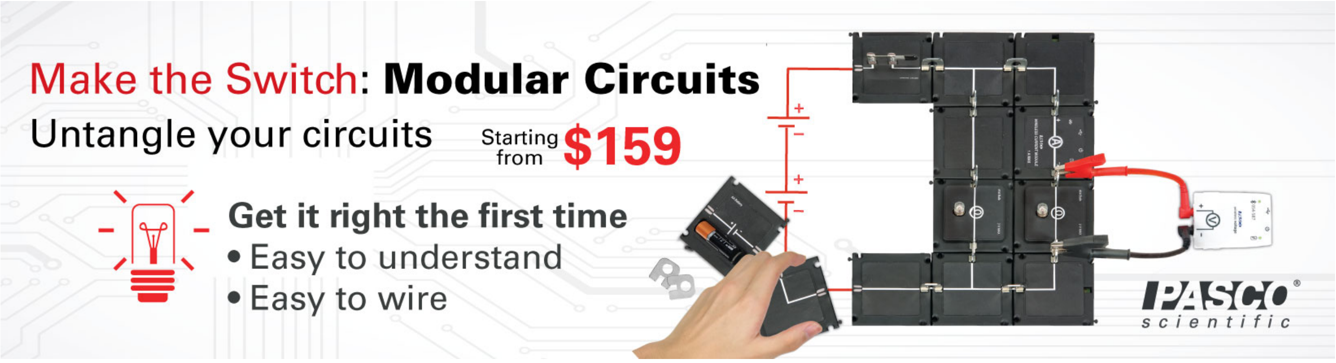
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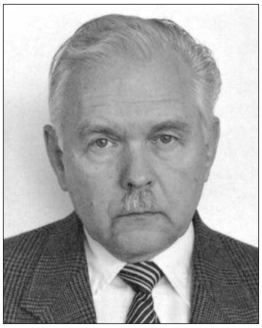
Dynamic Electric Field Maps

of Point Charge Moving with Constant Velocity

Oleg D. Jefimenko



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Conventional electric field maps of a moving point charge, such as the map shown in Fig. la, are based on the equation

$$\mathbf{E} = \frac{q(1 - v^2/c^2)}{4\pi\epsilon_0 r^3[1 - (v^2/c^2)\sin^2\theta]^{3/2}}\mathbf{r} \quad (1)$$

derived in 1888 by Oliver Heaviside.[1-3] In this equation, **E** is the electric field of a point charge q moving with constant velocity v along a straight line, c is the velocity of light, ϵ_0 is the permittivity of space, r is the distance between q and the point of observation, and θ is the angle between the velocity vector **v** and the radius vector **r** directed from q to the point of observation.

Heaviside noted that, according to Eq. (1), with increasing velocity of the charge the electric field of the charge concentrates itself more and more about the equatorial plane, $\theta = \pi/2$, and decreases along the line of motion, $\theta = 0$. This effect is shown in Fig. la by the density of the field lines. It should be noted, however, that field maps such as the map shown in Fig. la are somewhat misleading. First, it is impossible to represent the intensity of the electric field of a point charge (moving or stationary) on a two-dimensional map by the density of the field lines if *continuous* field lines are used. This is because on a two-dimensional map the radial lines diverge as $1/r$ rather than as $1/r^2$ as they really do in three dimensions.[4] Second, the field shown in Fig. la is an imaginary "snapshot" that cannot actually be observed (measured) by a single observer, moving or stationary. An observer co-moving with the charge would see only the ordinary electrostatic field of the charge at rest; a stationary observer would detect a time-dependent electric field rather than the time-independent field shown in Fig. la. The field shown in Fig. la could be observed only if many stationary observers (or field detecting instruments) located around the moving point charge would measure simultaneously the field of this charge at the respective points of their location.

An alternative way to represent graphically the electric field of a uniformly moving point charge is to use a map where the intensity of the field at the various points around the charge is represented by the length of the field vectors rather than by the density of the field lines. Such a map is shown in Fig. lb. However, this map, just as the map in Fig. la, represents the time-independent field that moves with the charge rather than the really important field that a single stationary observer would detect as the charge moves past the observer. To show the latter field, we have to construct a *dynamic field map* that depicts the electric field of a moving charge observed at a stationary point as a function of time.

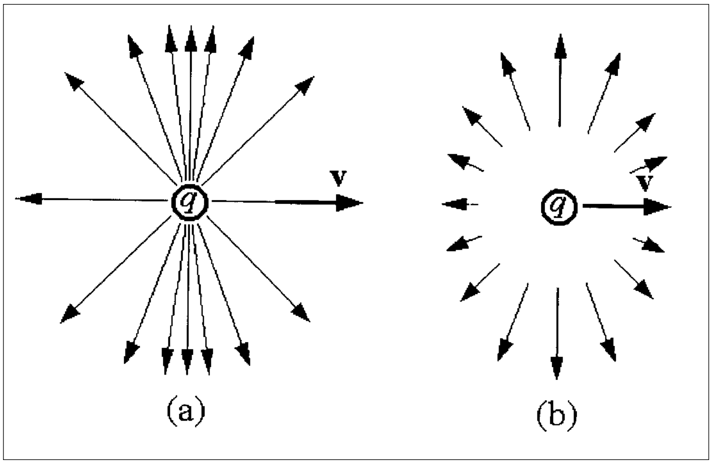

Fig. 1. (a) Conventional map of the electric field of a uniformly moving point charge indicates the magnitude of the electric field by the density of the field lines. (b) A more accurate way to show the magnitude of the electric field is to use field vectors of different lengths.

Constructing Dynamic Field Maps

To construct a dynamic field map of a uniformly moving point charge we proceed as follows. Writing Eq. (1) in terms of its Cartesian components (we only need two components for a two-dimensional map), we have

$$E_x = \frac{q(1 - v^2/c^2)}{4\pi\epsilon_0 r^3[1 - (v^2/c^2)\sin^2\theta]^{3/2}}x \tag{2}$$

$$E_y = \frac{q(1 - v^2/c^2)}{4\pi\epsilon_0 r^3[1 - (v^2/c^2)\sin^2\theta]^{3/2}}y \tag{3}$$

For the magnitude of **E** we have

$$E = \frac{q(1 - v^2/c^2)}{4\pi\epsilon_0 r^2[1 - (v^2/c^2)\sin^2\theta]^{3/2}} \tag{4}$$

Let us assume that the charge moves along the x-axis of rectangular coordinates and that the observer is located at a point P on the y-axis at a distance y from the origin. Noting that in this case $y/r = \sin\theta$ and $x/r = \cos\theta$ and using the abbreviations $q/4\pi\epsilon_0 = k$ and $v/c = \beta$, we can write Eqs. (2) through (4) as

$$E_x = k\frac{(1 - \beta^2)\sin^2\theta\cos\theta}{y^2(1 - \beta^2\sin^2\theta)^{3/2}} \tag{5}$$

$$E_y = k\frac{(1 - \beta^2)\sin^3\theta}{y^2(1 - \beta^2\sin^2\theta)^{3/2}} \tag{6}$$

and

$$E = k\frac{(1 - \beta^2)\sin^2\theta}{y^2(1 - \beta^2\sin^2\theta)^{3/2}} \tag{7}$$

These equations give the components and the magnitude of the electric field that would be detected by the stationary observer at the point P as a function of the angle θ. Since θ changes as the charge moves past P, these equations also represent the electric field as a function of time. By using different θ's in Eqs. (5) through (7) and plotting the corresponding field vectors placing their common origin at P, we can obtain a dynamic field map of the moving charge.[a] However, for the actual construction of the field map, Eqs. (5) through (7) can be further simplified by noting that y in these equations can affect only the scale of the map but not its shape. Therefore we can replace k/y^2 in Eqs. (5) through (7) by a new factor K obtaining

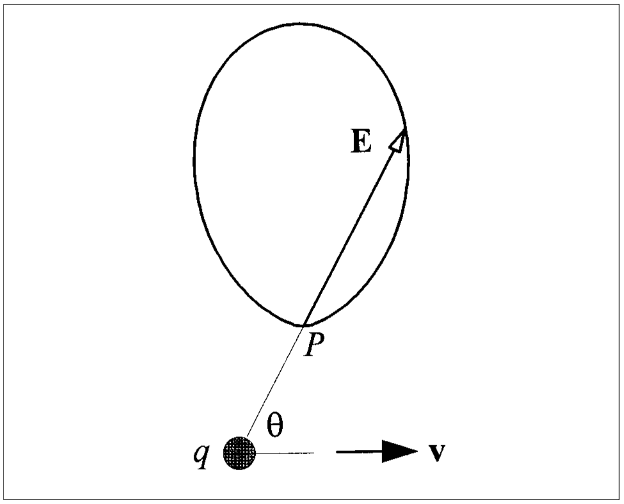

Fig. 2. Electric field contour curve of a point charge moving with constant velocity. When a point charge moves past a stationary point of observation, the charge creates a time-dependent electric field at this point of observation. The overall pattern of this field can be depicted by the electric field contour curve. A straight line drawn from the point of observation to a point of the contour curve gives the relative magnitude and the direction of the electric field ***E*** corresponding to an instantaneous position of the moving point charge. The charge q shown here moves along the x-axis. The point of observation P is on the y-axis. The angle θ is the angle between the radius vector joining the charge with the point of observation and the velocity vector of the charge. Contour curve was drawn for $v = 0.5\,c$.

$$E_x = K\frac{(1 - \beta^2)\sin^2\theta\cos\theta}{(1 - \beta^2\sin^2\theta)^{3/2}} \tag{8}$$

$$E_y = K\frac{(1 - \beta^2)\sin^3\theta}{(1 - \beta^2\sin^2\theta)^{3/2}} \tag{9}$$

and

$$E = K\frac{(1 - \beta^2)\sin^2\theta}{(1 - \beta^2\sin^2\theta)^{3/2}} \tag{10}$$

Before constructing a dynamic field map, observe that Eqs. (8) and (9) can be regarded as the parametric equations of the curve representing the locus of the end points of the electric field vectors of a moving point charge as these vectors would be measured by the stationary observer at the point P. Writing Eqs. (8) and (9) as

$$x = K\frac{(1 - \beta^2)\sin^2\theta\cos\theta}{(1 - \beta^2\sin^2\theta)^{3/2}} \tag{11}$$

$$y = K\frac{(1 - \beta^2)\sin^3\theta}{(1 - \beta^2\sin^2\theta)^{3/2}} \tag{12}$$

and using Eqs. (11) and (12) for drawing the corresponding curve, we obtain the electric field contour curve shown in Fig. 2 ($K = 1$ and $\beta = 0.5$ were used for this

[a] We need only any two of Eqs. (5) through (7) for plotting the map, but having all three equations simplifies the process of plotting.

drawing). A line segment drawn from the point P to any point of this curve gives the magnitude of the electric field vector $\mathbf{E}$ at P corresponding to a given θ (or to the instantaneous position of the moving point charge).

A dynamic field map based on Eqs. (8), (9), and (10) with $\beta = 0.5$ is shown in Fig. 3. Thirteen different angles θ were used for constructing this map. The 13 angles correspond to 13 instantaneous sequential positions occupied by the moving charge at the ends of equal time intervals Δt. The first 12 sequential positions of the charge are indicated in Fig. 3 by crosses; the charge is at the last position. Since the charge moves with constant speed, the instantaneous positions of the charge are separated by equal distances along the trajectory of the charge (the x-axis).

Whereas the maps shown in Fig. 1 are "snapshots" of the electric field co-moving with the charge producing this field, the map shown in Fig. 3 is a "multiple exposure" map where the individual field vectors as they would be measured by the stationary observer at time intervals Δt are shown all together. Of course, the entire map represents a very short event. For example, if the point P is located 1 m from the trajectory of the moving charge, the entire map represents an event that lasts only 10^{-8} s.

As can be seen from Eqs. (8) and (9) or (11) and (12), the electric field contour of a dynamic field map is strongly affected by the velocity of the charge (the β parameter). Three electric field contour curves for identical point charges moving with velocities $v = 0.01\ c$, $v = 0.70\ c$, and $v = 0.96\ c$, respectively, are shown in Fig. 4.

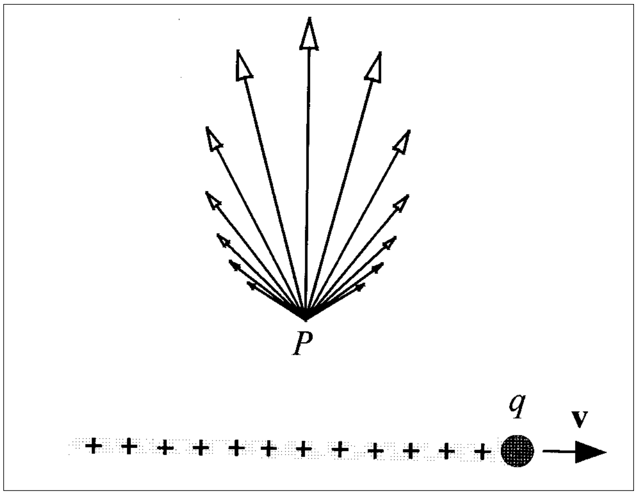

Fig. 3. Dynamic map of the electric field of the moving point charge *q*. Electric field vectors are shown at the stationary point of observation *P* as the charge moves past *P*. The field vectors correspond to the 13 sequential positions of the charge indicated on the map. An arbitrary scale was used for the vectors (actual numerical values depend on the constant *K*; see text). Map was drawn for *v* = 0.5 *c*.

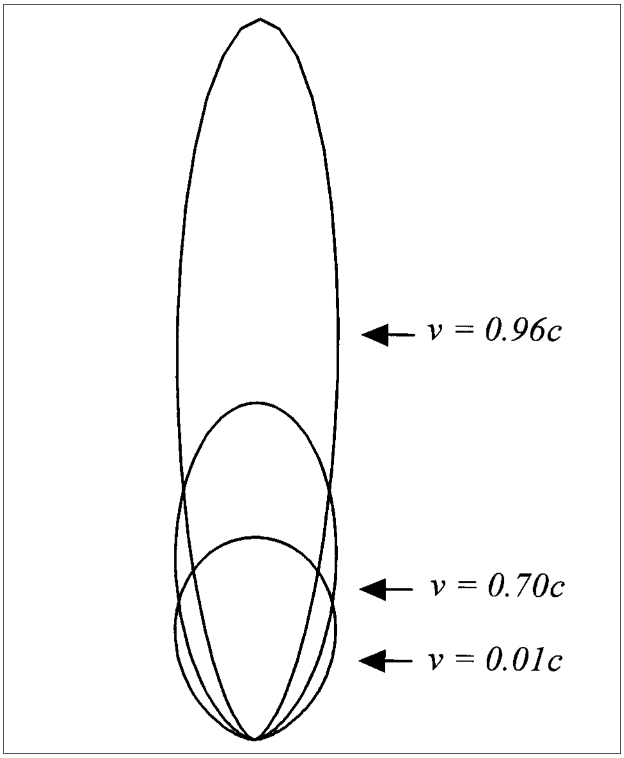

Fig 4. Heights of electric field contour curves are strongly affected by the velocity of the charge, but widths of the curves do not noticeably depend on *v*. The three contour curves shown here are for equal point charges moving at velocities *v* = 0.01 *c*, *v* = 0.70 *c*, and *v* = 0.96 *c*, as indicated.

Comments

Dynamic field maps and the corresponding contour curves provide a new way of depicting and analyzing the electric field of uniformly moving point charges and reveal several important properties of this field.

It is generally accepted that the field of a point charge moving with a velocity close to the velocity of light becomes a plane wave. The dynamic map shown in Fig. 3 indicates that this is not so. According to Fig. 3, the field of a fast-moving point charge, as seen by a stationary observer, is a momentary pulse, or burst, a sort of electric field explosion, but not a wave in the conventional sense.

It is also generally assumed that for a moving charge the electric field component in the direction of the motion of the charge rapidly diminishes and the component perpendicular to this direction rapidly increases with increasing velocity of the charge. This assumption is based on Eq. (1) with $v \to c$ and $\theta = 0$ or $\theta = \pi/2$. However, the contour curves shown in Fig. 4 indicate that this assumption is only partially correct. Note that whereas the heights of the curves in Fig. 4 are strongly affected by v, the widths of the curves do not noticeably depend on v. Since the half-width of a contour curve represents the maximum value of the field component parallel to the trajectory of the moving charge, it is clear that this value is hardly affected by the speed of the charge. Of course, if P were located on the trajectory of the charge (the x-axis), the only field component observed at P would be the x-component, and the value of this component would diminish with both the distance of the charge from P and the velocity of the charge, becoming zero for $v \to c$.

Another important effect revealed by the dynamic electric field map shown in Fig. 3 and by the contour curves shown in Fig. 4 concerns the force exerted by a moving point charge on a stationary charge when the moving charge passes the stationary charge. As is clear from Figs. 3 and 4, this force lasts only a very short time and is essentially normal to the trajectory of the moving charge. Therefore its main effect on the stationary charge is to give a sudden thrust to the stationary charge in the direction normal to the trajectory of the moving charge.[5] An interesting possible consequence of this effect is that a rapidly moving electric charge passing through a charged ring can cause a violent, explosion-like, destruction of the ring. As far as this author knows, this and similar force effects of rapidly moving point charges have not yet been discussed in the literature.

References

1. Oliver Heaviside, "The electromagnetic effects of a moving charge," *The Electrician* **22,** 147-148 (1888).
2. For a derivation of this equation from retarded field integrals see O. D. Jefimenko, "Direct calculation of the electric and magnetic fields of an electric point charge moving with constant velocity," *Am. J. Phys.* **62**, 79-85 (1994).
3. For a relativistic derivation of this equation see, for example, J. D. Jackson, *Classical Electrodynamics,* 3rd ed. (Wiley, New York, 1999), p. 559.
4. However, the electric field of a line charge whose axis is normal to the plane of the page can be correctly represented on a two-dimensional map. Heaviside found that the expression for the electric field of such a line charge moving in a direction perpendicular to the axis of the charge is [see Oliver Heaviside, "On the electromagnetic effects due to the motion of electricity through a dielectric," *Phil. Mag.*, **27,** 324-339 (1889)].

$$\mathbf{E} = \frac{\lambda(1 - v^2/c^2)^{1/2}}{2\pi\epsilon_0 r^2[1 - (v^2/c^2)\sin^2\theta]}\mathbf{r}.$$

The map shown in Fig. la is therefore more accurate as a map of the electric field of a line charge moving transversely rather than as a map of the electric field of a moving point charge.

5. For a discussion of this effect based on a different graphical representation of the field of a moving point charge, see Ref. 3. p. 560.