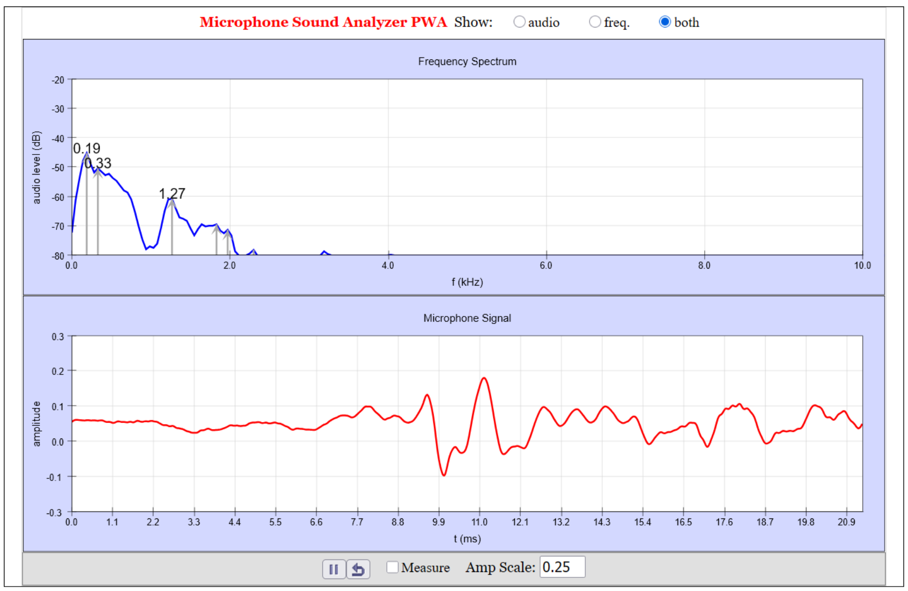
Fourier Analysis

by Dr. Colton, Physics 123 (last updated: Winter 2026)

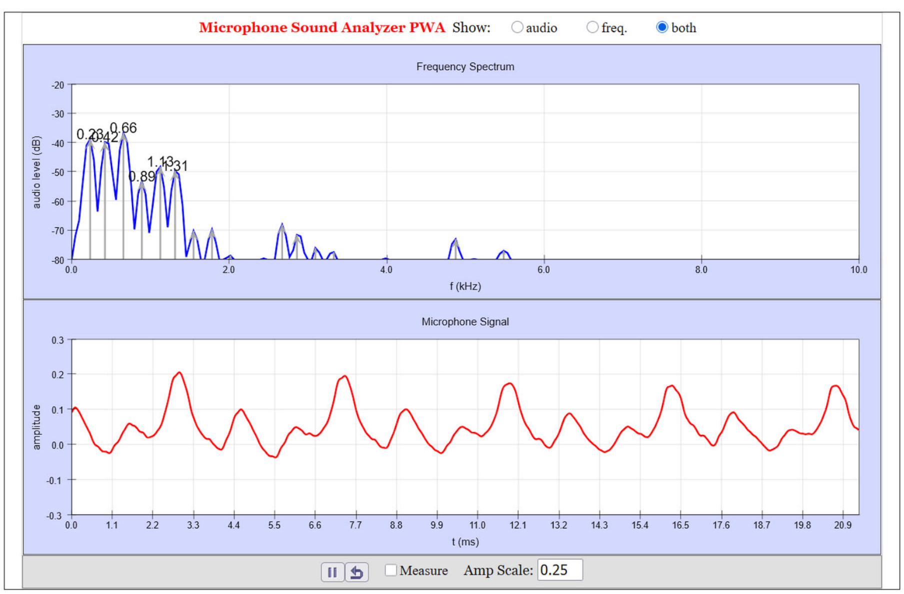
Introduction/Background

"Fourier analysis" involves the concept that complicated waves can be analyzed in terms of their component frequencies. This is familiar to anyone who has used an equalizer to boost the bass or tremble of their music: bass = low frequencies, treble = high frequencies. So somehow a complicated waveform with perhaps no noticeable regularity can be thought of as being comprised of a mixture of waves at various frequencies.

Using tools such as this one: https://www.compadre.org/osp/pwa/soundanalyzer, you can see the frequency (or "spectral") components of the sound wave. Here is me talking, with the frequency components plotted above a section of the sound wave in time:



And here is me singing; notice how regular the frequency components are. That is a hallmark of music.



Fourier Series

To reiterate: if the wave is more random, the frequency components are also random. But if the wave is more regular (more *periodic*), then the frequency components are more well defined. And if the wave were to be perfectly periodic, then the frequency components become perfectly sharp, and multiples of the "fundamental frequency:

$$\omega_0 = \frac{2\pi}{T} \quad (1)$$

Specifically, Fourier's theorem says that any (reasonably well-behaved) periodic function can be written as a sum of sines and cosines which are integer multiples of ω_0. That is called the "Fourier series" representation of the function.

Summary of formulas

If you write $f(t)$ as a series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \quad (2)$$

Then the "Fourier coefficients" a_n and b_n are calculated by the following.

- a_0 represents the average value of the function, and is calculated by:

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \quad (3)$$

- The other a_n coefficients represent the degree to which $f(t)$ contains cosines at frequencies $n\omega_0$:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\, dt \quad (4)$$

- The b_n coefficients represent the degree to which $f(t)$ contains sines at frequencies $n\omega_0$:

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\, dt \quad (5)$$

All integrals can be done from $-T/2$ to $-T/2$, or over any full period, instead of from 0 to T.

Symmetry notes:

- If the function $f(t)$ is even, only the cosine terms will be present, and possibly also a_0. The b_n coefficients will all be zero.
- If the function $f(t)$ is odd, only the sine terms will be present. The a_n coefficients will all be zero.

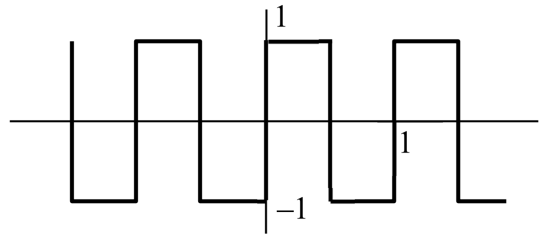
Additional notes:

The same equations hold true for periodic functions of x rather than t. In that case the spatial period L is used instead of the temporal period T, and the symbol k (rads/meter) is used in place of ω (rads/second).

Because sines and cosines relate to complex exponentials through Euler's identity $e^{ix} = \cos x + i \sin x$, (2), (4), and (5) are often written in terms of exponential functions. But that's beyond the scope of this course.

Example: Square wave (infinite, repeating)

As an example, consider this function, plotted as shown:



$$f(t) = \begin{cases} -1, & \text{for } -\frac{1}{2} < t < 0 \\ 1, & \text{for } 0 < t < \frac{1}{2} \end{cases}$$

(repeated with a period of 1)

In this case, the period is 1, so the fundamental frequency is $\omega_0 = 2\pi$. All of the terms in the series will have angular frequencies that are multiples of 2π. The average value of the function is 0, so $a_0 = 0$. Additionally, the function is odd, so the expansion will contain only sine terms. A formula for the coefficients of the sine terms for this specific case can be obtained by performing the b_n integral:

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt \\ &= \frac{2}{1}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\sin(2\pi n t)\,dt \quad \text{(do this integral piecewise)} \\ &= 2\left(\int_{-\frac{1}{2}}^{0}(-1)\sin(2\pi n t)\,dt + \int_0^{\frac{1}{2}}(1)\sin(2\pi n t)\,dt\right) \\ &= 2\left(\frac{\cos(2\pi n t)}{2\pi n}\bigg|_{-\frac{1}{2}}^{0} - \frac{\cos(2\pi n t)}{2\pi n}\bigg|_{0}^{\frac{1}{2}}\right) \\ &= \frac{4(1-\cos(\pi n))}{2\pi n} \\ &= \frac{2(1-\cos(\pi n))}{\pi n} \end{aligned}$$

That happens to be equal to 0 for even values of n, and equal to $\frac{4}{\pi n}$ for odd values of n, so with the terms explicitly written out, the series looks like this:

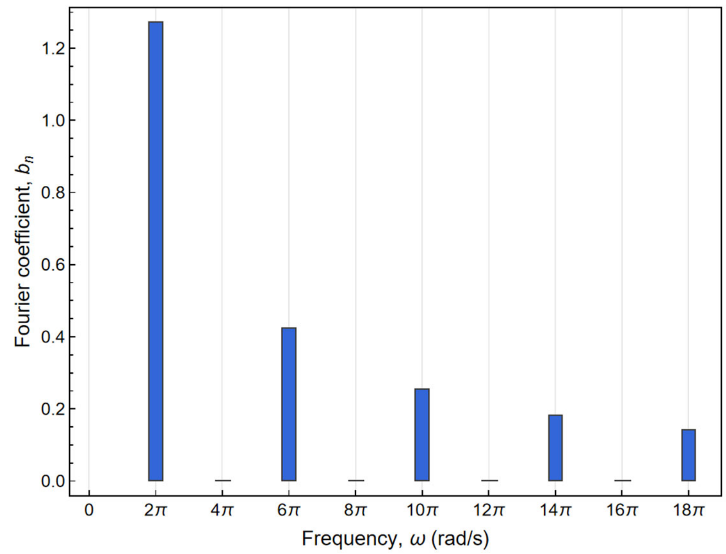
$$f(t) = \frac{4}{\pi}\sin(2\pi t) + \frac{4}{3\pi}\sin(6\pi t) + \frac{4}{5\pi}\sin(10\pi t) + \frac{4}{7\pi}\sin(14\pi t) + \cdots$$

The set of Fourier coefficients can be thought of as a list like this:

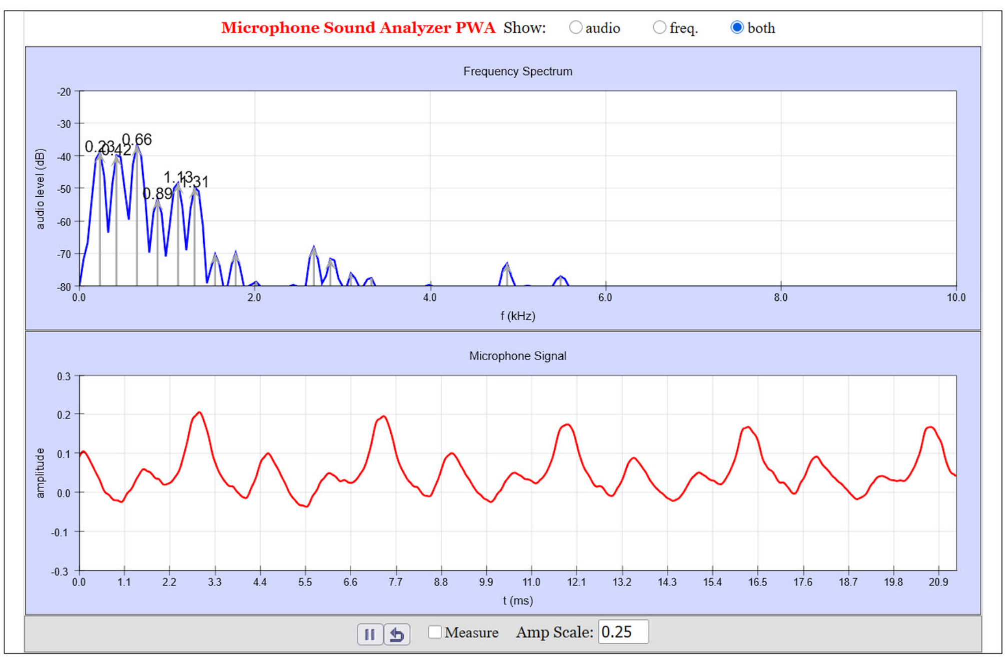
$$\left\{\frac{4}{\pi}, 0, \frac{4}{3\pi}, 0, \frac{4}{5\pi}, 0, \frac{4}{7\pi} \ldots\right\}$$

The Fourier coefficients can be thought of as a table of ordered pairs, or even as a plot:

$\omega = n\omega_0$	b_n
2π	$4/\pi$
4π	0
6π	$4/(3\pi)$
8π	0
10π	$4/(5\pi)$
12π	0
14π	$4/(7\pi)$
16π	0
18π	$4/(9\pi)$
…	…



When viewed as a plot, the set of coefficients form a function called the "Fourier transform" of $f(t)$, and is essentially the same sort of thing as what was plotted as the upper part of the lower graph on page 1, for singing:



The main differences are

- The shape of the periodic wave is different, so there are different allowed frequencies and different amplitudes for each of the frequencies.
- The plot from page 1 is a plot of *sound level*. Thus it involves intensity, so the b_n coefficients are squared (remember power of a wave is proportional to amplitude squared), and then converted to dB.
- The peaks on the plot from page 1 are not infinitely narrow like they are in the Fourier coefficient plot; that's because real world waves are never EXACTLY periodic. The closer to truly periodic a wave is, the sharper the peaks are in the Fourier transform.

Derivation of the formulas

Orthogonality. Equations (3)-(5) result from the *orthogonality* of sine and cosine functions. Functions are said to be orthogonal if when multiplied together, they integrate to 0. Here we are talking about sine and cosine functions which have frequencies that are multiples of ω_0. Sine and cosine functions with frequencies $n\omega_0$ and $m\omega_0$ are always orthogonal with each other, meaning that if you multiply $\sin(n\omega_0 t)$ by $\cos(m\omega_0 t)$ and integrate from 0 to T, then you will get a result of 0 for all combinations of integers n and m. But more than that, sine functions *with different integer values* are orthogonal to other sine functions, and cosine functions *with different integer values* are also orthogonal to each other.

For example, using $T = 1$ and $\omega_0 = 2\pi$ again for simplicity, you can check that the following are all true:

$$\int_0^1 \sin(4 \times 2\pi t) \cos(4 \times 2\pi t)\, dt = 0$$

$$\int_0^1 \sin(3 \times 2\pi t) \sin(8 \times 2\pi t)\, dt = 0$$

$$\int_0^1 \cos(1 \times 2\pi t) \cos(2 \times 2\pi t)\, dt = 0$$

But for the latter two examples, if you use the same integer, then the integral gives you $T/2$:

$$\int_0^1 \sin(3 \times 2\pi t) \sin(3 \times 2\pi t)\, dt = \frac{1}{2}$$

$$\int_0^1 \cos(1 \times 2\pi t) \cos(1 \times 2\pi t)\, dt = \frac{1}{2}$$

In general the orthogonality rules can be summarized like this:

$$\int_0^T \sin(n\omega_0 t) \cos(m\omega_0 t)\, dt = 0 \text{ for all } m, n \tag{6}$$

$$\int_0^T \sin(n\omega_0 t) \sin(m\omega_0 t)\, dt = \begin{cases} 0, & m \neq n \\ \frac{T}{2}, & m = n \end{cases} \tag{7}$$

$$\int_0^T \cos(n\omega_0 t) \cos(m\omega_0 t)\, dt = \begin{cases} 0, & m \neq n \\ \frac{T}{2}, & m = n \end{cases} \tag{8}$$

Derivation using orthogonality. Here's how the orthogonality integrals can be used to derive (3)-(5). I'll just do this for the b_n coefficients, but the derivation for the a_n coefficients is very similar.

Start with the definition of the series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Then multiply both sides by $\sin(m\omega_0 t)$, and integrate from 0 to T. We can interchange the order of summation vs integration for the two summation terms, because they act on different variables (summation over n vs. integral over t). Thus the equation becomes:

$$\int_0^T f(t) \sin(m\omega_0 t) dt = \int_0^T a_0 \sin(m\omega_0 t) dt + \sum_{n=1}^{\infty} \int_0^T a_n \cos(n\omega_0 t) \sin(m\omega_0 t) dt + \sum_{n=1}^{\infty} \int_0^T b_n \sin(n\omega_0 t) \sin(m\omega_0 t) dt$$

Consider the three terms on the right hand side of the equation, one at a time:

The first term integrates to 0 because a_0 can be pulled out of the integral and the resulting sine function integrates to zero for all value of m. That's because the interval 0 to T contains an integer number of periods of $\sin(m\omega_0 t)$, so there's an equal amount of positive vs negative area.

The second term ALSO integrates to zero, for all values of n and m, because sine and cosine functions are orthogonal.

The third term is an infinite summation. Nearly all of the terms in the summation will be zero, due to the orthogonality of cosine functions with themselves. The only exception is the particular term in the summation where n and m happen to be equal. THIS IS THE CRITICAL STEP, MAKE SURE IT MAKES SENSE TO YOU. In that single case, the integral gives $T/2$ by (7).

Thus the equation becomes

$$\int_0^T f(t)\sin(m\omega_0 t)dt = b_m \frac{T}{2}.$$

This means (renaming m to n because it's just an arbitrary index letter).

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt ,$$

which is (5).