Final Exam Solutions

(20 pts) **Problem 1**: Multiple choice conceptual questions. Choose the best answer. **Circle** the letter of your top choice. **Square** the letter of your 2nd choice (for half credit, if your top choice is incorrect).

1.1. I'm riding my bike at 1×10^8 m/s (relative to the ground). I turn on my bike light. How fast do I see the light waves travel away? Use $c = 3\times10^8$ m/s.
   a. Less than 3×10^8 m/s
   b. 3×10^8 m/s
   c. More than 3×10^8 m/s

1.2. Same situation. How fast will a person on the ground see the light waves travel away?
   a. Less than 3×10^8 m/s
   b. 3×10^8 m/s
   c. More than 3×10^8 m/s

1.3. Which of the following is the best resolution of the twin paradox, as discussed in class? Robert = the twin that goes on the rocket; Henry is the twin that stays home. Robert goes on a trip to a distant star, then returns to Earth.
   a. Each twin will be older than the other, when they meet up again.
   b. Robert accelerates for part of his trip, so the time dilation equation does not apply in a simple fashion to him.
   c. Robert will be older than Henry, because he has a larger proper time. (and Robert is younger)
   d. The two twins end up the same age, because each ages the slowest during half the total trip.

1.4. Which of the following is the best resolution of the barn paradox, as discussed in class and analyzed for homework? (Lee is the one running with the ladder; Cathy is the one at rest relative to the barn.)
   a. Cathy sees the ladder fit entirely within the barn, but Lee does not.
   b. Lee sees the ladder fit entirely within the barn, but Cathy does not.
   c. Each of them sees the ladder fit entirely within the barn.

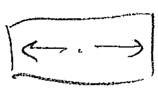
Lee sees the barn being shorter than the ladder.

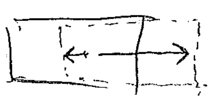
1.5. In the relativity context, which of the following would be an example of an "event"?
   a. A light beam hits a sensor.
   b. A light beam travels through space.
   c. Bill travels on a very fast train.
   d. Ted observes Bill traveling on a fast train.

1.6. Bill is eating breakfast on a train which moves at 2×10^8 m/s. Ted is sitting at a picnic table near the train tracks, also eating breakfast. Which pair of statements is correct (note, "slow motion" refers to the eating motions only, not to the overall train speed of 2×10^8 m/s, which is obviously anything but slow):
   a. To Bill, it looks like Ted is eating in fast motion. To Ted, it looks like Bill is eating in slow motion.
   b. To Bill, it looks like Ted is eating in fast motion. To Ted, it looks like Bill is eating in fast motion.
   c. To Bill, it looks like Ted is eating in slow motion. To Ted, it looks like Bill is eating in slow motion.
   d. To Bill, it looks like Ted is eating in slow motion. To Ted, it looks like Bill is eating in fast motion.

They both see the other one slowed down (time dilation)

1.7. While standing exactly in the middle of a train car, Bill shines flashlights at the right and left walls. (The train is moving to the right relative to the ground. That is, "right" = "forward", if you prefer that label.) He turns on the flashlights at exactly the same time (in his frame of reference). Ted is again watching the train from the ground nearby. Which pair of statements is correct:
   a. In Bill's frame, light hits the left wall first. In Ted's frame, light hits the walls simultaneously.
   b. In Bill's frame, light hits the left wall first. In Ted's frame, light hits the left wall first.
   c. In Bill's frame, light hits the left wall first. In Ted's frame, light hits the right wall first.
   d. In Bill's frame, light hits the right & left walls simultaneously. In Ted's frame, light hits the walls simultaneously.
   e. In Bill's frame, light hits the right & left walls simultaneously. In Ted's frame, light hits the left wall first.
   f. In Bill's frame, light hits the right & left walls simultaneously. In Ted's frame, light hits the right wall first.
   g. In Bill's frame, light hits the right wall first. In Ted's frame, light hits the walls simultaneously.
   h. In Bill's frame, light hits the right wall first. In Ted's frame, light hits the left wall first.
   i. In Bill's frame, light hits the right wall first. In Ted's frame, light hits the right wall first.


Bill's frame simultaneous


Ted's frame: left hits first

1.8. Three cubes of the same size and shape are put in water. They all sink. One is lead, one is steel and one is a dense wood (ironwood). $\rho_{lead} > \rho_{steel} > \rho_{ironwood}$. The buoyant force is greatest on the _______ cube

a. lead
b. steel
c. wood
d. same buoyant force

$B = \rho g V_{object}$ same

1.9. A boat is on a lake. If an anvil (that sinks) is pushed from the boat into the water, will the overall water level of the lake rise, fall or stay the same? (compared to when the anvil was in the boat)

a. Rise
b. Fall
c. Stay the same

See class discussion

1.10. Water flows from a little pipe into a big pipe with no friction or height change. The volume flow rate (m^3/s) in the little pipe will be _______ in the big pipe.

a. greater than
b. the same as
c. less than

as long as the water is "incompressible"

1.11. Same situation. The flow speed (m/s) in the little pipe will be ______ in the big pipe.

a. greater than
b. the same as
c. less than

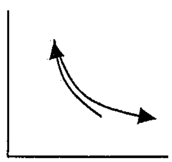
constant VFR means $A_1 v_1 = A_2 v_2$

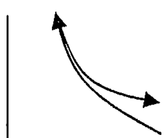
1.12. You have two jars of gas: helium and neon. Both have the same volume, same pressure, same temperature. Which jar contains the greatest number of gas molecules? (The mass of a neon molecule is greater than the mass of a helium molecule.)

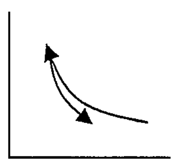
a. jar of helium
b. jar of neon
c. same number

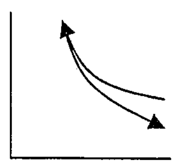
$PV = Nk_B T$ if P, V, T = same then N = same

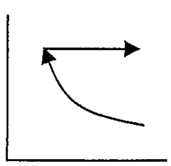
1.13. First, heat is removed from a gas while it is being compressed. This is done in such as way as to keep its temperature constant, while its pressure increases to 2.5× the original value. Next, more heat is removed from the gas, this time without letting its volume change, until it returns to its original pressure. Which of the following diagrams best represents the two processes on a standard P-V diagram?

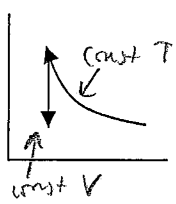
a)


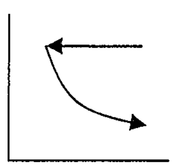
b)


c)


d)


e)


f)


g)


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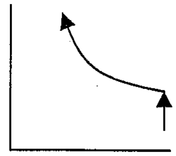


1.14. For waves on a string, the amplitude coefficients r and t must add up to 1.

a. True
b. False

as was seen in the problem on Exam 2

1.15. For waves on a string, the power/intensity coefficients R and T must add up to 1.

a. True
b. False

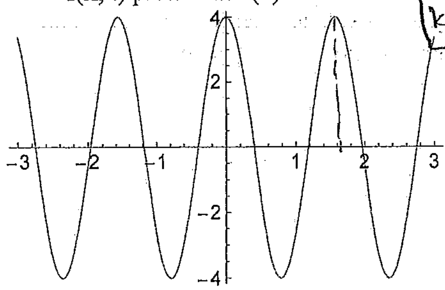
Energy is conserved

$\lambda \stackrel{\Delta}{=} 1.6 \rightarrow k = \frac{2\pi}{\lambda}$ $\boxed{k \approx 3.9}$

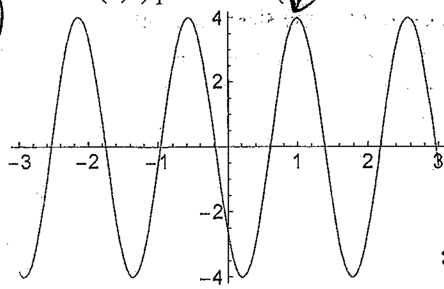
peak moves to $x=1$ in $\Delta t=1$ so $v = \frac{\Delta x}{\Delta t}$, $v = \frac{1}{1}$, $\boxed{v = 1\frac{m}{s}}$

amplitude $= 4$

f(x, t) plotted as f(x) for t=0



f(x, t) plotted as f(x) for t=1



since $v = \frac{\omega}{k}$, $\boxed{\omega = k}$

1.16. Which wave function f(x,t) is represented by the two graphs displayed? The left-hand graph is the wave function plotted vs. x, for t=0; the right hand graph is the wave function plotted for t=1.

a. $f(x,t) = 2\cos(3x - 5t)$
b. $f(x,t) = 2\cos(4x - 4t)$
c. $f(x,t) = 2\cos(5x - 3t)$
d. $f(x,t) = 4\cos(3x - 5t)$
e. $f(x,t) = 4\cos(4x - 4t)$ ← only one with amplitude = 4 and $\omega = k = 4$.
f. $f(x,t) = 4\cos(5x - 3t)$

1.17. For diffraction from a circular aperture, which is true of the central point (r=0)?

a. The intensity at y=0 is infinite
b. The intensity is finite, but typically much larger than the surrounding maxima
c. The intensity is finite, and about the same magnitude as the surrounding maxima
d. The intensity is finite, and substantially smaller than the surrounding maxima
e. The intensity is zero

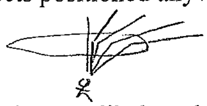
central max, surrounding max

circular diff. pattern

1.18. From a given location beneath the surface of water, it is possible to see objects positioned anywhere above the water.

a. True
b. False

Like fish "circle of light" problem



all of these rays reach person

1.19. If you look at sunlight reflected off of a lake near Brewster's angle, the light is more likely to be:

a. horizontally polarized
b. vertically polarized

Only the "S" polarization reflects, which (to you) will be horizontally polarized

1.20. In transparent glass, which goes faster: red (λ=650 nm) or violet light (λ=400 nm)?

a. red light
b. violet light
c. both travel at the same speed

smaller λ = more energy per photon = more interaction = slower speed.

(8 pts) **Problem 2.**

(a) Astronauts travel at 0.90c from Earth to a star which is 3 light years away, as measured by people on the Earth, and at rest with respect to the Earth. How long does it take them to reach the star as observed by people on Earth?

$$v = \frac{x}{t} \rightarrow t = \frac{x}{v}$$

$$t = \frac{3\ \cancel{c}\cdot yr}{.9\cancel{c}} = \boxed{3.33\ yr}$$

(b) How long does the trip take from the perspective of the astronauts? I.e., how much do the astronauts age on the trip?

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-.9^2}} = 2.294$$

time dilation: age less by factor of γ

$$\text{answer} = \frac{3.33\ yr}{2.294} = \boxed{1.45\ yr}$$

(c) How far apart are Earth and the star from the perspective of the astronauts as they travel?

length contraction: distance less by factor of γ

$$\text{answer} = \frac{3\ lt\ yr}{2.294} = \boxed{1.31\ l.y.}$$

(d) One their way, the astronauts fire a very small projectile ahead of them, at 0.90c (relative to the astronauts). How fast will people on the Earth see the projectile going?

$$\beta_{13} = \frac{\beta_{12} + \beta_{23}}{1 + \beta_{12}\beta_{13}} = \frac{.9 + .9}{1 + (.9)(.9)}$$

$$= .99448$$

$$\boxed{\text{answer} = .99448c}$$

(6 pts) **Problem 3**. A particle accelerator accelerates an electron up to 0.99*c*.

(a) How many joules of kinetic energy does the electron have?

$\gamma = \frac{1}{\sqrt{1-.99^2}} = 7.089$

$KE = (\gamma - 1) mc^2$

$= (7.089 - 1)(9.11 \times 10^{-31}\ kg)(3 \times 10^8 \frac{m}{s})^2$

$= \boxed{4.992 \times 10^{-13}\ J}$

(b) How many kg·m/s of momentum does the electron have? (kg·m/s is the SI unit of momentum; there's no special name for it)

$p = \gamma m v$

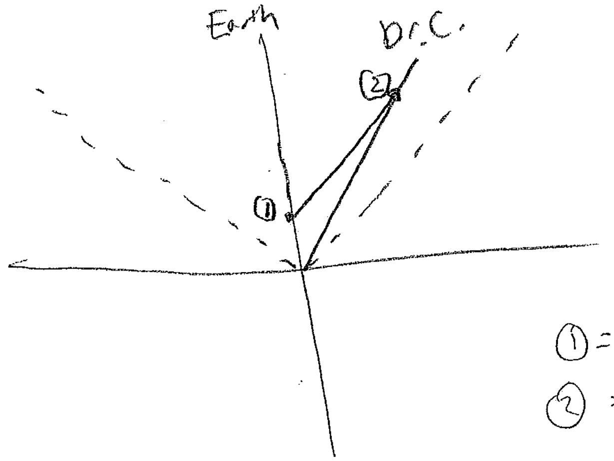
$= (7.089)(9.11 \times 10^{-31}\ kg)(.99 \times 3 \times 10^8 \frac{m}{s})$

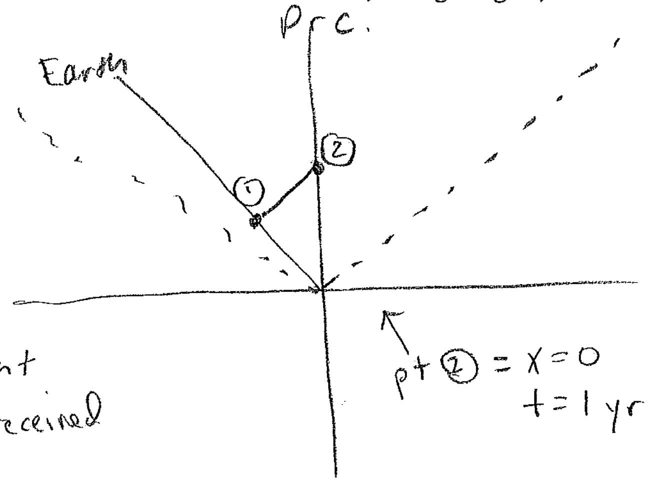
$= \boxed{1.918 \times 10^{-21}\ kg \frac{m}{s}}$

$\beta = .6,\ \gamma = 1.25,\ \gamma\beta = .75$

(12 pts) **Problem 4**. Dr. Colton is flying past Earth on a rocket going at a constant 0.6*c* in the positive x-direction. At some time later, the Earth sends a microwave signal to Dr. Colton (microwaves travel at *c*). Scientists on the Earth want to time it such that Dr. Colton receives the signal exactly one year (as measured by him) after he passes the Earth.

(a) Draw two fairly accurate space-time diagrams representing the situation, one from the perspective of the Earth, the other from the perspective of Dr. Colton. On each diagram label the light-cone, the world-lines of Earth and Dr. Colton, the light signal, and the "signal sent" & "signal received" events.





① = signal sent
② = signal received

pt ② = $x = 0$
$t = 1$ yr

(b) Where & when (as measured by the Earth) will Dr. Colton receive the signal?

$$\begin{pmatrix} x_2 \\ ct_2 \end{pmatrix}_{Earth} = \begin{pmatrix} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x_2 \\ ct_2 \end{pmatrix}_{Colton} = \begin{pmatrix} 1.25 & .75 \\ .75 & 1.25 \end{pmatrix} \begin{pmatrix} 0 \\ c \cdot 1\,yr \end{pmatrix}$$

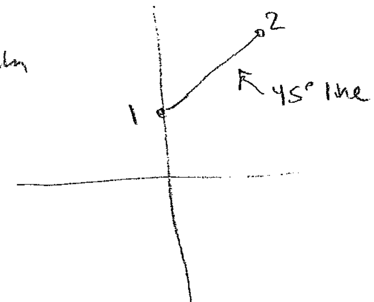
+, because it rotates to right

$$= \begin{pmatrix} c \times .75\,yr \\ c \times 1.25\,yr \end{pmatrix}$$

$\boxed{x_2 = .75\ l.y. \quad t_2 = 1.25\ yr}$ for Earth

(c) When (as measured by Earth) should the scientists on Earth send the signal?

Earth diagram



if ② is at $(.75, 1.25)$
then ① must be $(0, .5)$

$\boxed{t_1 = .5\,yr}$ for Earth

(d) Where/when (as measured by Dr. Colton) will the scientists on Earth send the signal?

$$\begin{pmatrix} x_1 \\ ct_1 \end{pmatrix}_{Colton} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ ct_1 \end{pmatrix}_{Earth} = \begin{pmatrix} 1.25 & -.75 \\ -.75 & 1.25 \end{pmatrix} \begin{pmatrix} 0 \\ .5c \end{pmatrix} = \begin{pmatrix} -.375 \\ +.625 \end{pmatrix}$$

negative, to rotate left

$\boxed{x_1 = -.375\ ly \quad t_1 = .625\ yr}$ for Colton

(8 pts) **Problem 5.** An aluminum rod is exactly 20 cm long at 20°C, and has a mass of 300 g. If 10.0 kJ of energy is added to the rod by heat, what will be the change in length of the rod? (Hint: the heat causes a change in temperature, which in turn causes a change in length.)

heat

Change in temp:

$$Q = mc\,\Delta T$$

$$10\times10^{3}\,J = (.3\,kg)\left(900\,\frac{J}{kg\,°C}\right)\Delta T$$

$$\underline{\underline{\Delta T = 37.04\,°C}}$$

change in length:

$$\Delta L = \alpha L \Delta T$$

$$= \left(24\times10^{-6}\,\frac{1}{°C}\right)(.2\,m)(37.04°C)$$

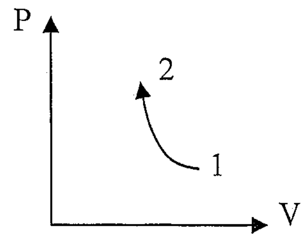
$$= \boxed{1.78\times10^{-4}\,m}$$

or .0178 cm

or 178 μm

$\gamma = 7/5$

(7 pts) **Problem 6.** A diatomic gas (0.2 moles) is compressed adiabatically from 200 kPa to 400 kPa. The initial volume and temperature of the gas are 0.002493 m3 and 300K, respectively. Find the final volume and temperature of the gas.



$$P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$$

$$V_2 = \left(\frac{P_1}{P_2}\right)^{1/\gamma} \times V_1$$

$$V_2 = \left(\frac{200}{400}\right)^{5/7} \times (.002493\ m^3)$$

$$V_2 = \boxed{.001520\ m^3}$$

Ideal gas:

$$P_2 V_2 = nRT_2$$

$$T_2 = \frac{P_2 V_2}{nR}$$

$$= \frac{(400 \times 10^3\ Pa)(.001520\ m^3)}{(.2\ mol)(8.31\ \frac{J}{mol\,K})}$$

$$= \boxed{365.7\ K}$$

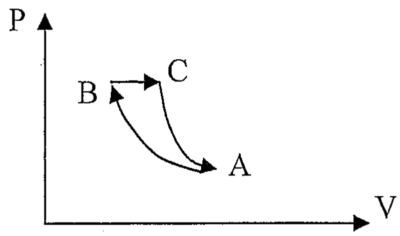
V_2 = ____________ m^3

T_2 = ____________ K

$\rightarrow C_p = \frac{7}{2}R$

(10 pts) **Problem 7.** An engine using 0.08 moles of a diatomic ideal gas is driven by this cycle: starting from state A, the gas is compressed isothermally until it reaches state B. Then, ~~the gas~~ is heated at constant pressure until it reaches state C. Finally, the gas is expanded adiabatically back to the original state. The pressures, volumes, and temperature of all three states are given in the table.

	P (kPa)	V (m^3)	T (K)
A	100	0.001994	300
B	400	0.0004986	300
C	400	0.0007409	445.8



First Law: $\Delta U = Q_{added} + W_{on}$

(a) Find the heat added to the gas during each of the three legs.

A-B

Isothermal: $\Delta U = 0$

So $Q = -W_{on} = +W_{by} = nRT \ln \frac{V_B}{V_A}$

(work $= \int p\,dV$, if you forgot / didn't have written down the work for isotherma!)

$Q = (.08)(8.31)(300) \ln \left(\frac{.0004986}{.001994}\right)$

$\boxed{Q_{AB} = -276.4 \text{ J}}$

B-C

Isobaric: $\Delta U = n C_p \Delta T$

$= (.08)\left(\frac{7}{2} \times 8.31\right)(445.8 - 300)$

$\boxed{Q_{BC} = +339.2 \text{ J}}$

C-A

Adiabatic: $\boxed{Q_{CA} = 0}$

(b) How much net work is done by the gas each cycle?

$W_{net} = Q_h - Q_c = 339.2 - 276.4$

$= \boxed{62.8 \text{ J}}$

(c) What is the efficiency of the engine?

$e = \frac{W_{net}}{Q_h} = \frac{62.8}{339.2} = \boxed{18.51\%}$

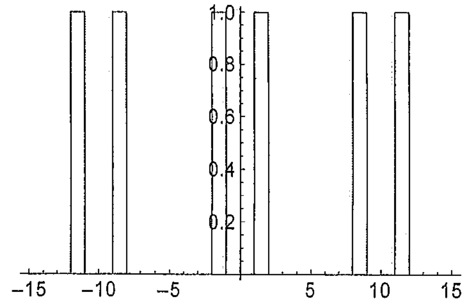
(d) What is the maximum theoretical efficiency for an engine operating between the same minimum and maximum temperatures?

$e_{max} = 1 - \frac{T_c}{T_h} = 1 - \frac{300}{445.8} = \boxed{32.7\%}$

(8 pts) **Problem 8.** The function f(x), graphed on the right, is defined as follows:

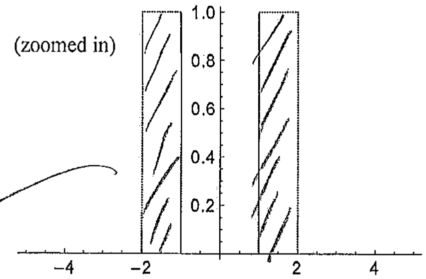
$$f(x)=\begin{cases}1, & \text{for x between -2 and -1}\\ 1, & \text{for x between +1 and +2}\\ 0, & \text{otherwise}\end{cases}$$

(repeated with a period of L=10)

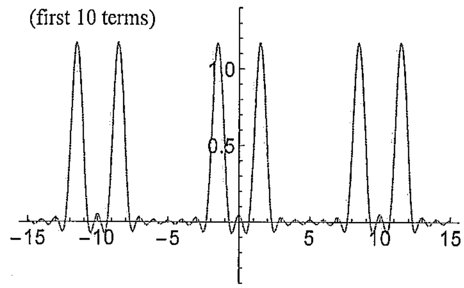


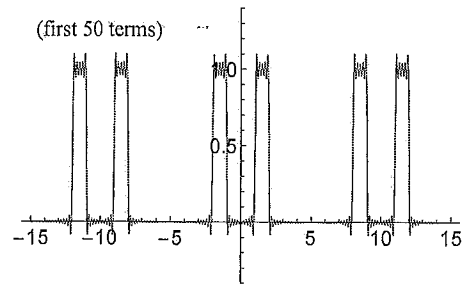
I worked out the Fourier coefficients for this function, and found the following:

$$f(x)=0.2+\sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\sin\left(\frac{4n\pi}{10}\right)-\sin\left(\frac{2n\pi}{10}\right)\right)\cos\left(\frac{2n\pi x}{10}\right)$$



Plots of f(x) for the first 10 terms, and for the first 50 terms, are also shown on the right.





(a) Why are there no sin(nπx/L) terms in the series?

$2n\pi x/L \rightarrow$ typo

function is even, so no sine terms required

(b) Prove that the constant term in my expression is correct.

a_0 = ave value

$= \frac{1}{10} \times$ (area in one period)

$= \frac{1}{10} \times (2)$

$a_0 = .2$ ✓

(c) Prove that the cosine coefficients in my expression are correct. Hint: integrate from -5 to 5 instead of from 0 to 10.

$$a_n = \frac{2}{L}\int f(x)\cos\frac{2n\pi x}{L}\,dx$$

$$= \frac{2}{10}\left[\int_{-2}^{-1}(1)\cos\frac{2n\pi x}{10}\,dx + \int_{1}^{2}(1)\cos\frac{2n\pi x}{10}\,dx\right]$$

$$= \frac{2}{10}\left(\left.\frac{\sin\frac{2n\pi x}{10}}{\frac{2n\pi}{10}}\right|_{x=-2}^{-1} + \left.\frac{\sin\frac{2n\pi x}{10}}{\frac{2n\pi}{10}}\right|_{1}^{2}\right)$$

$$= \frac{1}{n\pi}\left[\sin\frac{4n\pi}{10} - \sin\frac{2n\pi}{10} + \sin\frac{-2n\pi}{10} - \sin\frac{-4n\pi}{10}\right]$$

$$\boxed{a_n = \frac{2}{n\pi}\left[\sin\frac{4n\pi}{10} - \sin\frac{2n\pi}{10}\right]}$$ ✓

(9 pts) **Problem 9**. My trumpet, for a particular configuration of values, acts very similarly to an "open-open" pipe with a length of 1.0 meters. Suppose I have been playing various sweet melodies for several minutes; my breath has warmed up the air inside the trumpet so that it has a speed of sound of 350 m/s instead of the standard 343 m/s.

$f_n = n \times f_1$

(a) If I play the fifth harmonic, what frequency will it have?

$$f_5 = 5f_1 = 5\left(\frac{v}{\lambda_1}\right) = 5\left(\frac{v}{2L}\right)$$

$$f_5 = \frac{5 \times (350 \frac{m}{s})}{2 \times (1.0 m)}$$

$$\boxed{f_5 = 875 \text{ Hz}}$$

(b) On an equal temperament scale referenced to A (above middle C) = 440 Hz, what note is that frequency closest to?

f_5 is very close to 880 Hz, which would be an **A one octave higher** than the reference A.

For more details:

$$\left.\begin{aligned} A\# \text{ or } Bb &= 880 \times \sqrt[12]{2} = 932.3 \text{ Hz} \\ A &= 880 \text{ Hz} \\ Ab \text{ or } G\# &= \frac{880}{\sqrt[12]{2}} = 830.6 \text{ Hz} \end{aligned}\right\} \text{clearly it's closest to an A}$$

(c) A fellow trumpet player tries to play the same note on his trumpet (an exact duplicate of mine). However, because he has not warmed up his instrument, the speed of sound is 343 m/s in his trumpet. How many beats occur between the tones made by his trumpet and mine? Both of us are playing the fifth harmonics. (Note: I believe this is in fact the dominant source of brass instruments playing out of tune when not warmed up first. You can completely neglect the thermal expansion/contraction of the metal itself.)

$$f_5 = 5 \times \frac{343}{2(1)} = \underline{\underline{857.5 \text{ Hz}}}$$

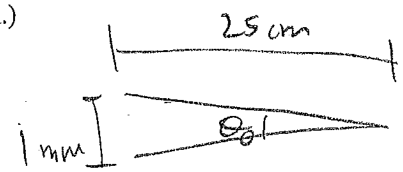
$$f_{beat} = |f_1 - f_2|$$

$$= (875 - 857.5) \text{ Hz}$$

$$\boxed{f_{beat} = 17.5 \text{ Hz}}$$

(8 pts) **Problem 10**. You are trying to look at an ant, height $h = 1$ mm.

(a) What is the maximum viewing angle you can use to look at the ant, without using any magnifying lenses? (Your near point is 25 cm.)



$$\theta_0 = \frac{1\text{ mm}}{250\text{ mm}} = \boxed{.004 \text{ radians}}$$

(b) You now introduce a magnifying glass, f = +5 cm. You place the magnifying glass into your line of sight, ~~then you~~ adjust both the magnifying glass and your head until you have a clear view of the image of the ant, which has formed 40 cm behind the lens. How far is the (actual) ant from the lens?

$q = -40$

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$p = \left(\frac{1}{f} - \frac{1}{q}\right)^{-1} = \left(\frac{1}{5} - \frac{1}{-40}\right)^{-1}$$

$$\boxed{p = 4.44 \text{ cm}}$$

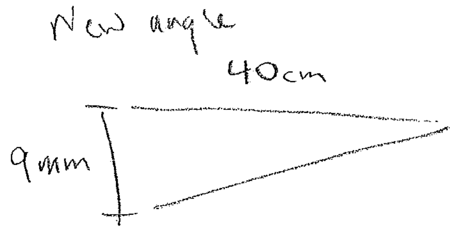
(c) What is the height of the image?

$$M = -\frac{q}{p} = -\left(\frac{-40}{4.44}\right) = \underline{+9}$$

$$h_{image} = 1\text{ mm} \times (+9)$$

$$= \boxed{9 \text{ mm}}$$

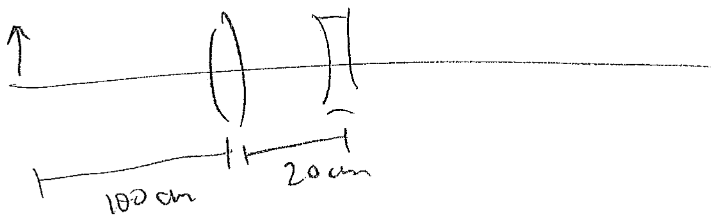
(d) What angular magnification have you obtained via the magnifying glass?



$$\theta = \frac{9\text{ mm}}{400\text{ mm}} = \underline{.0225 \text{ radians}}$$

$$m = \frac{\theta}{\theta_0} = \frac{.0225}{.004} = \boxed{5.625}$$

(4 pts, no partial credit) **Problem 11.** An object is placed 100 cm to the left of lens 1 (converging, f = +70 cm). Lens 1 is placed 20 cm to the left of lens 2 (diverging, f = –40 cm). How far (in magnitude) from lens 2 will the final image be formed? Will the image be to the left or the right of lens 2? Will it be real or virtual? You do not have to provide any ray diagrams for this problem.



lens 1 : $q = \left(\frac{1}{f} - \frac{1}{p}\right)^{-1} = \left(\frac{1}{70} - \frac{1}{100}\right)^{-1} = 233.3 \text{ cm}$

lens 2 : relative to lens 2, that distance is 213.3 cm to the right

so $p_2 = -213.3$

$q_2 = \left(\frac{1}{f_2} - \frac{1}{p_2}\right)^{-1} = \left(\frac{1}{-40} - \frac{1}{-213.3}\right)^{-1}$

$\boxed{q_2 = -49.23}$

$|q_2| = 49.23$
and it's to the left of lens 2

Because it's to the left, it must be virtual, not real

You must get all three of these answers right to get any credit for this problem.

|q| = 49.23 cm

Final image will be to the left (left/right) of lens 2

real vs. virtual: virtual