Advanced Circuits Topics - Part 1
by Dr. Colton (Fall 2017)

Part 1: Some things you should already know from Physics 220 and 145

These are all things that you should have learned in Physics 220 and/or 145. This section is organized like this so you can see the parallels between resistor problems and complex impedance problems:

Outline:

1. Voltage and Current
2. Resistors and Ohm's Law
3. Series and Parallel Resistors
4. Solving Circuit Problems With Equivalent Resistances
5. Interlude: A Quick Summary of Complex Numbers

6. Complex Voltage and Current
7. Complex Impedances and Ohm's Law
8. Series and Parallel Impedances
9. Solving Circuit Problems With Equivalent Impedances

1. Voltage and Current

The basic water analogy works well for me: if electricity is like water flowing in pipes, then the current is the rate at which the water is flowing in the pipe (kg/s) and the voltage is the height of the pipe (potential energy of the water). Batteries which raise the potential of the charges are like pumps pumping the water up to a higher elevation. The "potential difference" of a circuit element is the difference in height from one side of the element to the other.

2. Resistors and Ohm's Law

Resistors and conductors are the pipes themselves. A conductor is like a very large diameter pipe that allows for a lot of current flow. A resistor is a like a small diameter pipe which limits the flow: the larger the resistor, the smaller the diameter and the smaller the flow. The "resistor pipe" is angled downwards, so that the end at which the water comes out is lower than the end at which the water goes in.

The potential difference of a resistor is given by Ohm's law: $\Delta V_R = IR$

3. Series and Parallel Resistors

A combination of 2 resistors can be labeled either as "series" or "parallel", depending on how they are connected: pipes can either be one after the other (series) or side-by-side (parallel). Series resistors share the same current; parallel resistors share the same potential difference.

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SERIES
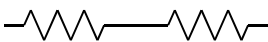
$$R_{total} = R_1 + R_2$$

PARALLEL
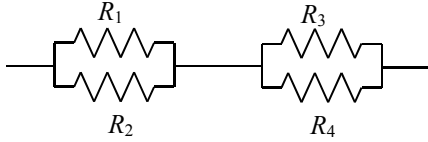

$$R_{total} = (1/R_1 + 1/R_2)^{-1}$$

Shortcut notation: $R_1 \text{ // } R_2 = (1/R_1 + 1/R_2)^{-1}$

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4. Solving Circuit Problems With Equivalent Resistances

Networks of resistors can often be decomposed into combinations of resistors that are in series and parallel with each other, called equivalent resistances of the resistor combinations.

Example Problem 1*: What's the total resistance of the 4 resistors?*

Solution: $R_{total} = R_1 \,//\, R_2 \;+\; R_3 \,//\, R_4 \;=\; (R_1 + R_2)^{-1} + (R_3 + R_4)^{-1}$

Suppose the four resistors were hooked up to a battery with voltage V_B. The total current flowing from the battery would be given by Ohm's law for the circuit as a whole: $I = V_b/R_{total} = V_b/((R_1 + R_2)^{-1} + (R_3 + R_4)^{-1})$.

The current and voltage of any of the resistors in this problem can be obtained by application of series/parallel resistor formulas along with Ohm's law. The same is true for nearly any circuit problem involving a single voltage source. For more complicated situations, Kirchhoff's two circuit laws can be used to obtain simultaneous equations, but that's beyond the scope of this document.

5. Interlude: A Quick Summary of Complex Numbers

We will be using complex numbers in Physics 441 and 442 as a tool for describing oscillating electromagnetic phenomena: AC circuits in Physics 441, and electromagnetic waves in Physics 442.

- A complex number $x + iy$ can be written either in rectangular form (x, y) or polar form (A, ϕ).
- The polar form can equivalently be expressed as a complex exponential $Ae^{i\phi}$. The complex exponential form follows directly from Euler's formula: $e^{i\phi} = \cos\phi + i\sin\phi$, and by looking at the x- and y-components of the polar coordinates, $(A\cos\phi, A\sin\phi)$.
    - A is called the amplitude, magnitude, or modulus of the complex number. In Mathematica it's obtained via the command "Abs".
    - ϕ is called the phase angle, phase factor, phase, angle, argument, or amplitude[1] of the complex number. In Mathematica it's obtained via the command "Arg".
- For example, consider the complex number $3 + 4i$:
    - $= (3, 4)$ in rectangular form,
    - $= (5, 53.13°)$ in polar form, and
    - $= 5e^{i53.13°}$ or $5e^{0.9273i}$ in complex exponential form, since $53.13° = 0.9273$ rad.
- I often write the polar form using this shortcut notation: $A\angle\theta$. The "$\angle$" symbol can be read as, "at an angle of".
- By the rules of exponents, when you multiply two complex numbers in polar form, the amplitudes multiply and the angles add. Similarly, when you divide in polar form, the amplitudes divide and the angles subtract. Thus you can write:
    - $(3 + 4i) \times (5 + 12i) \;=\; 5\angle53.13° \;\times\; 13\angle67.38° \;=\; 65\angle120.51°$
    - $(3 + 4i) / (5 + 12i) \;=\; 5\angle53.13° \;/\; 13\angle67.38° \;=\; 0.38462\angle{-}14.25°$

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[1] Yes, unfortunately the word "amplitude" can at times refer to either A or ϕ. I myself will reserve "amplitude" for A, though.

6. **Complex Voltage and Current**

Suppose you have an oscillation such as an AC voltage or current that you want to represent via complex numbers. The equation for a generic oscillation would be this, with arbitrary amplitude V_0 and phase ϕ:

$$V = V_0\cos(\omega t + \phi)$$

One represents the oscillatory function via complex numbers like this:

$$V = V_0\cos(wt + \phi) \quad \rightarrow V = \text{Re}\{V_0 e^{i(\omega t + \phi)}\}$$
$$\rightarrow V = V_0 e^{i(\omega t + \phi)} \qquad \text{Starting with this step, there's an implied "take}$$
the real part" of everything on the right hand side

$$\rightarrow V = V_0 e^{i\omega t} e^{i\phi}$$
$$\rightarrow V = V_0 e^{i\phi}\ e^{i\omega t} \qquad \text{There's often an implied } e^{i\omega t} \text{ as well, so this can}$$
become:

$$V = V_0 e^{i\phi} = V_0 \angle \phi$$

Thus a voltage oscillating in time can be represented as a single complex number. This trick will make the math much easier for many calculations.

Therefore, if for example you see $3.2V\angle 30°$, what it means is that a voltage is oscillating in time as $3.2V\cos(\omega t + \frac{\pi}{6})$. Presumably the oscillation frequency ω would be given elsewhere in the problem, or maybe you'd be looking for things as a function of ω. Similarly, if you see $1.5A\angle 120°$ it means that a current is oscillating as $1.5A\cos(\omega t + \frac{2\pi}{3})$.

In summary:
$$V_0\cos(\omega t + \phi_1) \leftrightarrow V_0\angle\phi_1$$
$$I_0\cos(\omega t + \phi_2) \leftrightarrow I_0\angle\phi_2$$

7. **Complex Impedances and Ohm's Law**

If you have a circuit with a sinusoidal driving function (e.g. voltage) and if you are only concerned about the steady-state (long term) response, then the concept of complex impedances simplifies matters considerably. Actually, any periodic driving functions can be handled this way since with Fourier analysis you can decompose any periodic signal into a sum of sinusoidal ones.

If you are supplying a sinusoidal voltage at frequency ω, after a long time all currents and potential differences of individual circuit elements will also be sinusoidal and oscillating at the same frequency ω. As discussed in the last section, we will use complex numbers to represent these voltages and currents, and we will restrict ourselves to circuits made of resistors, capacitors, and inductors. As you will see, complex numbers will allow us to easily handle phase shifts between circuit elements.

To solve such systems we want to define an "Ohm's Law"-like quantity for capacitors and inductors; then we can use the series and parallel resistance formulas—which were derived via Ohm's Law—to describe complicated circuits. That quantity is called the *impedance*, and will be a complex number Z: $\Delta V = IZ$ is like $\Delta V = IR$ for a resistor.

We will set the zero of time such that the driving battery voltage is a cosine function with no phase shift, i.e. it reaches its maximum at $t = 0$. Let's call the amplitude V_0:

$$V_b = V_0 \cos(\omega t) = V_0 \angle 0$$

The total current supplied by the battery will also be sinusoidal but it may have a different phase than the voltage. Let's call the current's amplitude I_0, and say that it lags the voltage by a phase ϕ (if the current is actually leading the voltage, then ϕ will be negative):

$$I = I_0 \cos(\omega t - \phi) = I_0 \angle -\phi$$

Keep in mind that the voltages and currents are *real* quantities, not complex; the point of the complex exponentials is just to simplify the mathematics of sinusoidal signals with phase shifts. At the end of a problem, if you want to know the actual voltages or currents just convert back from the complex numbers using the $V \angle \phi \leftrightarrow V\cos(\omega t + \phi)$ types of correspondences.

Impedance of Resistor. Resistors are easy: $Z_R = R$. Ohm's Law applies directly with no weird modifications.

Impedance of Capacitor. $\Delta V_C = \frac{Q}{C} = \frac{1}{C}\int I\, dt$, and if $I = I_0 e^{i(\omega t - \phi)}$, then when you integrate I you get a factor of $1/i\omega$ from the exponential, times the current I itself. Thus the capacitor equation becomes: $\Delta V_C = \frac{1}{i\omega C}I$, and the resistance-like quantity is $Z_C = \frac{1}{i\omega C}$ (also often written as $Z_C = \frac{-i}{\omega C}$).

Impedance of Inductor. As is later in Chapter 7 of Griffiths, and you learned in Phys 220, $\Delta V_L = L\frac{dI}{dt}$. When you take the derivative of I, you get a factor of $i\omega$ from the exponential, times the current I itself. Thus the inductor equation becomes: $\Delta V_L = i\omega L I$, and the resistance-like quantity is $Z_L = i\omega L$.

In summary:

$$
\boxed{
\begin{aligned}
Z_R &= R \\
Z_C &= \frac{1}{i\omega C} = -\frac{i}{\omega C} \\
Z_L &= i\omega L
\end{aligned}
}
$$

If you use these complex impedances (and have sinusoidal responses), then Ohm's law works for capacitors and inductors—and for the circuit as a whole—just like it does for resistors:

$$\boxed{\Delta V = IZ}$$

8. Series and Parallel Impedances

Because the series and parallel resistor formulas were derived via Ohm's law, they carry over completely with complex impedances.

$$
\boxed{
\begin{aligned}
&\text{Series: } Z_{total} = Z_1 + Z_2 \\
&\text{Parallel: } Z_{total} = (1/Z_1 + 1/Z_2)^{-1} = Z_1 \;//\; Z_2
\end{aligned}
}
$$

These equations are very powerful because they can be used with combinations of any types of impedances.

9. Solving Circuit Problems With Equivalent Impedances

Using these impedances to determine the amplitude and phase of the current from the power supply is straightforward:

Step 1. Find the total impedance of the circuit, Z_{tot}, using series/parallel rules. It's a complex number.

*Step 2.* Write Z_{tot} in polar form: $|Z_{tot}|\angle\phi$

Step 3. Use Ohm's law to get the total current, also a complex number: $I = \dfrac{V}{Z} = \dfrac{V_0\angle 0}{|Z_{eq}|\angle\phi} = \dfrac{V_0}{|Z_{eq}|}\angle-\phi$

Step 4. Convert back to a cosine oscillation to get your "real" answer if desired: $I = \dfrac{V_0}{|Z_{eq}|}\cos(\omega t - \phi)$,

using the same phase convention as written in section 7 above (I lags V when ϕ is positive). Note that if the phase angle of the complex impedance is negative, then the phase angle of the complex current will be positive, and vice versa.

Once the total current is known in complex form, other quantities of interest can generally also be found using Phys 220 techniques involving Ohm's law and series/parallel impedances.
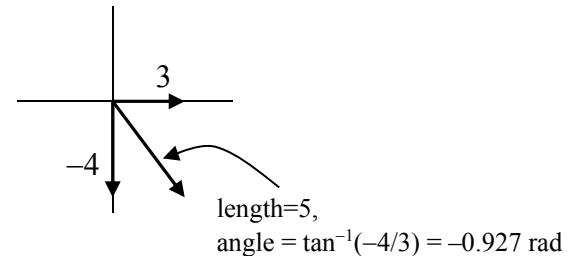
Example Problem 2

What current is supplied here?

0.25 F   3 ohm

1V cos(1 rad/s * t)
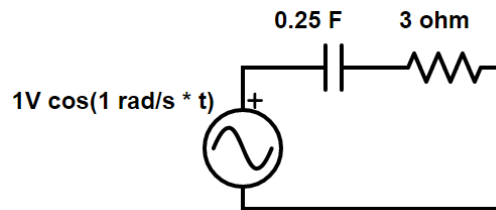
Solution:

Step 1: $Z_{eq} = Z_R + Z_C = 3 + \dfrac{1}{0.25i} = 3 - 4i$

Step 2: Now write $3 - 4i$ in polar form: $Z_{eq} = 5\angle - 0.927$
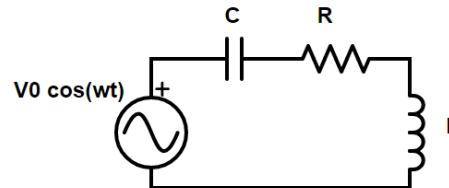
length=5,
angle $= \tan^{-1}(-4/3) = -0.927$ rad

Step 3: $I = \dfrac{V_0}{|Z_{eq}|}\angle-\phi = 0.2\angle 0.927$

Step 4: If desired, take the real part to get the actual formula for the current: $I = 0.2\text{A}\cos(t + 0.927)$
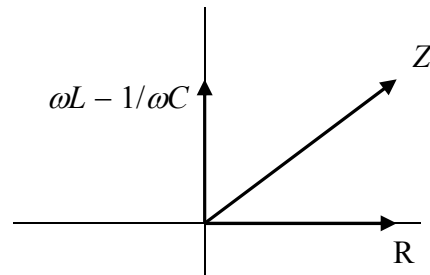
Example Problem 3

For this circuit, what is (a) ΔV_R and (b) ΔV_L?

C   R

V0 cos(wt)

L

Step 1: $Z_{eq} = R + i\omega L + \dfrac{1}{i\omega C}$

$$Z_{eq} = R + i\left(\omega L - \dfrac{1}{\omega C}\right)$$

Step 2: Put in polar form, $|Z_{eq}|\angle\phi$:

$$|Z_{eq}| = \sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$$

$$\phi = \tan^{-1}\left(\dfrac{1}{R}\left(\omega L - \dfrac{1}{\omega C}\right)\right)$$

(I assumed that $\omega L > 1/\omega C$ so the vector shown above is indeed in the first quadrant. <u>Caution</u>: if not in the first or fourth quadrants the $\tan^{-1}$ function will give wrong answers. As mentioned above Mathematica's **Arg** command is helpful, and much better than the regular $\tan^{-1}$ function.)

Steps 3 & 4: $I = V/Z$, and convert to a real oscillation:

$$I = \dfrac{V_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}\cos\left(\omega t - \tan^{-1}\left(\dfrac{1}{R}\left(\omega L - \dfrac{1}{\omega C}\right)\right)\right)$$

(a) Everything is in series, so this same current goes through the resistor. Use $\Delta V_R = I R$ by Ohm's Law, and the answer is:

$$\boxed{\Delta V_R = \dfrac{V_0 R}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}\cos\left(\omega t - \tan^{-1}\left(\dfrac{1}{R}\left(\omega L - \dfrac{1}{\omega C}\right)\right)\right)}$$

(b) Use the complex Ohm's law on the inductor:

$$\Delta V_L = I\, Z_L$$

$$\Delta V_L = \left(\dfrac{V_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}\right)\angle\left(\tan^{-1}\dfrac{1}{R}\left(\omega L - \dfrac{1}{\omega C}\right)\right)\times(i\omega L)$$

$$\Delta V_L = \left(\dfrac{V_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}\right)\angle\left(\tan^{-1}\dfrac{1}{R}\left(\omega L - \dfrac{1}{\omega C}\right)\right)\times\left(\omega L\angle\dfrac{\pi}{2}\right)$$

That last step was done by writing i itself in polar form. From there it's just complex number multiplication and converting to real numbers if you want:

$$\Delta V_L = \left(\frac{V_0 \, \omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \right) \angle \left(\tan^{-1} \frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right) + \frac{\pi}{2} \right)$$

$$\Delta V_L = \left(\frac{V_0 \, \omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \right) \cos\!\left(\omega t - \left(\tan^{-1}\left(\frac{\omega L}{R} - \frac{1}{\omega C R}\right) + \frac{\pi}{2} \right)\right)$$
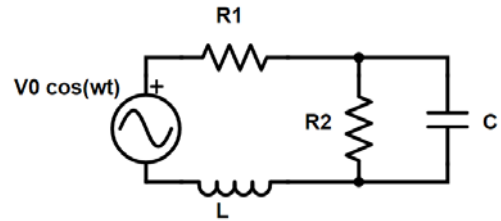
Example Problem 4

What's the current coming from the power supply, in terms of V_0, ω, t, R_1, R_2, L, and C?

First find the total impedance of the circuit:

$$Z_{eq} = Z_{R1} + \left(Z_{R2} \, /\!/ \, Z_C\right) + Z_L$$

$$= R_1 + \left(\frac{1}{R_2} + \frac{1}{\frac{1}{i\omega C}} \right)^{-1} + i\omega L$$

I'll use Mathematica to help with the algebra:

In[1]:= **Z = R1 + (1 / R2 + 1 / (I w C)) ^ -1 + I w L // ComplexExpand**

Out[1]= $R1 + \dfrac{1}{R2 \left(\frac{1}{R2^2} + \frac{1}{c^2 w^2}\right)} + \mathtt{i}\left(\dfrac{1}{C\left(\frac{1}{R2^2} + \frac{1}{c^2 w^2}\right)w} + L\,w\right)$

It needs to be put into polar form in order to find its magnitude and phase. The polar coordinates can easily be found in Mathematica with these commands:

$$|Z| = \text{Abs}[Z]$$
$$\text{and} \qquad \phi = \text{Arg}[Z]$$

Then the current is given by $I = \frac{V_0}{|Z|}\cos(\omega t - \phi)$ for that $|Z|$ and ϕ. It's probably not worth the space to write out the full-blown answer.