We can see that if we make the

time of flight between the field and detector τ longer or if we made the beam more narrow, i.e. smaller Δr_0, this spreading would be more pronounced.

Now, in order to assure ourselves that the spreading from the spin interactions is observable against the inevitable spreading from collimation we require that

$$R > R'. \tag{7.111}$$

Without this condition observation of the ISGE would be swallowed up in a blur.

Using our definitions for R and R' from eq. (7.106) and eq. (7.110) we get the inequality

$$b\Delta r_0 T > \frac{m}{e} \tag{7.112}$$

where all parameters that can be experimentally adjusted have been consolidated to the left side. Only physical constants are on the right. The semi-classical description tells us that this is the condition for an observable effect.

We discussed the field locally in the previous section. At $\mathbf{r}_0$ there is apparently a local SGE with a precession frequency of

$$\omega = \frac{e|\mathbf{B}|}{2m} = \frac{ebr_0}{2m} \tag{7.113}$$

about the local field direction $\hat{r}$.

This could average away the transverse deflections due to the transverse components of the spin if it is allowed to operate long enough. Avoiding this gives us one more experimental condition. We require

$$T \approx \frac{2\pi}{\omega} = \frac{4\pi m}{ebr_0} \tag{7.114}$$

where T is the time of interaction or the time the particle spends in the field. Putting this value for T into the condition eq. (7.112) we arrive at

$$r_0 < 4\pi\Delta r_0. \tag{7.115}$$

If this condition is met then (1) precession is not a valid argument to discount the measurement of transverse components in the field because it is too slow to sufficiently average them away and yet (2) the spreading from spin effects will be observable

despite the spreading due to collimation. If we recall our interpretation of r_0 as the initial location of a particle *within* a beam spot of width Δr_0 it is seen that this condition is *always* met.

This is an extremely interesting result but it must be qualified.

(1) It follows an approach that neglects the kinetic energy terms which amounts to dropping important derivatives from a differential equation in **x**-space.

(2) It is a semi-classical derivation so it is neither purely classical nor purely quantum. Its interpretation is therefore somewhat *ad hoc* and stands on unclear ground. Whatever concepts might be most useful from either of the two regimes can be borrowed at will despite their mutual inconsistencies.

(3) From (2) all variable definitions are vague.

(4) As an example of (3) the momentum p_r was used in one case as a definite value whereas in another case it was said to uphold the uncertainty principle. Therefore, perhaps the inconsistency of the result with the uncertainty principle is a function of the application of the semi-classical representation and not of the phenomenon itself.

(5) The **B**-field is only approximate.

(6) In conjunction with (1) the incident waves were assumed to be infinite plane waves as stated in sections 7.1.1 so in actuality, even if all spin "up" components go "out" and all "down" components go "in" nothing but an infinite blur will be detected.

There are many other concerns that might be mentioned. These are sufficient though to point out that this discussion provides an interesting test, namely the method of this section, that might be applied under any circumstance using any approach or interpretation. Even if physically wrong it is useful in teaching us about our own misunderstandings of both interpretations of the representations and of the representations themselves.

7.11   Other Representations

Amongst the representations explored in this chapter there are several other possible representations by which we can view and assess the question of the ISGE

and precession in the standard SGE. We mention here a few of those that may be beneficial.

7.11.1 Propagators

The Green's function method outlined in section 7.8 maps source points in space to field points in space. But this glossed over the time evolution of the system. We have yet to do a solution that does not trivialize the time-dependence but allows it to naturally evolve.[7] Propagators can do this. They are similar to Green's functions but they map temporally past points to temporally present or future points.

[31] gives analytic expressions for the states $\psi_{\uparrow\downarrow}$ of the ISGE in the coordinates (x, y, z) which supposedly justifies the claim quoted in section 7.9.2. However, as many of the technical details in its derivation are cited from other papers and have not been sufficiently verified or interpreted we only mention this for completeness and to point out the potential fruitfullness of the propagator method of [31] for describing the time evolution in the ISGE.

7.11.2 Numerical Methods

To this point no numerical investigation has been done. While representing these problems numerically gives concrete and definite results they do not offer the same sort of insight as does the explicit confrontation of detail that an analytic approach offers. However, for a computer the PDEs that we have dealt with here could be solved for specific choices of parameters. This would not only give us one more perspective on the phenomenon of the ISGE but one that focused on *results*. This could guide our work in other more revealing analytic approaches including the ones discussed in this chapter. For example, it could provide an estimate for how accurate eq. (7.45) is for the near-origin approximation of section 7.5.7. For this reason numerical methods should be pursued in the future.

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[7]Section 7.9 is the closest we have come to this so far. Eq. (7.85) might be generalized from assuming incident plane waves (setting $v_j = 0$ as we did) to assuming some sort of incident packet (a spread in v_j).
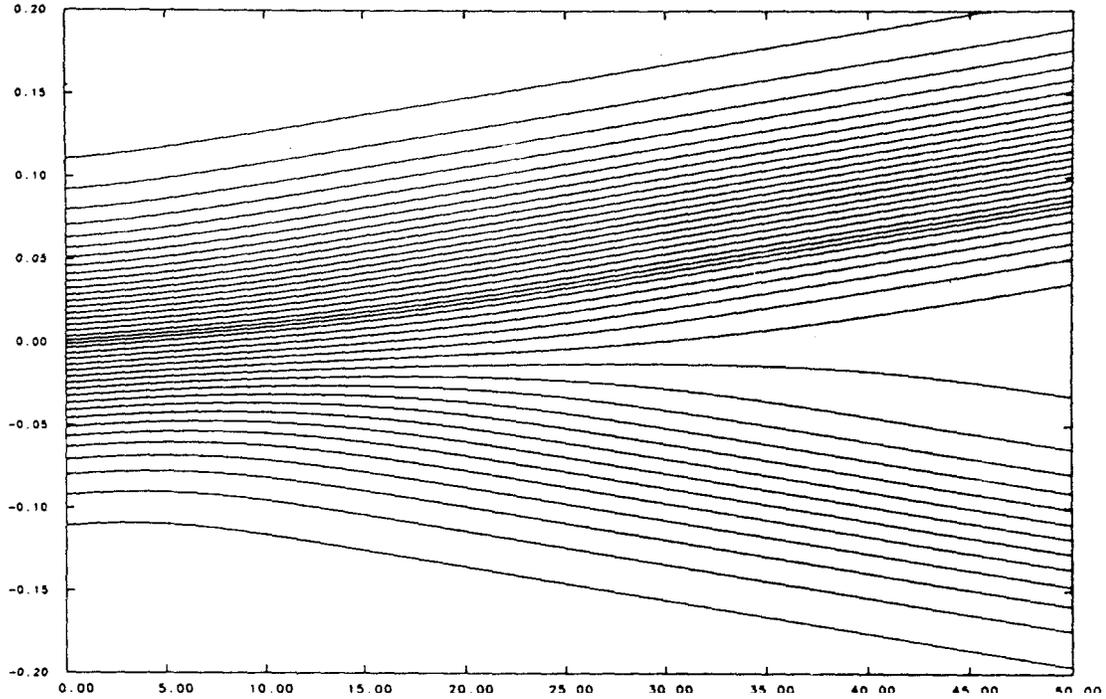
Figure 7.4: The trajectories of particles in a traditional Stern-Gerlach apparatus as calculated using the Bohmian techniques. 25% are initially chosen with spin "down" in the selected direction and 75% are chosen with spin "up."

7.11.3 Perturbation Theory

Often a portion of a system can be designated as small. Using perturbation methods these small effects can be eventually accounted for. This generally yields approximate but often sufficient results. Although one of the defining characteristics of the ISGE is the *equal* treatment of field components there are several other perturbation techniques that might be applied to solving the ISGE as was the Born method in section 7.8.2. For example, the series expansion eq. (7.7) could be perturbatively treated to slowly remove the relative strength of the homogeneous field B_0.

7.11.4 Bohmian Mechanics

Because we desire to understand the inner workings of the measurement process it is difficult to apply standard quantum techniques to gain understanding. In a
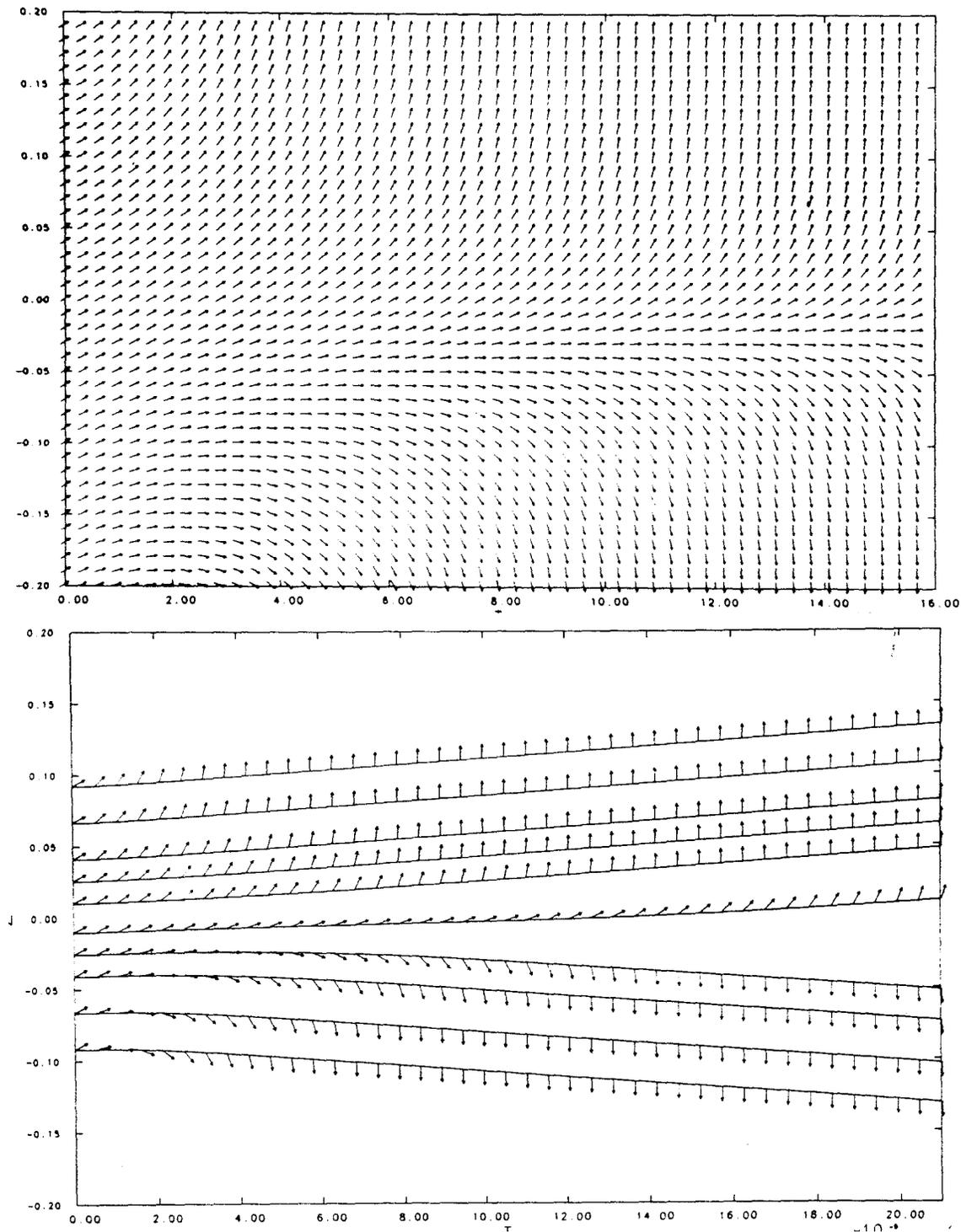
Figure 7.5: *Top* The spin field showing the orientation of the spin vectors associated with the conditions of fig. 7.4 at every point in physical space. *Bottom* If the trajectory and spin pictures of fig. 7.4 and *Top* are overlaid only those spins that are realized by the actual particle trajectories are emphasized and we obtain a model representative of the evolution of the spin vector during the process of a standard Stern-Gerlach measurement for given initial conditions. This point-by-point correlation allows a more ontological, or realist, interpretation of spin. Taken from [43].

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field that is formed by measurement axioms[8] how does one objectively study the nature of measurement itself? The methods of quantum mechanics have been effectively designed and interpreted to match experimental results or potential results, not *processes.*

There has been relatively little done with the other approaches that claim to give an accounting of the behind-the-scenes dynamics of the measurement process but they do exist. Many have applied the mechanics first introduced by de Broglie in 1927 and formalized by David Bohm in 1952 to inquiries of the SGE. In such cases the actual dynamics of both the position and spin orientation of the particles can be continuously and consistently described (see figs. 7.4 and 7.5. See also [14], [33], [43], and [44]). This has even been done using the representational algebra of section 7.7 to account for relativistic effects [45].

Insightful investigations of the sort proposed by Bohm should be extended to the case of the ISGE. Like many other representations they have the potential to offer unique insights into the operation of measurement and the SGE in general.

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[8]Note the discussion of the "collapse" and "cut" axioms in section 3.2.3.

Chapter 8

Conclusions

There has been an overarching tension in all that we have done in this thesis. It arises from the fact that while representations are necessary in order for rational communication and comprehension they also necessarily alter the perceived behavior of the phenomena they represent. They carry an accompanying value system and set of assumptions. In the case of the traditional SGE these values and assumptions have not been explicitly identified in the past because of its axiomatic role in the modern interpretation and practice of quantum mechanics. In Chapter 5 we attempted to show how this has limited our understanding, or possibility for understanding, in various ways and proposed a study of the ISGE. The question then becomes, "How does one study a phenomenon independent of its representation?" In this thesis we have used several different techniques in an effort to understand the SGE, and more particularly the ISGE, on a deeper level.

8.1 The Method of Relativity

Despite the multiplicity of specific methods that were used our overall approach is not that different from the general methodology of Einsteinian relativity. Instead of being confused by the number and relative nature of the several possible points of view that can be taken to solve a single problem we have deliberately moved between these "frames of reference" in order to study nature on a level independent of the frames themselves. That is, by honestly recognizing and comparing our *relative* knowledge we have hoped to approach more *absolute* knowledge.

In more concrete language we have approached the ISGE using several different methods. Each method necessarily involved some simplifying assumptions and approximations that allowed us a more or less explicit view of particular aspects of the ISGE. By comparing the effects of these simplifications, both on the results and on the methods themselves, we were able to round out our understanding of the SGE to a larger degree than had we merely applied just one result. Granted any one approach here could be pursued much further adding even more insight. In one sense this comparison method served to conceptually triangulate the one thing that was held constant in all approaches: the phenomenon. In another sense, the varied approaches can serve the same purpose as statistical sampling does in any study of complex behavior, though we have not used it so here.[1] In this sense we gain confidence and insight into both our descriptions of phenomena and into the phenomena themselves.

8.2 Our Results

For example, in our thematic account we saw how precession is often used as a physical argument by which to simplify the description of the SGE. Later we saw how the same mathematical result could be rigorously arrived at by treating the field with a series expansion eq. (7.7). This gave us a concrete way to study the nature of the precession argument, which had been difficult to approach until then, and the ISGE by perturbatively adding back in higher order corrections making precession less and less dominant. We also saw this happen with the assumption of incoming plane waves which is sometimes used in thematic approaches to the SGE. Although this led to the recognition of two distinct momenta it posed other problems. However, using the Intermediate or Heisenberg representations made this assumption more explicit so that one could perhaps lift the assumption in graded steps thus more closely returning to exact solutions.

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[1]In this sense the approach we used here mirrors the experimental method of statistically comparing large samples with "controls" and "variables" but with the analytical methods themselves as the object of study. Nature is only indirectly an object of study.

In our use of differential equations we did not make the assumption of incoming plane waves. It did however show us the effect of not having an asymptotically finite field. Using the integral equations provided by Green's functions provided for both these issues: we could address the asymptotic behavior of the field without having to assume incoming plane waves. Along this line there is much promise in pursuing the work of [31].

Using the Clifford algebra Cl_3 we found a result which in addition to emphasizing the apparent precession of the particle about the local field direction hinted at an interesting field of research as to the relation of the ISGE and spin to the classical and quantum regimes.

We also saw how it was precession that by and large justified the usual interpretation of the uncertainty principle in Stern-Gerlach measurements by averaging away incompatible components. However, it appears that the precessionless ISGE could require us to formulate a more definite notion of uncertainty as the appearance of *any localized* spot on the detecting plate in that case would seem to violate the usual interpretation (see section 5.5.3). Based on our work in section 7.10.2 this seems feasible.

Thus, the ISGE, which appears to be only a *local* version of the traditional SGE, has opened the door for a clearer study of both the theory and interpretation of physical science. We believe there are still several interesting and unanswered questions as to the correct interpretation of the SGE. Being very subtle and "relative" issues these may be pursued using the methodology of relativity as we have used it here.

8.3   The Dangers of an Inadequate Philosophy

Perhaps one of the most interesting conclusions of applying methods involving several representations is not the knowledge that comes out but the apparent fact that there *is* further knowledge to be gained. For this reason we cannot be too content with the prevalent, results-oriented, "it works" attitude although it may be necessary for the purposes of instruction. That is, "it works" should only serve as a temporary

justification for *pursuing* knowledge and not as its permanent *replacement*. Whether or not something "works," which is a relative term based on the *desired* ends of the investigator and her choice of assumptions, many different theoretical and analytical studies should be encouraged as only they *when taken together* provide explicit conceptual confrontation with *all* the details of nature. The process of understanding the differences and connecting the similarities of these details is a much less scientific process² but it is nonetheless *vital.* As we hinted at in section 2.1

> Without abstract ideas...'[we] would not be able to deal with concrete, particular, real-life problems. [We] would be in the position of a new-born infant, to whom every object is a unique, unprecedented phenomenon'...'As a human being [we] have no choice about the fact that [we] need a philosophy. [Our] only choice is whether [we] define [our] philosophy by a conscious, rational, disciplined process of thought and scrupulously logical deliberation - or let [our] subconscious accumulate a junk heap of unwarranted conclusions, false generalizations, undefined contradictions, undigested slogans, unidentified wishes, doubts and fears, thrown together by chance, but integrated by [our] subconscious into a kind of mongrel philosophy and fused into a single, solid weight: self-doubt, like a ball and chain in the place where [our] minds's wings should have grown.' (Rand in [5] p. 1-2)

Therefore, a carefully scrutinized conceptual scheme is necessary for a more effective and accurate picture of physical processes.

On a broader scale we finally note that our search for this consistent conceptual scheme has led us to an understanding, not only of the SGE and precession, but more generally of the process of searching itself. We have found that there is an inherent tension in this process that can contribute to our progress. It is demonstrated in the

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²[5] argues that this unscientific process of integration of diverse concepts can be nonetheless *objective.*

act of representing a phenomenon for communication or study, which is a necessary part of teaching and research. In research it was manifest as the tension between accurate historical accounts and clearly formed thematic ones, between the physical and analytical justifications of our approximations, between the realist and positivist interpretations of physics, between the $\mathbf{x}$ and $\mathbf{p}$ representations, between mutually inconsistent approximation methods, etc. In physics education and teaching this tension lies in the necessary balance of both clear and accurate communication; in both learning to answer questions and to question answers; in a sense, to both open and close the mind. There is a value in not only understanding these necessary disparities but also in accepting and using them for our advantage and progress. Thus, it seems that masterful and progress-oriented research and teaching will require that we master the art of mediation.

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