30071627

Physics 471 Exam 2 Winter 2008
Instructor: John Colton

Name: Solutions

I promise I do not have any "illegal" constants/formulas stored in my calculator:

(signed) ____________________

Instructions: Closed book. 3 hour time limit, 1% penalty per minute over. Calculators permitted. **Show your work.** Include units where appropriate. If additional space is needed, you may use the backs of pages.

Formulas:

Fresnel Eqns: $\alpha = \frac{\cos\theta_2}{\cos\theta_1}$, $\beta = \frac{n_2}{n_1}$

(p-polarization) $r = \frac{\alpha - \beta}{\alpha + \beta}$, $t = \frac{2}{\alpha + \beta}$

(s-polarization) $r = \frac{1 - \alpha\beta}{1 + \alpha\beta}$, $t = \frac{2}{1 + \alpha\beta}$

$$\frac{1}{n^2} = \frac{u_x^{\;2}}{n^2 - n_x^{\;2}} + \frac{u_y^{\;2}}{n^2 - n_y^{\;2}} + \frac{u_z^{\;2}}{n^2 - n_z^{\;2}}$$

Uniaxial, optic axis $\perp$ to surface:

$$n = n_o, \frac{n_o n_e}{\sqrt{n_o^{\;2}\sin^2\theta_2 + n_e^{\;2}\cos^2\theta_2}}$$

$$\tan\theta_2 = \frac{n_e}{n_o}\frac{\sin\theta_1}{\sqrt{n_e^{\;2} - \sin^2\theta_1}}$$

$$\tan\phi' = \frac{n_o}{n_e}\frac{\sin\theta_1}{\sqrt{n_e^{\;2} - \sin^2\theta_1}}$$

Two interfaces:

$$t_{13} = \frac{t_{12}t_{23}}{e^{-ik_2 d\cos\theta_2} - r_{21}r_{23}e^{ik_2 d\cos\theta_2}}$$

$$T_{13} = \frac{n_3}{n_1}\frac{\cos\theta_3}{\cos\theta_1}|t_{13}|^2 = \frac{T_{\max}}{1 + F\sin^2\frac{\Phi}{2}}$$

$$T_{\max} = \frac{n_3}{n_1}\frac{\cos\theta_3}{\cos\theta_1}\frac{|t_{12}|^2|t_{23}|^2}{\left(1 - |r_{21}||r_{23}|\right)^2} = \frac{T_{12}T_{23}}{\left(1 - \sqrt{R_{21}}\sqrt{R_{21}}\right)^2}$$

typo, should be R_{23}

$$F = \frac{4|r_{21}||r_{23}|}{\left(1 - |r_{21}||r_{23}|\right)^2}$$

$$\Phi = 2k_2 d\cos\theta_2 + \delta_{21} + \delta_{23}$$

$$\Delta\Phi_{FWMH} = \frac{4}{\sqrt{F}}$$

$$\Delta\lambda_{FWMH} = \frac{\lambda^2}{\pi n_2 d\cos\theta_2\sqrt{F}}$$

$$\Delta\lambda_{FSR} = \frac{\lambda^2}{2n_2 d\cos\theta_2}$$

Multilayers:

$t_{13} = 1/a_{11}$

$r = a_{21}/a_{11}$

$\beta_j = k_j l_j \cos\theta_j$

p-polar:

$$M_j = \begin{pmatrix} \cos\beta_j & \frac{-i\sin\beta_j\cos\theta_j}{n_j} \\ \frac{-in_j\sin\beta_j}{\cos\theta_j} & \cos\beta_j \end{pmatrix}$$

$$A = \frac{1}{2n_0\cos\theta_0}\begin{pmatrix} n_0 & \cos\theta_0 \\ n_0 & -\cos\theta_0 \end{pmatrix}\left(\prod_{j=1}^{N} M_j\right)\begin{pmatrix} \cos\theta_{N+1} & 0 \\ n_{N+1} & 0 \end{pmatrix}$$

s-polar:

$$M_j = \begin{pmatrix} \cos\beta_j & \dfrac{-i\sin\beta_j}{n_j\cos\theta_j} \\ -in_j\cos\theta_j\sin\beta_j & \cos\beta_j \end{pmatrix}$$

$$A = \frac{1}{2n_0\cos\theta_0}\begin{pmatrix} n_0\cos\theta_0 & 1 \\ n_0\cos\theta_0 & -1 \end{pmatrix}\left(\prod_{j=1}^{N} M_j\right)\begin{pmatrix} 1 & 0 \\ n_{N+1}\cos\theta_{N+1} & 0 \end{pmatrix}$$

<u>Linear dispersion</u>:

$$v_g \approx \left(\frac{d}{d\omega}\mathrm{Re}(k)\bigg|_{\omega=\omega_0}\right)^{-1}$$

$$t' \approx \frac{d}{d\omega}\mathrm{Re}(\vec{k})\bigg|_{\omega=\omega_0}\cdot\Delta\vec{r}$$

$$I \propto e^{-2\vec{k}_{imag}(\omega_0)\cdot\Delta\vec{r}}\left|E(t-t',\vec{r}_0)\right|^2$$

<u>Quadratic dispersion</u>:

$$k = k_0 + \frac{1}{v_g}(\omega-\omega_0) + \alpha(\omega-\omega_0)^2$$

$$\frac{1}{v_g} = \frac{1}{c}(n'\omega+n)\bigg|_{\omega=\omega_0}$$

$$\alpha = \frac{1}{2c}(n''\omega+2n')\bigg|_{\omega=\omega_0}$$

Gaussian wavepacket, through thickness z:

$$\Phi = 2\alpha z$$

$$T = \tau\sqrt{1+\Phi^2}$$

$$E(t,z) = \frac{E_0 e^{i(kz-\omega_0 t)}}{\left(1+\Phi^2\right)^{1/4}} e^{\frac{i}{2}\tan^{-1}\Phi - \frac{i\,\Phi}{2T^2}\left(t-\frac{z}{v_g}\right)^2} e^{-\frac{1}{2T^2}\left(t-\frac{z}{v_g}\right)^2}$$

<u>Michelson</u>:

Single ω: $I_{\det}(\tau) = 2I_0(1+\cos\omega\tau)$

Band of ω's:

$$\varepsilon_0 = \int_{-\infty}^{\infty} I(\omega)d\omega$$

$$\gamma(\tau) = \frac{1}{\varepsilon_0}\int_{-\infty}^{\infty} I(\omega)e^{-i\omega\tau}d\omega$$

$$\int_{-\infty}^{\infty} I_{\det}(\tau)dt = 2\varepsilon_0\left(1+\mathrm{Re}\,\gamma(\tau)\right)$$

$$\langle I_{\det}(\tau)\rangle = 2\langle I_{onebeam}\rangle\left(1+\mathrm{Re}\,\gamma(\tau)\right)$$

$$t_c = \int_{-\infty}^{\infty}|\gamma(\tau)|^2 d\tau$$

$$\frac{FT(NormSig)}{\sqrt{2\pi}} = 2\varepsilon_0\delta(\omega) + I(\omega) + I(-\omega)$$

<u>Young</u>:

Point source: $I_{\det}(h) = 2I_0\left(1+\cos\left(\frac{kyh}{D}+\Delta\phi\right)\right)$

Extended source:

$$\varepsilon_0 = \int_{-\infty}^{\infty} I(y')dy'$$

Not clear space

$$\gamma(h) = \frac{e^{\frac{ikyh}{D}}}{\varepsilon_0}\int_{-\infty}^{\infty} I(y')e^{-\frac{ikhy'}{R}}dy'$$

$$\langle I_{\det}(h)\rangle = 2\langle I_{oneslit}\rangle\left(1+\mathrm{Re}\,\gamma(h)\right)$$

$$h_c = \int_{-\infty}^{\infty}|\gamma(h)|^2 dh$$

<u>Integrals</u>:

$$\int_{-\infty}^{\infty} e^{i\omega(t-t_0)}d\omega = 2\pi\delta(t-t_0)$$

$$\int_{-\infty}^{\infty} e^{-Ax^2+Bx+C}dx = \sqrt{\frac{\pi}{A}}\,e^{\frac{B^2}{4A}+C}, \quad \mathrm{Re}(A) > 0$$

True/False and Multiple Choice. Please circle the correct answer. (2 pts each)

1. T or F: It is always possible to *completely eliminate* reflections with a single-layer antireflection coating as long as the right thickness is chosen for a given real index

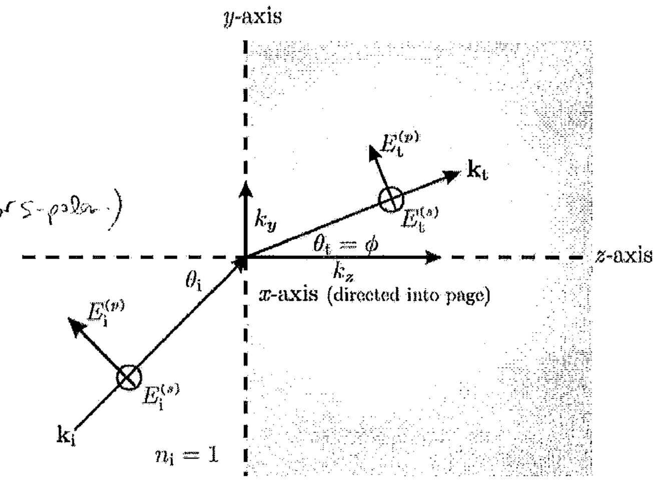
   If material 3 ≠ material 1 then T_{13} can be [illegible]

2. T or F: When coating each surface of a lens with a single-layer antireflection coating, the thickness of the coating on the exit surface will need to be *the same as* the thickness of the coating on the entry surface. As long as $\Delta OPL = n\lambda$, it'll be OK

3. T or F: The Michelson interferometer is ideal for measuring the *temporal* coherence of light.

4. T or F: The integral of $I(t)$ over all t equals the integral of $I(\omega)$ over all ω.

5. T or F: p-polarized light entering a uniaxial crystal as shown in the figure (optic axis in z-direction) sees $n = n_o$ inside the crystal, regardless of incident angle. (would only be true for s-polar.)



6. The term describing a low-symmetry crystal is called
   a. antisymmetric
   b. homogeneous
   c. non-isotropic
   d. rarified

7. In a uniaxial crystal, which vector always obeys Snell's law?
   a. k
   b. S
   c. both k and S
   d. neither k nor S

8. Which of the following was *not* mentioned in the book as something that could be varied to allow you to see Fabry-Perot fringes (transmission peaks)?
   a. angle going through the etalon
   b. diameter of the light beam
   c. spacing between partial reflectors
   d. wavelength of the light

9. The etalons with the narrowest transmission peaks are those with:
   a. small R
   b. large R
   c. Peak width is independent of R

   $\Delta\lambda_{FWHM} \sim \frac{1}{\sqrt{F}}$ and F grows with R

10. The etalons with the greatest resolving power have
    a. small R
    b. large R
    c. Resolving power is independent of R

    $RP = \frac{\lambda}{\Delta\lambda_{FWHM}} = \frac{\lambda}{\lambda^2 / (\pi n d \cos\theta \sqrt{F})} = \frac{\pi n d \cos\theta \sqrt{F}}{\lambda}$

    large F → large RP

11. In Michelson interferometer experiment, the light will produce the best fringes if the movable arm of the interferometer is:
    a. shorter than the length of the fixed arm
    b. the same length as the fixed arm $|\gamma| =$ maximum when $\tau = 0$
    c. longer than the length of the fixed arm
    d. The fringes do not depend on the relative lengths of the two arms.

Problems. Please answer the following questions/solve the following problems.

$\beta_j = \pi/2$ s vs p doesn't matter; $\theta_j = 0$

12. (5 pts) For the structure as shown, light comes in at normal incidence. The n_1 and n_2 layers have the right thickness to make them $\lambda/4$ for the wavelength of interest. (a) Write down the matrix equation for the matrix A. (Please don't multiply the matrices together.) (b) Explain what you would do in order to find R, after you multiplied the matrices together.

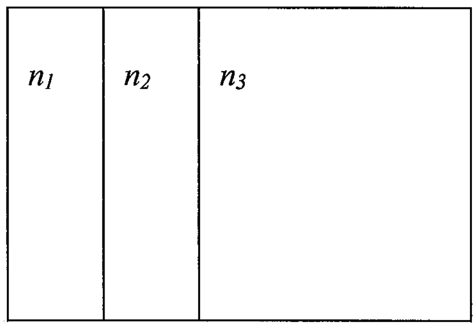
(a)

Using eq. p pola eqn:

$$A = \frac{1}{2 n_0 \cos\theta_0}\begin{pmatrix} n_0 & \cos\theta_0 \\ n_0 & -\cos\theta_0 \end{pmatrix} \prod M_j \begin{pmatrix} \cos\theta_{N+1} & 0 \\ n_{N+1} & 0 \end{pmatrix}$$

$$M_j = \begin{pmatrix} \cos\beta_j & \frac{-i\sin\beta_j\cos\theta_j}{n_j} \\ \frac{-i n_j \sin\beta_j}{\cos\theta_j} & \cos\beta_j \end{pmatrix}$$

air, $n=1$
n_0



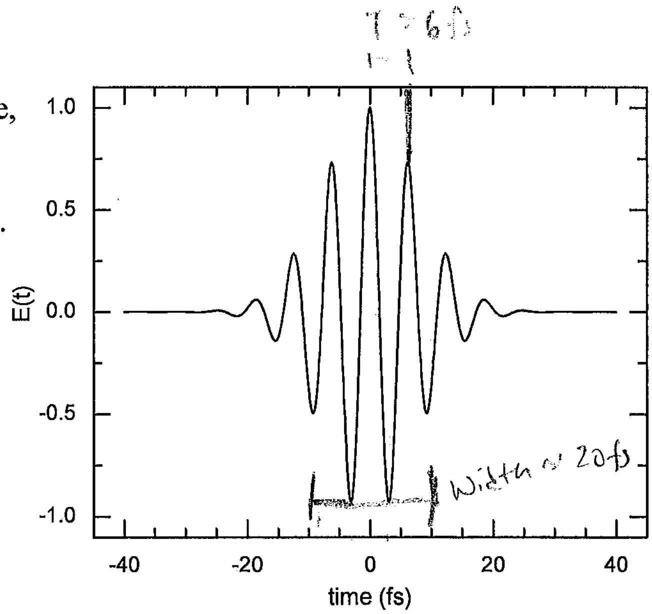
$$\boxed{A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i/n_1 \\ -in_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i/n_2 \\ -in_2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ n_3 & 0 \end{pmatrix}}$$

(b) Multiply together, get 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Then $r = \dfrac{a_{21}}{a_{11}}$

and $R = |r|^2$

13. (5 pts) For the normalized $E(t)$ in the figure,
   a. What is the carrier wavelength?
   b. Sketch $|E(\omega)|$ for $\omega>0$; be fairly precise about the position and width of the peak.



a. period ≈ 6 fs

$f = \frac{1}{T}$ and $\lambda f = c$

$\lambda = \frac{c}{f} = cT$

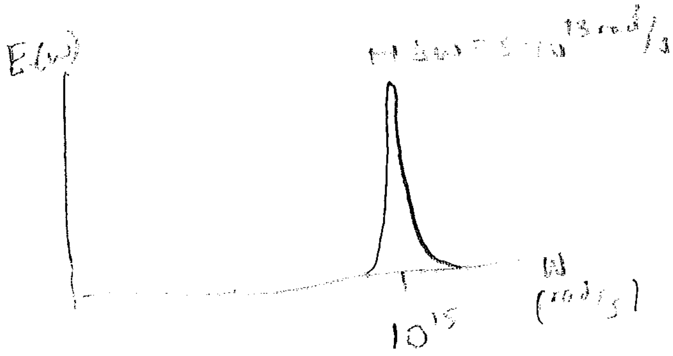
$\lambda = (3\cdot10^{8})(6\cdot10^{-15})$

$\boxed{\lambda = 1.8\times10^{-6}\ m}$

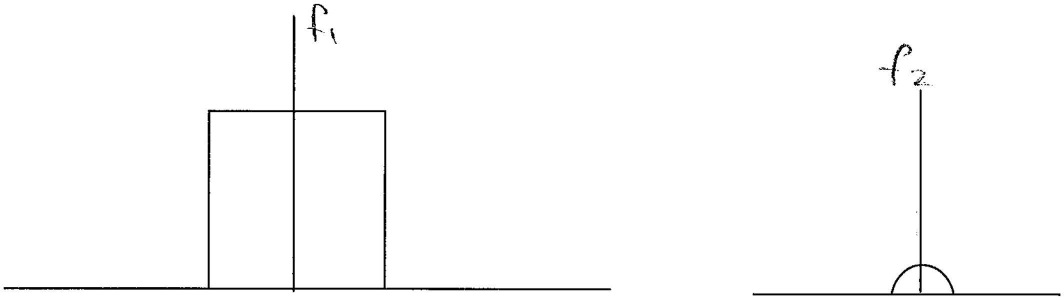
b. peak will be at $\omega_0 = 2\pi f_0 = 2\pi/T$

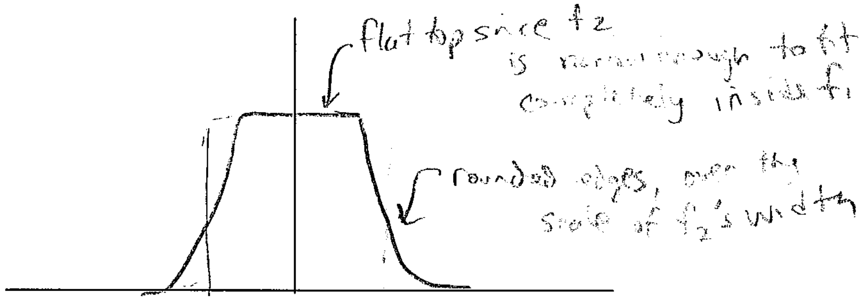
$\omega_0 = \frac{2\pi}{6\cdot10^{-15}} = 1.047\cdot10^{15}$ rad/s

width ... approx $1/\text{time width} = \frac{1}{20\cdot10^{-15}} = 5\cdot10^{13}$ rad/s



14. (5 pts) Sketch the convolution of these two functions, in a reasonably accurate fashion.





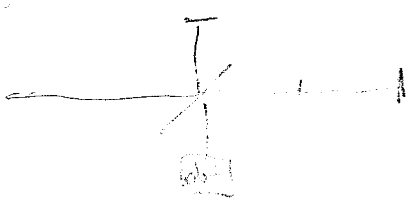
15. (7 pts) In class I derived this equation for a Michelson interferometer with single-frequency light that is split equally into two beams: $I_{det} = 2\, I_0\, (1+\cos\omega\tau)$. (a) Show that the phase shift $k\Delta x$ really is equivalent to $\omega\tau$, as I claimed in class. (b) Derive this equation.

(a) $k\Delta x = \left(\frac{2\pi}{\lambda}\right)(c\tau)$ where τ is time to travel Δx

$= 2\pi f \tau$ since $f = c/\lambda$

$= \omega\tau$ ✓

(b)



$$\tilde{E}_{det} = \tilde{E}_{path\,1} + \tilde{E}_{path\,2}$$

$$= \tilde{E}_0 e^{i(kx-\omega t)} + \tilde{E}_0 e^{i(kx-\omega(t-\tau))}$$

$$= \tilde{E}_0 e^{i(kx-\omega t)}\left[1 + e^{i\omega\tau}\right]$$

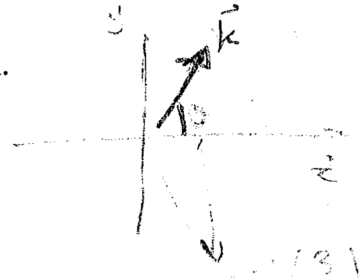
$$|E_{det}|^2 = |E_0|^2 \underbrace{(1+e^{i\omega\tau})(1+e^{-i\omega\tau})}_{1+1+e^{i\omega\tau}+e^{-i\omega\tau}}$$

$$= 2 + 2\cos\omega\tau$$

$$\boxed{I_{det} = 2I_0(1+\cos\omega\tau)}$$

16. (12 pts) Light ($\lambda_{vac} = 500$ nm) enters a uniaxial crystal with $n_o = 1.4$ and $n_e = 1.8$. The optic axis of the crystal is in the z direction, and the light's k-vector <u>inside the crystal</u> points in the direction $3\hat{y}+\hat{z}$.

a. If the E-field is polarized in the $\hat{x}$ direction, find λ inside the crystal.
b. If the E-field is polarized in the y-z plane, find λ inside the crystal.
c. Identify the direction of the Poynting vector for the case in (a).
d. Identify the direction of the Poynting vector for the case in (b).



$\theta = \tan^{-1}\left(\frac{3}{1}\right)$

$= 71.57°$

a. It's is s-polar, Nothing unusual. $n = n_o = 1.4$

$$\lambda = \lambda_{vac}/n = \frac{500}{1.4} = \boxed{357\text{ nm}}$$

b. This is p-polar

$$n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta_2 + n_e^2 \cos^2\theta_2}} = \frac{1.4 \cdot 1.8}{\sqrt{1.4^2 \sin^2 71.57° + 1.8^2 \cos^2 71.57}}$$

$$n = 1.744$$

$$\lambda = \frac{\lambda_{vac}}{n} = \frac{500}{1.744} = \boxed{287\text{ nm}}$$

c. Poynting vector in k-direction $\rightarrow$ $\boxed{71.57° \text{ above } z \text{ axis}}$ (towards y-axis)

d. Poynting vector given by ϕ'

$$\tan\phi' = \frac{n_o}{n_e}\frac{\sin\theta_1}{\sqrt{n_e^2 - \sin^2\theta_1}} = \frac{n_o}{n_e}\left[\frac{n_o}{n_e}\tan\theta_2\right]$$

$$= \left(\frac{1.4}{1.8}\right)^2 \tan(71.57°)$$

$$\phi' = 61.1° \rightarrow \boxed{61.1° \text{ above } z \text{ axis}}$$ (towards y-axis)

17. (12 pts) A Gaussian wavepacket, $E(z=0,t) = E_0 e^{-t^2/2\tau^2} e^{-i\omega_0 t}$ travels through a 1 cm piece of material with $n(\omega) = 1 + \frac{\omega}{10\omega_o}$. The pulse width is such that $\tau = 20/\omega_o$.

a. Find $k(\omega)$.
b. Find $v_p(\omega)$.
c. Find $v_g(\omega)$.
d. How much longer does it take the pulse peak to get through the glass than if it traveled through vacuum?

a. $\frac{\omega}{k} = \frac{c}{n} \rightarrow k = \frac{n\omega}{c} \qquad k = \frac{\omega}{c}\left(1 + \frac{\omega}{10\omega_o}\right)$

$$\boxed{k = \frac{\omega^2}{10c\omega_o} + \frac{\omega}{c}}$$

b. $v_p = \frac{\omega}{k} = \frac{c}{n} = \boxed{c\left[1 + \frac{\omega}{10\omega_o}\right]^{-1}}$

c. $v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} = \left(\frac{\omega}{5c\omega_o} + \frac{1}{c}\right)^{-1}$

$$\boxed{v_g = c\left[1 + \frac{\omega}{5\omega_o}\right]^{-1}} = c\,\frac{5\omega_o}{5\omega_o + \omega} = \boxed{5c} \,.f$$

you [illegible] at ω_o

d. For this particular [illegible] must evaluate v_g at $\omega = \omega_o$ to figure out how fast pulse peak is moving

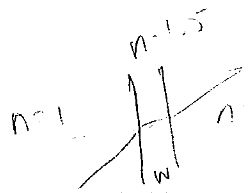
$$v_g\big|_{\omega_o} = c\left(1 + \frac{1}{5}\right)^{-1} = \frac{5}{6}c$$

$$t = \frac{d}{v_g} \Rightarrow (t)_{glass} = \frac{d}{5/6\,c} = \frac{6}{5}\frac{d}{c}$$

$$(t)_{vacuum} = \frac{d}{c}$$

$$\Delta t = \frac{1}{5}\frac{d}{c}$$

$$\Delta t = \frac{1}{5}\,\frac{.01}{3\cdot 10^8} = \boxed{6.67\cdot 10^{-12}\ s}$$



$n_1 \sin\theta_1 = n_2 \sin\theta_2$

$1 \sin 45° = 1.5 \sin\theta_2$

$\theta_2 = 28.13°$ (= .4998 rads)

18. (16 pts) A thin glass film is suspended in air similar to a soap bubble film, with index $n = 1.5$ and thickness w. It has s-polarized light of λ_0 incident on it at an angle of 45°.
    a. What is the smallest thickness w that will give a maximum in the reflectance? → minimum in transmission
    b. Evaluate R for this thickness.

a. Fabry - Perot $T = \dfrac{T_{max}}{1 + F \sin^2 \frac{\Phi}{2}}$

T will be a minimum when denominator = maximum $\Rightarrow \frac{\Phi}{2} = 90°$

$\Phi = 180° = \pi$

Assume δ_{21} and $\delta_{23} = 0°$ (will check later)

then $\Phi = 2 k_2 d \cos\theta_2 = \dfrac{4\pi n_2 d \cos\theta_2}{\lambda_0}$

$d = \dfrac{\pi \lambda_0}{4\pi (1.5) \cos 28.13°} = \boxed{.189\,\lambda_0}$

b. $T_{max} = \dfrac{T_{12} T_{23}}{\left(1 - \sqrt{R_{21} R_{23}}\right)^2} = 1$ since no absorption
• material 3 = material 1

$F = \dfrac{4 |r_{21}||r_{23}|}{\left(1 - |r_{21}||r_{23}|\right)^2}$ $\Rightarrow$ need some Fresnel coeff's

s-polar. $r_{21} = \dfrac{1 - \frac{\cos\theta_1}{\cos\theta_2}\frac{n_1}{n_2}}{1 + \frac{\cos\theta_1}{\cos\theta_2}\frac{n_1}{n_2}} = \dfrac{1 - \frac{\cos 45}{\cos 28.13}\frac{1}{1.5}}{1 + \frac{\cos 45}{\cos 28.13}\frac{1}{1.5}} = .3033$ (no phase shift, as expected)

$r_{23} = r_{21}$ because of symmetry

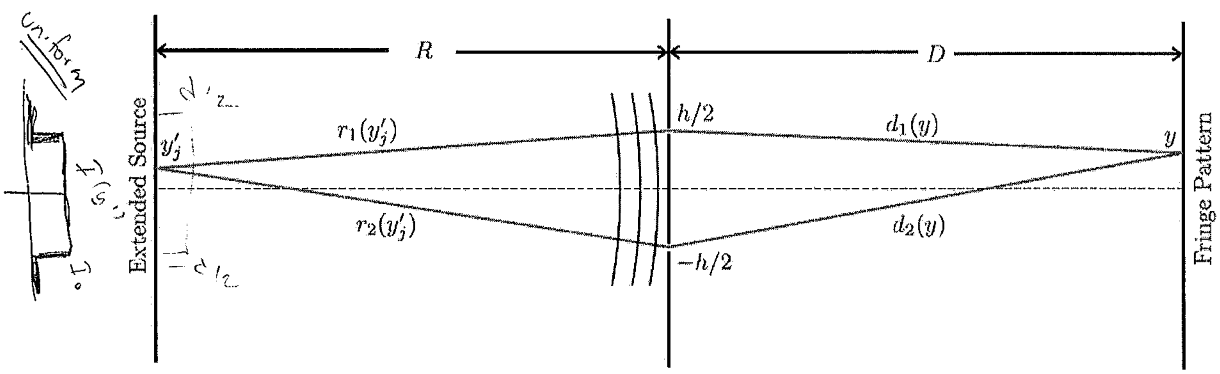
$F = \dfrac{4 (.3033)^2}{\left[1 - (.3033)^2\right]^2} = .4464$

then $T = \dfrac{T_{max}}{1 + F \sin^2\frac{\Phi}{2}} = \dfrac{1}{1 + .4464 (1)^2} = .6914$

$R = 1 - T$

$\boxed{R = 30.9\%}$

19. (16 pts) An extended light source of fixed wavelength λ and length d (extending from $y' = -d/2$ to $d/2$) is located a distance R away from two narrow slits. The light has a randomly-varying phase across its surface. The slits are separated by a variable distance, h. A screen is placed a distance D away from the slits in order to view the interference pattern. As h is varied, the intensity at point y on the screen oscillates.



a. For a *uniform* light source, determine the degree of coherence function, $\gamma(h)$, and from that deduce the intensity $I_{screen}(h)$ in terms of λ, d, h, y, D, and R, and the intensity you'd get from a single slit, $I_{oneslit}$.

b. Suppose $\lambda = 700$ nm, $d = 1$ mm, $D = 100$ cm, and $R = 50$ cm, what will be the fringe visibility of the oscillations produced at location $y = 3$ mm when h is around 1 mm?

$$\gamma = \frac{e^{-ikyh/D}}{\mathcal{E}_0} \int_{-\infty}^{\infty} I(y') e^{-ikhy'/R} dy' \qquad \Bigg| \qquad \mathcal{E}_0 = \int_{-\infty}^{\infty} I(y')\, dy' = \int_{-d/2}^{d/2} I_0\, dy' = I_0 d$$

$$= \frac{e^{-iky h/D}}{\cancel{I_0} d} \underbrace{\int_{-d/2}^{d/2} \cancel{I_0}\, e^{-ikhy'/R} dy'}_{\left.\frac{e^{-ikhy'/R}}{-ikh/R}\right|_{y'=-d/2}^{d/2}}$$

$$= \frac{R}{ikh}\underbrace{\left(e^{+ikhd/2R} - e^{-ikhd/2R}\right)}_{2i\sin\frac{khd}{2R}}$$

$$\gamma = \frac{e^{-iky h/D}}{d}\, \frac{2R}{kh} \sin\frac{khd}{2R}$$

$$= e^{-2\pi i y h/\lambda D}\, \frac{\cancel{2} R\lambda}{dh\cancel{2}\pi} \sin\frac{\cancel{2}\pi h d}{\cancel{2}\lambda R}$$

$$\boxed{\gamma = \frac{R\lambda}{\pi d h}\, e^{-2\pi i y h/\lambda D} \sin\frac{\pi h d}{\lambda R}} \qquad = e^{-2\pi i y h/\lambda D} \operatorname{sinc}\frac{\pi h d}{\lambda R}$$

if you like

Problem 19, cont.

$$\text{Re}\,\gamma = \frac{R\lambda}{\pi d h} \cos \frac{2\pi y h}{\lambda D} \sin \frac{\pi h d}{\lambda R}$$

$$\langle I_{det} \rangle = 2 \langle I_{one\ slit} \rangle (1 + \text{Re}\,\gamma)$$

$$\boxed{\langle I_{det} \rangle = 2 \langle I_{one\ slit} \rangle \left(1 + \frac{R\lambda}{\pi d h} \cos \frac{2\pi y h}{\lambda D} \sin \frac{\pi h d}{\lambda R}\right)}$$

b. Fringe visibility $\rightarrow V = |\gamma| = \frac{R\lambda}{\pi d h} \sin \frac{\pi h d}{\lambda R}$

Plug in quantities given

$$V = \frac{(0.5)(700 \cdot 10^{-9})}{\pi (.001)(.001)} \sin\left(\frac{\pi (.001)(.001)}{(700 \cdot 10^{-9})(.5)}\right)$$

$$\boxed{V = .0483}$$