

# Circuits – Advanced Topics

## by Dr. Colton (Fall 2016)

### 1. Time dependence of general RC and RL problems

General RC and RL problems can always be cast into first order ODEs. You can solve these via the “particular solution” and “complementary solution” method. The first step is to use KVL to write a differential equation for  $V$  or  $I$  (whichever is easier). Put the equation into this standard form:

$$(stuff) \frac{dV}{dt} + V = g(t)$$

$$(stuff) \frac{dI}{dt} + I = g(t)$$

Note that “stuff” has units of time. Let’s call it  $\tau$ . The equation for  $V$ , for example, then becomes:

$$\tau \frac{dV}{dt} + V = g(t)$$

The **particular solution** is something that solves the equation with the “forcing function”  $g(t)$  in place, but doesn’t necessarily match the boundary conditions. The **complementary solution** which is something that solves the equation with  $g(t)$  set equal to 0, but with one or more unknown parameters. The **total solution** is the sum of the particular and complementary solutions. The total solution still solves the equation with  $g(t)$  in place, and the unknown parameter(s) can be used to satisfy the initial condition(s).<sup>1</sup> Once you have the best of both worlds, namely a function which solves the equation with  $g(t)$  in place and which also matches the initial conditions, by the uniqueness theorem, that is then *the* solution to the problem.

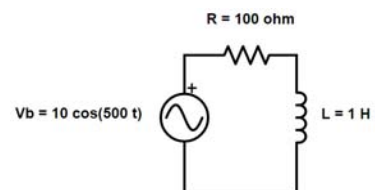
Finding the particular solution: Guess an overall form for the answer, using intuition/experience and the forcing function as a guide. For example, if  $g(t)$  is a constant, guess a constant for the particular solution. If  $g(t)$  is an exponential then guess an exponential (or maybe a sum of decaying and growing exponentials). If  $g(t)$  is a sine or cosine, then guess a sine or cosine (or sum of the two).

Finding the complementary solution: For first order equations, the complementary solution will always be  $Ke^{-t/\tau}$ , where  $\tau$  is from the standard form equation and  $K$  is an unknown constant; notice that this function will always solve the equation with  $g(t)$  set to 0, regardless of the value of  $K$ .

Finding the total solution: Add the two together, then use the initial condition to determine  $K$ .

#### Worked problem 1

An AC voltage is connected to an RC circuit. The current is zero at  $t = 0$ . What is  $I(t)$ ?



Starting out. To solve this problem we first use KVL to obtain the ODE, then put it in standard form. For simplicity I’m not going to worry about units.

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<sup>1</sup> First order equations will have one unknown parameter and one initial condition; second order equations will have two of each.

$$10 \cos(500t) - 100I - 1 \frac{dI}{dt} = 0$$

$$0.01 \frac{dI}{dt} + I = 0.1 \cos(500t)$$

So  $\tau = 0.01$  seconds.

Finding the particular solution. Make a guess:

$$I(t) = A \cos(500t) + B \sin(500t)$$

Plug into the ODE.

$$0.01(-500A \sin(500t) + 500B \cos(500t)) + (A \cos(500t) + B \sin(500t)) = 0.1 \cos(500t)$$

Solve for  $A$  and  $B$  (by equating cosine and sine coefficients, separately).

$$\sin(500t)(-5A + B) + \cos(500t)(5B + A) = 0.1 \cos(500t)$$

So  $-5A + B = 0$  and  $5B + A = 0.1$ . Those are easy to solve simultaneously, yielding

$$A \approx 0.003846 \text{ and } B \approx 0.01923$$

Our particular solution is therefore

$$I(t) = 0.003846 \cos(500t) + 0.01923 \sin(500t)$$

Finding the complementary solution. The complementary solution is  $Ke^{-t/\tau}$  ( $= Ke^{-t/0.01}$ ). That was easy.

Combining, and matching initial condition:

Our total solution is

$$I(t) = 0.003846 \cos(500t) + 0.01923 \sin(500t) + Ke^{-t/0.01}$$

Enforcing the initial condition of  $I(t = 0) = 0$ , we have

$$0 = 0.003846 + 0 + K$$

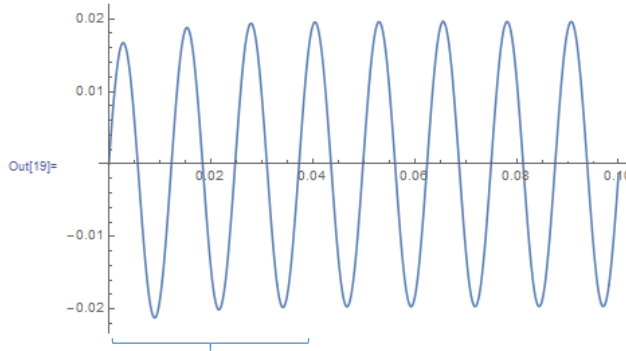
$$K = -0.003846$$

Our final, total solution is therefore:

$$I(t) = 0.003846 \cos(500t) + 0.01923 \sin(500t) - 0.003846 e^{-t/0.01}$$

Additional details (e.g. the inductor voltage) are readily obtained if desired now that we have  $I(t)$ . I have plotted  $I(t)$  on the next page.

```
In[18]:= i[t_] = 0.003846 Cos[500 t] + 0.01923 Sin[500 t] - 0.003846 Exp[-t/0.01]
Plot[i[t], {t, 0, .1}]
Out[18]:= -0.003846 e-100. t + 0.003846 Cos[500 t] + 0.01923 Sin[500 t]
```

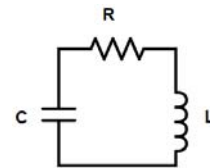


The transient region in which the complementary solution affects things.

## 2. Simple RLC series circuit (no power supply)

For a simple RLC series circuit KVL gives this as the differential equation:

$$V + LC \frac{d^2 V_c}{dt^2} - IR = 0$$



where  $V$  is the capacitor voltage. Put  $I$  in terms of the capacitor voltage, namely  $I = -C \frac{dV}{dt}$  (from  $Q = VC$ , take the derivative and add a negative sign because the capacitor is discharging, i.e.  $\frac{dV}{dt}$  is negative when  $I$  is positive). Then we have:

$$V + LC \frac{d^2 V_c}{dt^2} + RC \frac{dV}{dt} = 0$$

$$\frac{d^2 V}{dt^2} + \frac{R}{L} \frac{dV}{dt} + \frac{1}{LC} V = 0$$

$$\frac{d^2 V}{dt^2} + 2\alpha \frac{dV}{dt} + \omega_0^2 V = 0$$

where the last step involved defining the damping constant  $\alpha = \frac{R}{2L}$  and the resonant frequency  $\omega_0 = \frac{1}{\sqrt{LC}}$ .

We may rightly expect oscillatory behavior if  $\alpha$  is small, since when  $\alpha = 0$  the solutions are purely sinusoidal. But if  $\alpha$  is large, we don't know what to expect... maybe a decaying exponential since there's a lot of damping. Therefore let's guess a solution of the form  $V = Ke^{st}$ ; we expect  $s$  to be complex if  $\alpha$  is small, and possibly not if  $\alpha$  is larger.

Plugging that guess into the circuit equation we have:

$$s^2 K e^{st} + 2\alpha s K e^{st} + \omega_0^2 K e^{st} = 0$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

(which is called the “characteristic equation”).

We can use the quadratic formula to solve it, namely:

$$s = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega_0^2}}{2}$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

There are two values of  $s$  that work, namely two independent solutions of the 2<sup>nd</sup> order ODE. Linear combinations are also solutions

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$V = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

The coefficients  $K_1$  and  $K_2$  must be chosen to match the given initial conditions (there must be two).

There are three distinct regimes:

Case 1: overdamped,  $\alpha > \omega_0$ .

The general solution for this case is:

$$V = K_1 e^{(-\alpha + \sqrt{\alpha^2 - \omega_0^2}) t} + K_2 e^{(-\alpha - \sqrt{\alpha^2 - \omega_0^2}) t}$$

These are decaying exponentials, with two different decay rates.

Case 2: underdamped,  $\alpha < \omega_0$ .

The general solution for this case is:

$$V = K_1 e^{(-\alpha + \sqrt{\alpha^2 - \omega_0^2}) t} + K_2 e^{(-\alpha - \sqrt{\alpha^2 - \omega_0^2}) t}$$

$$V = K_1 e^{(-\alpha + i\sqrt{\omega_0^2 - \alpha^2}) t} + K_2 e^{(-\alpha - i\sqrt{\omega_0^2 - \alpha^2}) t}$$

$$V = K_1 e^{-\alpha t} e^{i\sqrt{\omega_0^2 - \alpha^2} t} + K_2 e^{-\alpha t} e^{-i\sqrt{\omega_0^2 - \alpha^2} t}$$

$$V = K_1 e^{-\alpha t} \cos\left(\sqrt{\omega_0^2 - \alpha^2} t\right) + K_2 e^{-\alpha t} \sin\left(\sqrt{\omega_0^2 - \alpha^2} t\right)$$

(The  $K$ 's in the last step are different than the  $K$ 's in the preceding step, but they are linear combinations of them. Since the  $K$ 's represent arbitrary constants anyway, it doesn't matter.)

These are damped oscillations.

Case 3: critically damped,  $\alpha = \omega_0$ .

For this case the two solutions become identical:

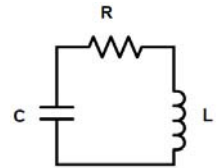
$$V = Ke^{-\alpha t}$$

However, a second order ODE must have two independent solutions! What's happened? Well, if we were to have made a *different* guess from the outset, namely a solution of the form  $V = Kte^{st}$ , we would have seen that it works for this case (and this case only), with  $s = -\alpha$  as above. I'll skip the math on that. Therefore the general solution for this case is:

$$V = K_1e^{-\alpha t} + K_2te^{-\alpha t}$$

**Worked problem 2**

In the circuit,  $C = 1 \mu\text{F}$ ,  $L = 10 \text{ mH}$ , and  $R$  is variable. The capacitor has an initial voltage of 1 V, and no initial current. What is  $V_{cap}(t)$  for (a)  $R = 100 \Omega$ , (b)  $R = 200 \Omega$ , and (c)  $R = 400 \Omega$ ?



Starting out. Doing a quick calculation,  $\omega_0 = \frac{1}{\sqrt{LC}} = 10000 \text{ rad/s}$ . Also note that the second initial condition means that  $dV/dt = 0$  at  $t = 0$ , since  $V = Q/C \rightarrow dV/dt = I/C$ .

Part (a).  $\alpha = \frac{R}{2L} = 5000$ , is the damping constant. This is less than  $\omega_0$ , so it's underdamped.  
 $\sqrt{\omega_0^2 - \alpha^2} = \sqrt{10000^2 - 5000^2} = 8660 \text{ rad/s}$ , is the frequency of oscillation.

The general formula is

$$V = K_1e^{-5000t} \cos(8660 t) + K_2e^{-5000t} \sin(8660 t)$$

but we need to determine  $K_1$  and  $K_2$  from the initial conditions.

$$\begin{aligned} V(t = 0) &= 1 \\ 1 &= K_1 + 0 \\ K_1 &= 1 \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt}(t = 0) &= 0 \\ 0 &= -5000K_1 + 8660K_2 \\ K_2 &= \frac{5000K_1}{8660} = 0.5774 \end{aligned}$$

So the solution is:

$$V = 1e^{-5000t} \cos(8660 t) + 0.5774e^{-5000t} \sin(8660 t)$$

(Without that second term, the initial slope would not be equal to zero.)

Part (b).  $\alpha = \frac{R}{2L} = 10000$ , is the damping constant. This is equal to  $\omega_0$ , so it's critically damped.

The general formula is

$$V = K_1 e^{-10000t} + K_2 t e^{-10000t}$$

but we need to determine  $K_1$  and  $K_2$  from the initial conditions.

$$\begin{aligned} V(t=0) &= 1 \\ 1 &= K_1 + 0 \\ K_1 &= 1 \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt}(t=0) &= 0 \\ 0 &= -10000K_1 + K_2 \\ K_2 &= 10000K_1 = 10000 \end{aligned}$$

So the solution is:

$$\boxed{V = 1e^{-10000t} + 10000te^{-10000t}}$$

Part (c).  $\alpha = \frac{R}{2L} = 20000$ , is the damping constant. This is more than  $\omega_0$ , so it's overdamped.

The two decay rates are  $-\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2679$  and  $-\alpha - \sqrt{\alpha^2 - \omega_0^2} = -37321$ .

The general formula is

$$V = K_1 e^{-2679t} + K_2 e^{-37321t}$$

but we need to determine  $K_1$  and  $K_2$  from the initial conditions.

$$\begin{aligned} V(t=0) &= 1 \\ 1 &= K_1 + K_2 \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt}(t=0) &= 0 \\ 0 &= -2679K_1 + 37321K_2 \end{aligned}$$

Those two equations can be solved simultaneously to find that

$$\begin{aligned} K_1 &= 1.0774 \\ K_2 &= -0.0774 \end{aligned}$$

So the solution is:

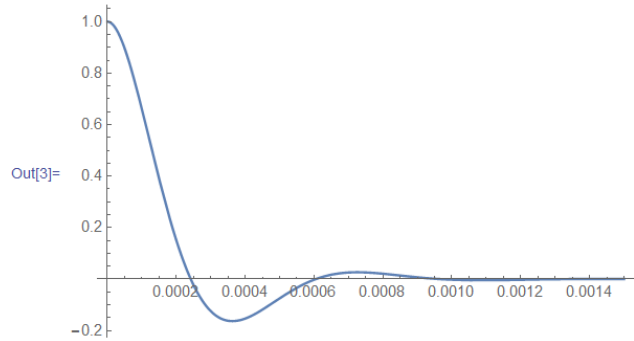
$$\boxed{V = 1.0774e^{-2679t} - 0.0774e^{-37321t}}$$

The solutions to parts (a), (b), and (c) are plotted on the next page.

In[1]= (\* RLC circuit for three different values of R \*)  
 (\* notice how for all three cases at the start V=1 and the slope = 0 \*)

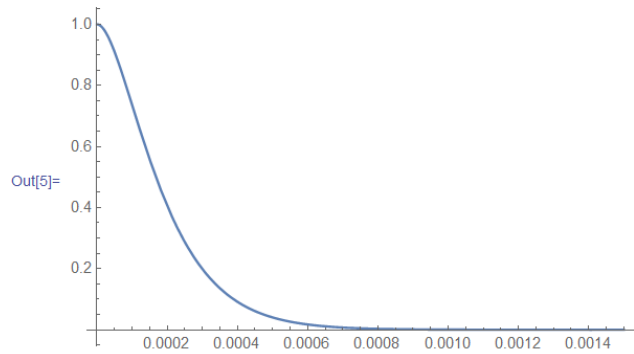
In[2]= (\* part a, R = 100 ohm, underdamped \*)  
**V[t\_] = Exp[-5000 t] Cos[8660 t] + 0.5774 Exp[-5000 t] Sin[8660 t]**  
**Plot[V[t], {t, 0, .0015}, PlotRange -> All]**

Out[2]=  $e^{-5000 t} \cos[8660 t] + 0.5774 e^{-5000 t} \sin[8660 t]$



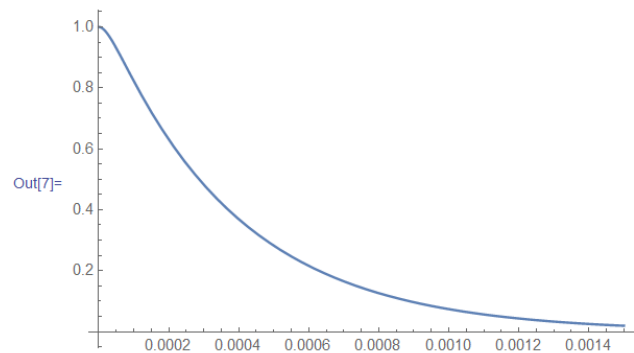
In[4]= (\* part b, R = 200 ohm, critically damped \*)  
**V[t\_] = Exp[-10000 t] + 10000 t Exp[-10000 t]**  
**Plot[V[t], {t, 0, .0015}, PlotRange -> All]**

Out[4]=  $e^{-10000 t} + 10000 t e^{-10000 t}$



In[6]= (\* part c, R = 400 ohm, overdamped \*)  
**V[t\_] = 1.0774 Exp[-2679 t] - 0.0774 Exp[-37321 t]**  
**Plot[V[t], {t, 0, .0015}, PlotRange -> All]**

Out[6]=  $-0.0774 e^{-37321 t} + 1.0774 e^{-2679 t}$



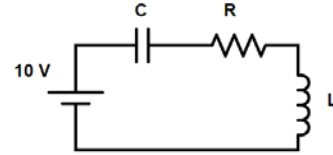
In[8]= (\* takes longer to decay than the critically damped situation \*)

### 3. More complicated RLC series circuit (e.g. with power supplies)

More complicated RLC time-dependent circuit problems can be solved via the particular + complementary solution technique. The “simple RLC series circuit” solution just presented is the complementary solution. The particular solution should be guessed according to the form of the forcing function, as in the 1<sup>st</sup> order problems.

#### Worked problem 3

A 10 V battery is connected to a series RLC circuit as shown. In the circuit,  $C = 1 \mu\text{F}$ ,  $L = 10 \text{ mH}$ , and  $R = 10 \Omega$ . The current and capacitor voltage are initially both zero. What is  $I(t)$ ?



Starting out. The circuit differential equation is:

$$V + LC \frac{d^2V_c}{dt^2} - IR = V_b$$

Similar to before, we put  $I$  in terms of the capacitor voltage, and use the constants  $\alpha$  and  $\omega_0$ , to obtain:

$$\frac{d^2V}{dt^2} + 2\alpha \frac{dV}{dt} + \omega_0^2 V = \omega_0^2 V_b$$

Doing some quick calculations,  $\omega_0 = \frac{1}{\sqrt{LC}} = 10000$  and  $\alpha = \frac{R}{2L} = 500$ , so it's underdamped with a natural frequency of oscillation of  $\sqrt{\omega_0^2 - \alpha^2} = \sqrt{10000^2 - 500^2} = 9987 \text{ rad/s}$ .

Particular solution. Since the forcing function is a constant, let's guess a constant for the particular solution,  $V = C$ .

$$0 + 0 + \omega_0^2 C = \omega_0^2 V_b$$

$$C = V_b$$

$$C = 10$$

Complementary solution. The general formula for the underdamped case is

$$V = K_1 e^{-500t} \cos(9987 t) + K_2 e^{-500t} \sin(9987 t)$$

Total solution. Putting the two together gives:

$$V = K_1 e^{-500t} \cos(9987 t) + K_2 e^{-500t} \sin(9987 t) + 10$$

but we need to determine  $K_1$  and  $K_2$  from the initial conditions.

$$V(t = 0) = 0$$

$$0 = K_1 + 10$$

$$K_1 = -10$$



$$\begin{aligned}\frac{dV}{dt}(t=0) &= 0 \\ 0 &= -500K_1 + 9987K_2 \\ K_2 &= \frac{500K_1}{9987} = -0.5007\end{aligned}$$

So the solution is:

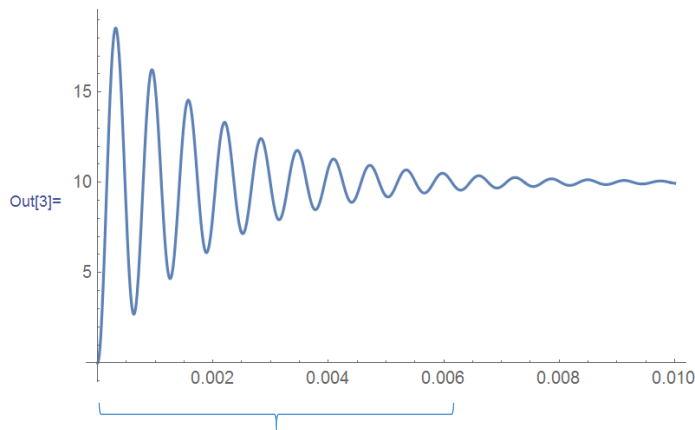
$$V = -10e^{-500t} \cos(9987t) - 0.5007e^{-500t} \sin(9987t) + 10$$

Here's a plot:

```
In[1]:= (* RLC circuit attached to battery *)
(* initial conditions V=0, dV/dt=0 *)

In[2]:= V[t_] = -10 Exp[-500 t] Cos[9987 t] - 0.5007 Exp[-500 t] Sin[9987 t] + 10
Plot[V[t], {t, 0, .01}, PlotRange -> All]

Out[2]:= 10 - 10 e^{-500 t} Cos[9987 t] - 0.5007 e^{-500 t} Sin[9987 t]
```



Eventually the capacitor charges up to 10 V

The transient region in which there is substantial “ringing” due to the complementary solution.

Part of the moral of the story here is that if you don't want ringing, you should shoot for critical damping, in any second order system (possibly robot arms, shock absorbers, etc.).

#### 4. AC voltage sources: steady state solutions

If you are only concerned about the steady state solution to a circuit situation, then the concept of **complex impedances** simplifies matters considerably.

Complex impedances arise when you assume a sinusoidal signal. This is OK to do because with Fourier series you can decompose any periodic signal into a sum of sine/cosine waves—thus, once you've figured out the problem for an arbitrary sine wave, the problem is solved for all periodic signals.

If you are supplying a sinusoidal voltage, it's reasonable to assume that the output current will also be sinusoidal. However, it will (a) have its own amplitude, and (b) likely be changed in phase. The new amplitude and phase will depend on the details of the R/L/C network you connect to the power supply.

To simplify solving such systems and to avoid the types of differential equations discussed above (and harder ones!), we're going to define an "Ohm's Law"-like quantity for capacitors and inductors; then we can use the series and parallel resistance formulas (which were derived with Ohm's Law) to describe complicated circuits. That quantity is called the *impedance*, and will be a complex number  $Z$ :  $\Delta V = IZ$  is like  $\Delta V = IR$  for a resistor.

The complex nature of the impedance is how we will handle phase shifts between circuit elements. We'll also assume a complex current  $I$  while we're working through the math, but will always take the real part of  $I$  before giving our final answer.

We will always assume that the "zero" of time is chosen such that the battery voltage is a cosine function with amplitude  $V_0$ :

$$V_b = V_0 \cos(\omega t) = \text{Re}\{V_0 e^{i\omega t}\}$$

We will suppose that the current is also sinusoidal with amplitude  $I_0$ , lagging the voltage by a phase  $\phi$  (if the current is actually leading the voltage, then  $\phi$  will be negative.):

$$I = I_0 \cos(\omega t - \phi) = \text{Re}\{I_0 e^{i(\omega t - \phi)}\}$$

I'll now leave off the  $\text{Re}\{\}$  symbols; just keep in mind that the voltages and currents are *real* quantities, not complex; the point of the complex exponentials is to simplify the mathematics of sinusoidal signals with phase shifts. At the end of a problem, if you want to know the actual current and voltage just convert back from the complex exponential to a cosine by taking the real part.

Impedance of Resistor. Resistors are easy:  $Z_R = R$ . Ohm's Law applies directly with no weird modifications.

Impedance of Capacitor.  $V_C = \frac{1}{C} \int I$ , and if  $I = I_0 e^{i(\omega t - \phi)}$ , when you integrate  $I$  you get a factor of  $1/i\omega$  from the exponential, times the current  $I$  itself. Thus the capacitor equation becomes:  $V_C = \frac{1}{i\omega C} I$ , and the resistance-like quantity is  $Z_C = \frac{1}{i\omega C}$  (also often written as  $Z_C = \frac{-i}{\omega C}$ ).

Impedance of Inductor.  $V_L = L dI/dt$ . When you take the derivative of  $I$ , you get a factor of  $i\omega$  from the exponential, times the current  $I$  itself. Thus the inductor equation becomes:  $V_L = i\omega L I$ , and the resistance-like quantity is  $Z_L = i\omega L$ .

In summary:

$Z_R = R$ $Z_C = \frac{1}{i\omega C} = -\frac{i}{\omega C}$ $Z_L = i\omega L$
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If you use these impedances, then

- Series/parallel rules work exactly as they do with resistors
- $V = IZ$  (or  $I = V/Z$ ) gives you the current through power supply

Using these impedances to determine the amplitude and phase of the current from the power supply is straightforward:

Step 1. Find  $Z_{eq}$  using series/parallel resistor rules. It's a complex number.

Step 2. Write  $Z_{eq}$  in polar form:  $|Z_{eq}|e^{i\phi}$

Step 3. Then  $I = \frac{V}{Z} = \frac{V_0 e^{i\omega t}}{|Z_{eq}|e^{i\phi}} = \frac{V_0}{|Z_{eq}|} e^{i(\omega t - \phi)}$ .

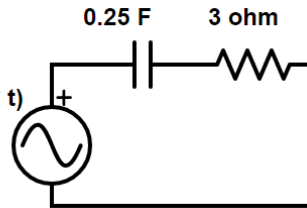
Step 4. Take the real part to get your final answer:  $I = \frac{V_0}{|Z_{eq}|} \cos(\omega t - \phi)$ .

In other words, its amplitude  $I_0 = \frac{V_0}{|Z_{eq}|}$  and its phase is  $-\phi$  (as assumed from the start).

#### Worked Problem 4

What current is supplied here?

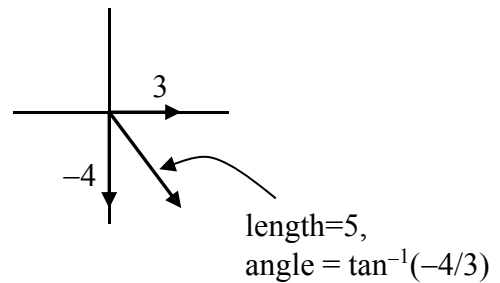
$1V \cos(1 \text{ rad/s} * t)$



Step 1:  $Z_{eq} = Z_R + Z_C = 3 + \frac{1}{0.25i} = 3 - 4i$

Step 2: Now write  $3 - 4i$  in polar form,  $3 - 4i = 5 e^{-0.927i}$

$$Z_{eq} = 5 e^{-0.927i}$$



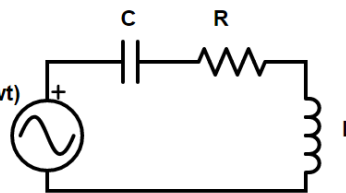
Step 3:  $I = \frac{V_0}{|Z_{eq}|} e^{i(\omega t - \theta)} = 0.2 e^{i(t + 0.927)}$

Step 4: Take real part to get the actual current:  $I = 0.2 \cos(t + 0.927)$

#### Worked problem 5

For the following circuit, what is  $\Delta V_R$ ?

$V_0 \cos(\omega t)$

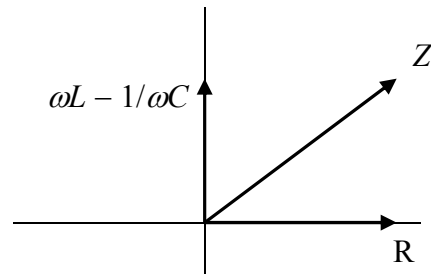


Step 1:  $Z_{eq} = R + i\omega L + \frac{1}{i\omega C}$

$$Z_{eq} = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

Step 2: polar form is  $Z_{eq} = |Z_{eq}|e^{i\phi}$ , so we have:

$$|Z_{eq}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$



$$\phi = \tan^{-1}\left(\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right)\right)$$

(I assumed that  $\omega L > 1/\omega C$  so the vector shown above is indeed in the first quadrant. If not in the first quadrant the  $\tan^{-1}$  function can be wrong answers. Mathematica's **Arg** command is helpful.)

Step 3 & 4:  $I = \text{Re}\{V/Z\}$

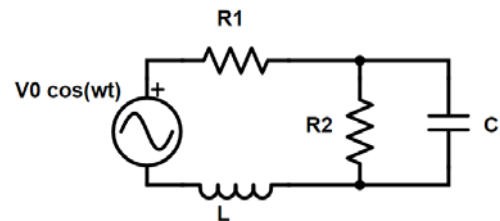
$$I = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right)\right)\right)$$

Then  $\Delta V_R = I R$  by Ohm's Law, so the answer is:

$$\Delta V_R = \frac{V_0 R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right)\right)\right)$$

### Worked problem 6

What's the current coming from the power supply, in terms of  $V_0 \cos(\omega t)$ ,  $R_1$ ,  $R_2$ ,  $L$ , and  $C$ ?



Step 1:  $Z_{eq} = Z_{R1} + (Z_{R2} // Z_C) + Z_L$

$$= R_1 + \left(\frac{1}{R_2} + \frac{1}{1/i\omega C}\right)^{-1} + i\omega L$$

I'll use Mathematica to help with the algebra:

```
In[1]= Z = R1 + (1/R2 + 1/(I w C))^-1 + I w L // ComplexExpand
```

$$\text{Out[1]= } R_1 + \frac{1}{R_2 \left(\frac{1}{R_2^2} + \frac{1}{C^2 \omega^2}\right)} + i \left(\frac{1}{C \left(\frac{1}{R_2^2} + \frac{1}{C^2 \omega^2}\right) \omega} + L \omega\right)$$

Steps 2, 3, and 4:

From there, the polar coordinates can be found in Mathematica with

$$\begin{aligned} |Z| &= \text{Abs}[Z] \\ \text{and } \phi &= \text{Arg}[Z] \end{aligned}$$

and the current via  $I = \frac{V_0}{|Z|} \cos(\omega t - \phi)$  for that  $|Z|$  and  $\phi$ .

It's probably not worth the space to write out the full-blown answer.

## 5. Power in AC problems

The instantaneous power supplied by the power supply is  $P(t) = \text{Re}\{V\}\text{Re}\{I\}$  —you must multiply the *actual* voltage and *actual* currents together to get the *actual* power.

However, power as a function of time is often not a quantity of interest. Typically a more useful quantity is the *average* rather than instantaneous power. The power averaged over time is often written like this  $\langle P \rangle$ , and is given by any of the following formulas. As above,  $\phi$  is the complex phase angle of the circuit's complex impedance  $Z$ :

$$\langle P \rangle = \frac{1}{2} V_0 I_0 \cos \phi = \frac{1}{2} \left( \frac{V_0^2}{|Z|} \right) \cos \phi \quad (\text{use the real amplitudes } V_0 \text{ and } I_0)$$

$$\langle P \rangle = V_{rms} I_{rms} \cos \phi \quad (\text{use the real rms values, where e.g. } V_{rms} = \frac{V_0}{\sqrt{2}})$$

$$\langle P \rangle = \text{Re} \left\{ \frac{1}{2} V I^* \right\} \quad (\text{use the complex } V \text{ and } I; I^* \text{ means the complex conjugate of } I)$$

The first equation is proved like this:

$$\begin{aligned} P(t) &= \text{Re}\{V\}\text{Re}\{I\} \\ &= V_0 \cos(\omega t) I_0 \cos(\omega t - \phi) \\ &= V_0 I_0 \cos(\omega t) (\cos(\omega t) \cos(\phi) + \sin(\omega t) \sin(\phi)) \\ &= V_0 I_0 (\cos^2(\omega t) \cos(\phi) + \sin(\omega t) \cos(\omega t) \sin(\phi)) \\ \langle P \rangle &= V_0 I_0 \left( \frac{1}{2} \cos(\phi) + 0 \sin(\phi) \right) \\ \langle P \rangle &= \frac{1}{2} V_0 I_0 \cos \phi \end{aligned}$$

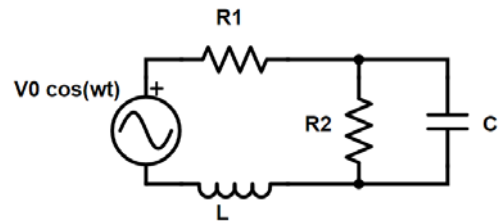
The factor of  $\frac{1}{2}$  arises from the time averaging of  $\cos^2 \omega t$ .

Note that:

- $\cos \phi$  is often called the **power factor** of the circuit
- $\phi$  itself is sometimes called the “power angle” of the circuit

### Worked problem 7

In the previous problem (Worked problem 6), suppose  $V_0 = 1$  V,  $R_1 = 10 \Omega$ ,  $R_2 = 20 \Omega$ ,  $C = 0.1 \mu\text{F}$ , and  $L = 0.1$  mH. What is the average power supplied by the power supply as a function of  $\omega$ ? Make a plot of  $\langle P \rangle$  vs  $\omega$ .



*Solution:* I'll use Mathematica to make my life easy.

```

In[1]:= Z[w_] =
  R1 + (1/R2 + 1/(I w C))^-1 + I w L /.
  {R1 -> 10, R2 -> 20, C -> 0.1*^-6, L -> 0.1*^-3} // ComplexExpand

Out[1]= 10. +  $\frac{0.05}{0.0025 + \left(0. - \frac{1. \times 10^7}{w}\right)^2}$  + i  $\left(0. + \frac{1. \times 10^7}{\left(0.0025 + \left(0. - \frac{1. \times 10^7}{w}\right)^2\right) w} + 0.0001 w\right)$ 

```

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In[2]:= V0 = 1;
IO[w_] = V0 / Abs[Z[w]];
power[w_] = 1 / 2 V0 IO[w] Cos[Arg[Z[w]]]
Plot[power[w], {w, 0, 200000}, PlotRange -> All]

```

$$\text{Out[4]= } \frac{\text{Cos}\left[\text{Arg}\left[10. + \frac{0.05}{0.0025 + \left(0. - \frac{1. \times 10^7}{w}\right)^2} + i \left(0. + \frac{1. \times 10^7}{\left(0.0025 + \left(0. - \frac{1. \times 10^7}{w}\right)^2\right) w} + 0.0001 w\right)\right]\right]}{2 \text{ Abs}\left[10. + \frac{0.05}{0.0025 + \left(0. - \frac{1. \times 10^7}{w}\right)^2} + i \left(0. + \frac{1. \times 10^7}{\left(0.0025 + \left(0. - \frac{1. \times 10^7}{w}\right)^2\right) w} + 0.0001 w\right)\right]}$$

