

What You Should Already Know About Circuits

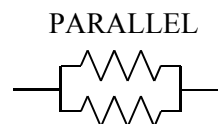
by Dr. Colton (Fall 2016)

We're only going to spend a brief amount of time reviewing things that you should have learned in Physics 220. Here's a run-down of some of the things that you should already know. (*Yes, you were supposed to remember these things!*) Review these items and if there are things you do not recall, review the relevant sections of your old Phys 220 textbook. (*Yes, you were supposed to keep your old textbook!*)

1. Series and Parallel Resistors

The voltage (or more specifically, voltage difference of one side relative to the other) of a resistor is given by Ohm's law: $\Delta V_R = IR$.

A combination of 2 resistors can be labeled either as "series" or "parallel", depending on how they are connected.



Series resistors share the same current; parallel resistors share the same potential difference (ΔV). The equivalent resistance for two series and parallel resistors is given by these formulas:

$$\text{SERIES: } R_{eq} = R_1 + R_2 \quad (1)$$

$$\text{PARALLEL: } \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{or} \quad R_{eq} = (1/R_1 + 1/R_2)^{-1} \quad (2)$$

I will often use the symbol "//" to indicate a parallel resistance combination:

$$R_1 // R_2 = (1/R_1 + 1/R_2)^{-1} \quad (\text{definition of "//" symbol}) \quad (3)$$

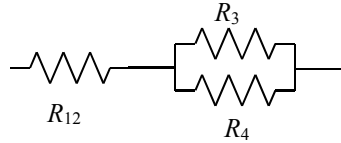
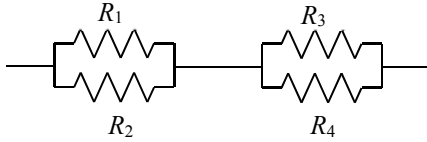
It can easily be proven that the series and parallel formulas extend as you would expect to any number of resistors, such as $R_{eq} = R_1 + R_2 + R_3 + \dots$ for series resistors and $R_{eq} = (1/R_1 + 1/R_2 + 1/R_3 + \dots)^{-1}$ for parallel resistors.

2. How to solve circuits with one battery and a mess of resistors

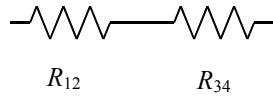
The series and parallel formulas are extremely powerful because networks of resistors can often be decomposed into combinations of resistors that are in series and parallel with each other.

The way to tackle networks with many resistors is to slowly simplify the circuit by substituting in equivalent resistors for combinations of series and/or parallel resistors. Then use repeated applications of Ohm's law to solve for the quantity you are looking for. For example, using " R_{12} " to represent the equivalent resistance of R_1 and R_2 , etc., here's how I would go about figuring out the equivalent resistance of this network:

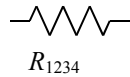
Sample Problem 1: What's the equivalent resistance of the 4 resistors?



R_{12} is given by parallel resistance of R_1 and R_2 , $R_{12} = (1/R_1 + 1/R_2)^{-1}$



R_{34} is given by parallel resistance of R_3 and R_4 , $R_{34} = (1/R_3 + 1/R_4)^{-1}$

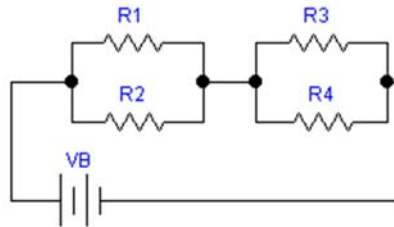


R_{1234} is given by series resistance of R_{12} and R_{34} , $R_{1234} = (1/R_1 + 1/R_2)^{-1} + (1/R_3 + 1/R_4)^{-1}$

Shortcut notation:

$$R_{1234} = R_1 // R_2 + R_3 // R_4$$

Now suppose the four resistors were hooked up to a battery with voltage V_B such that current flows through the circuit from left to right. The total current flowing from the battery is $I_{tot} = V_B/R_{1234}$. You could then use that information to determine currents through other parts of the circuit.



Sample Problem 2: What is the current through R_3 ?

One way to solve this problem would be to proceed as follows:

1. First, break down the whole circuit in terms of series and parallel equivalent resistances and solve for the total current flowing from the battery (as already done).
2. Treating R_1 and R_2 as a single resistor through which I_{tot} is flowing, Ohm's Law says that the ΔV for R_{12} is equal to $I_{tot}R_{12}$. Thus the voltage just to the right of R_1 and R_2 (and just to the left of R_3 and R_4) must be: $V_B - I_{tot}R_{12}$.
3. Because the right hand side of R_3 is at 0 V, we can apply Ohm's law to find the ΔV across R_3 :

$$I_{R_3} = \frac{\Delta V_{R_3}}{R_3} = \frac{(V_B - I_{tot}R_{12}) - 0}{R_3}$$

where I_{tot} and R_{12} are as solved for above. Problem solved!

To reiterate, the technique of using Ohm's law for various parts of the circuit, together with the series and parallel formulas, can be used to solve a number of circuit problems as long as you only have a bunch of resistors together with a single battery.

3. Kirchhoff's Laws

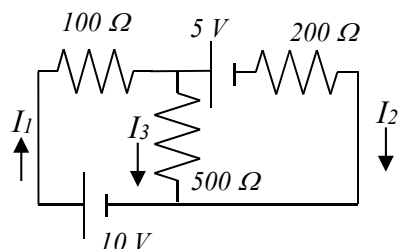
Kirchhoff's two circuit laws are this: (KCL = Kirchhoff's current law and KVL = Kirchhoff's voltage law)

$$\text{KCL: } \sum_{\text{junction}} I = 0 \quad (6)$$

$$\text{KVL: } \sum_{\text{closed loop}} \Delta V = 0 \quad (7)$$

Kirchhoff's laws can frequently be used to solve circuit problems in situations that are too complicated for series/parallel analysis such as done above.

Sample Problem 3: What is the current through the 500 Ω resistor?



For problems like this, the typical method taught in physics books is to use Kirchhoff's laws to obtain simultaneous equations using the currents I_1 , I_2 , and I_3 as unknowns. (Engineers learn a similar but slightly quicker method called the "mesh current" method; we'll discuss that in class.)

The "junction rule" and "loop rule" equations are:

$$\begin{aligned} I_1 &= I_2 + I_3 && \text{(junction equation)} \\ -100 I_1 - 500 I_3 + 10 &= 0 && \text{(left hand loop)} \\ -100 I_1 - 5 - 200 I_2 + 10 &= 0 && \text{(outer loop)} \end{aligned}$$

You could also do a loop rule equation for the right-hand loop, but it would be redundant with the left and outer loop equations (it's the outer loop equation minus the left hand loop equation, to be precise).

Once you have enough linearly independent simultaneous equations, solve them via your favorite method. My own favorite method is to turn it into a matrix equation and use the matrix inverse to solve for the "vector" of currents:

$$\begin{pmatrix} 1 & -1 & -1 \\ -100 & 0 & -500 \\ -100 & -200 & 0 \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ -5 \end{pmatrix} \quad \text{matrix equivalent of the three equations}$$

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ -100 & 0 & -500 \\ -100 & -200 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ -10 \\ -5 \end{pmatrix} \quad \text{multiply by the matrix inverse on both sides}$$

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0.026471... \\ 0.011765... \\ 0.014706... \end{pmatrix} \quad \text{the answers}$$

Here's the code I'd use in *Mathematica* for that type of thing:

```
m = {{1, -1, -1}, {-100, 0, -500}, {-100, -200, 0}};
v = {0, -10, -5};
currents = Inverse[m].v // MatrixForm // N
```

If you have a very large number of simultaneous equations there are more efficient ways to solve them than calculating the matrix inverse, but for anything we'd get in this class this method will work just fine.

4. Series and Parallel Capacitors

The voltage of a capacitor depends on its charge: $\Delta V_C = Q/C$.

When capacitors are connected together, they can be said to be in series or parallel just like resistors. However, the formulas for the "equivalent capacitance" of the series/parallel combination are the reverse of the resistor formulas.

$$\text{SERIES:} \quad \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{or} \quad C_{eq} = (1/C_1 + 1/C_2)^{-1} \quad (4)$$

$$\text{PARALLEL:} \quad C_{eq} = C_1 + C_2 \quad (5)$$

Circuits having just one battery and a network of capacitors can often be analyzed using only the series and parallel formulas. This is very analogous to the resistor networks discussed above.

Due to the opposite nature of the parallel capacitors vs. parallel resistors formulas, I never use the symbol "/" to indicate parallel capacitances. (I do, however, use it to indicate parallel *impedances* of capacitors, which we'll talk about in class.)

5. Time dependence of simple RC circuits

When a capacitor is discharged through a resistor, the charge on the capacitor (and hence the voltage of the capacitor) decays exponentially.

$$Q(t) = Q_0 e^{-t/\tau} \quad \text{discharging capacitor} \quad (8)$$

$$\Delta V_C(t) = \Delta V_0 e^{-t/\tau} \quad \text{discharging capacitor} \quad (9)$$

where Q_0 and $\Delta V_0 (= Q_0/C)$ are the capacitor's initial charge and voltage respectively, and the "time constant" τ is given by:

$$\tau = RC \quad (10)$$

After a time equal to τ , the charge on the capacitor is equal to 36.8% ($\approx e^{-1}$) of its initial value.

When an initially uncharged capacitor is connected to a battery, similar equations hold:

$$Q(t) = Q_{max}(1 - e^{-t/\tau}) \quad \text{charging capacitor} \quad (11)$$

$$\Delta V_C(t) = \Delta V_{max}(1 - e^{-t/\tau}) \quad \text{charging capacitor} \quad (12)$$

where Q_{max} and $\Delta V_{max} (= Q_{max}/C)$ represent the final charge and voltage of the capacitor, respectively, and $\tau = RC$ is the same time constant as for discharging capacitors. After a time equal to τ , the charge on the capacitor is equal to 63.2% ($\approx 1 - e^{-1}$) of its maximum value. The reason for this nonlinear charging is that the more "full" the capacitor is, the harder it gets to add more charge.

Proof of the time dependent RC equations. Eqs. (8) and (11) can be proved via KVL and the "guess and check" method of solving differential equations, and Eqs. (9) and (12) follow by the voltage of a capacitor $\Delta V_C = Q/C$. For example, for a charging series RC circuit, KVL yields:

$$V_{battery} - IR - \frac{Q}{C} = 0$$

which by using $I = dQ/dt$ can be turned into this:

$$V_{battery} - \frac{RdQ}{dt} - \frac{Q}{C} = 0$$

If we guess $Q(t) = Q_{max}(1 - e^{-t/\tau})$, we find

$$\begin{aligned} V_{battery} - R \left(+ \frac{Q_{max}}{\tau} e^{-t/\tau} \right) - \frac{Q_{max}(1 - e^{-t/\tau})}{C} &= 0 \\ \left(V_{battery} - \frac{Q_{max}}{C} \right) + Q_{max} \left(-\frac{R}{\tau} + \frac{1}{C} \right) e^{-t/\tau} &= 0 \end{aligned}$$

That equation is true if $Q_{max} = V_{battery}C$ and $\tau = RC$. Moreover our guess satisfies the initial condition of $Q(t = 0) = 0$. Therefore by the uniqueness theorem of first order differential equations, Eq. (11) is *the* solution for this situation. (Eq. 8 is proved similarly for a discharging capacitor.)

6. Inductors

The voltage of an inductor depends on the rate of change of its current: $\Delta V_L = -L dl/dt$. The negative sign follows from Lenz's law. Inductors can be thought of as current-storing devices, similar to how capacitors are charge-storing devices.

Inductors use the same series and parallel formulas as resistors, with the caveat that if two inductors are placed too near each other, their mutual inductance must also be taken into account.

7. Time dependence of simple RL circuits

When the current stored in an inductor is discharged through a resistor, the current and the voltage of the inductor both decay exponentially.

$$I(t) = I_0 e^{-t/\tau} \quad \text{discharging inductor} \quad (13)$$

$$\Delta V_L(t) = \Delta V_0 e^{-t/\tau} \quad \text{discharging inductor} \quad (14)$$

where I_0 and $\Delta V_0 (= I_0 L/\tau)$ are the inductor's initial current and voltage respectively, and the "time constant" τ is given by:

$$\tau = L/R \quad (15)$$

After a time equal to τ , the current through the inductor is equal to 36.8% ($\approx e^{-1}$) of its initial value.

When an inductor is connected to a battery with no initial current, similar equations hold:

$$I(t) = I_{max}(1 - e^{-t/\tau}) \quad \text{charging inductor} \quad (16)$$

$$\Delta V_L(t) = -\Delta V_{max} e^{-t/\tau} \quad \text{charging inductor} \quad (17)$$

where I_{max} and $\Delta V_{max} (= I_{max} L/\tau)$ represent the final, maximum current and the maximum voltage magnitude (which incidentally occurs at $t = 0$) of the inductor, respectively. The time constant $\tau = L/R$ is the same as for discharging inductors. After a time equal to τ , the current through the inductor is equal to 63.2% ($\approx 1 - e^{-1}$) of its maximum value. The reason for this nonlinear charging is that the more "full" the inductor is in terms of allowable current, the harder it gets to add more current.

Proof of the time dependent RL equations. Eqs. (13) and (16) can be proved via KVL and the "guess and check" method of solving differential equations, and Eqs. (14) and (17) follow by the voltage of an inductor $\Delta V_L = -L di/dt$. For example, for a charging series RL circuit, KVL yields:

$$V_{battery} - IR - L \frac{dI}{dt} = 0$$

If we guess $I(t) = I_{max}(1 - e^{-t/\tau})$, we find

$$\begin{aligned} V_{battery} - I_{max}R \left(1 - e^{-\frac{t}{\tau}}\right) - \frac{L I_{max}}{\tau} e^{-t/\tau} &= 0 \\ (V_{battery} - I_{max}R) + I_{max} \left(R - \frac{L}{\tau}\right) e^{-t/\tau} &= 0 \end{aligned}$$

That equation is true if $I_{max} = V_{battery}/R$ and $\tau = L/R$. Moreover our guess satisfies the initial condition of $I(t = 0) = 0$. Therefore by the uniqueness theorem of first order differential equations, Eq. (16) is *the* solution for this situation. (Eq. 13 is proved similarly for a discharging inductor.)