Advanced Circuits Topics – Part 2
by Dr. Colton (Fall 2017)

Part 2: Some Possibly New Things

These are some topics that you may or may not have learned in Physics 220 and/or 145. This handout continues where Part 1 leaves off. We will continue to use complex numbers to represent oscillating voltages and currents, as well as impedances for circuit elements that create phase shifts.

<u>Outline:</u>

1. Power in AC problems
2. Transfer Functions
3. Voltage Dividers, Real and Complex
4. Specific Applications
    a. High Pass Filter
    b. Low Pass Filter
    c. Band Pass Filter

1. Power in AC problems

Instantaneous Power: The instantaneous power supplied by a power supply (or consumed by a circuit element) at time *t* is $P(t) = Re\{V(t)\}Re\{I(t)\}$ —you must multiply the *actual* voltage and *actual* currents together to get the *actual* power. That is, you must take the real parts of the complex voltage and complex current *before* multiplying them together.

For example, if the voltage supplied by a power supply is given by $1\text{V} \angle 0$ and the current is given by $0.2\text{A} \angle -0.927$, then the power as a function of time is:

$$P(t) = \text{actual } V \times \text{actual } I$$
$$P(t) = 1\text{V}\cos(\omega t) \;\; 0.2\text{A}\cos(\omega t - 0.927)$$
$$P(t) = 0.2\text{W}\cos(\omega t)\cos(\omega t - 0.927)$$

Caution: That is *not* the same thing you would get if you multiplied the two complex quantities together before converting to the real values:

$$P(t) \neq \text{complex } V \times \text{complex } I$$
$$P(t) \neq (1\text{V} \angle 0)\,(0.2\text{A} \angle -0.927)$$
$$P(t) \neq 0.2\text{W} \angle -0.927$$
$$P(t) \neq 0.2\text{W}\cos(\omega t - 0.927)$$

As you can see, $0.2\text{W}\cos(\omega t)\cos(\omega t - 0.927)$ (correct answer) is *not* the same as $0.2\text{W}\cos(\omega t - 0.927)$ (incorrect answer)! So taking the real parts before multiplying is critical here.

Average Power: Power as a function of time is often not a quantity of interest. Typically a more useful quantity is the *average* rather than instantaneous power. The power averaged over time is often written as <*P*>, and is given by any of the following formulas, where ϕ is the complex phase angle of the circuit's complex impedance *Z*:

$$\boxed{\langle P\rangle = \frac{1}{2}V_0 I_0 \cos\phi = \frac{1}{2}\left(\frac{V_0^{\ 2}}{|Z|}\right)\cos\phi}$$

(use the real amplitudes V_0 and I_0)

$$\boxed{\langle P\rangle = V_{rms} I_{rms} \cos\phi}$$

(use the real rms values, where e.g. $V_{rms} = \frac{V_0}{\sqrt{2}}$)

$$\boxed{\langle P\rangle = \mathrm{Re}\left\{\frac{1}{2}VI^*\right\}}$$

(use the complex V and I; I^* means the complex conjugate of I)

The first equation can be proved like this:

$$\begin{aligned}
P(t) &= Re\{V(t)\}Re\{I(t)\} \\
&= V_0 I_0 \cos(\omega t)\cos(\omega t - \phi) \\
&= V_0 I_0 \cos(\omega t)\,(\cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi)) \\
&= V_0 I_0 (\cos^2(\omega t)\cos(\phi) + \sin(\omega t)\cos(\omega t)\sin(\phi)) \\
\langle P\rangle &= V_0 I_0 \left(\frac{1}{2}\cos(\phi) + 0\sin(\phi)\right) \\
\langle P\rangle &= \frac{1}{2}V_0 I_0 \cos\phi
\end{aligned}$$

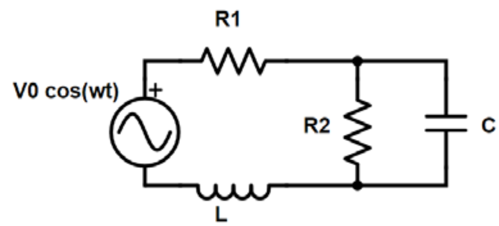
In the second-to-last step, the factor of $\frac{1}{2}$ arises from the time averaging of $\cos^2(\omega t)$ and the factor of 0 arises from the time averaging of $\sin(\omega t)\cos(\omega t)$.

The derivation of the others is left as an exercise for the reader.

Note that:

- $\cos\phi$ is often called the **power factor** of the circuit
- ϕ itself is sometimes called the "power angle" of the circuit

Worked problem 1



In this circuit, suppose $V_0 = 1$ V, $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $C = 0.1$ μF, and $L = 0.1$ mH. What is the average power supplied by the power supply as a function of ω? Make a plot of $\langle P\rangle$ vs ω.

Solution: I'll use Mathematica to make my life easy.

```
In[1]:= Z[w_] =
  R1 + (1/R2 + 1/(I w C))^-1 + I w L  /.
   {R1 → 10, R2 → 20, C -> 0.1*^-6, L -> 0.1*^-3} // ComplexExpand
```

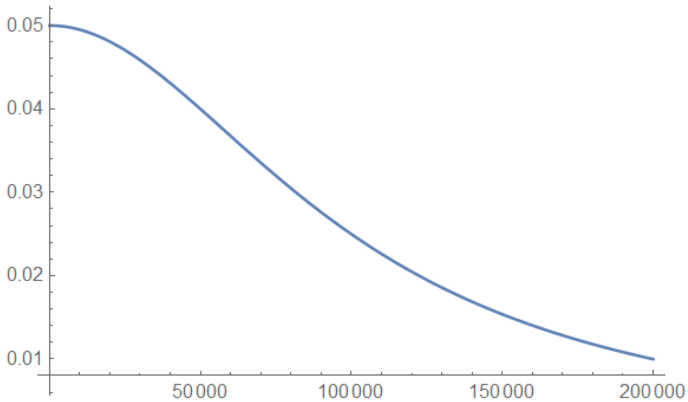
$$\text{Out[1]=}\ 10. + \frac{0.05}{0.0025 + \left(0. - \frac{1.\times10^7}{w}\right)^2} + i\left(0. + \frac{1.\times10^7}{\left(0.0025 + \left(0. - \frac{1.\times10^7}{w}\right)^2\right)w} + 0.0001\,w\right)$$

```
In[2]:= V0 = 1;
I0[w_] = V0 / Abs[Z[w]];
power[w_] = 1 / 2 V0 I0[w] Cos[Arg[Z[w]]]
Plot[power[w], {w, 0, 200000}, PlotRange → All]
```

Out[4]=

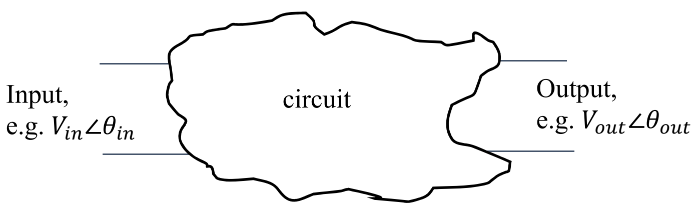
$$\frac{\mathrm{Cos}\left[\mathrm{Arg}\left[10.+\frac{0.05}{0.0025+\left(0.-\frac{1.\times10^{7}}{w}\right)^{2}}+i\left(0.+\frac{1.\times10^{7}}{\left(0.0025+\left(0.-\frac{1.\times10^{7}}{w}\right)^{2}\right)w}+0.0001\,w\right)\right]\right]}{2\,\mathrm{Abs}\left[10.+\frac{0.05}{0.0025+\left(0.-\frac{1.\times10^{7}}{w}\right)^{2}}+i\left(0.+\frac{1.\times10^{7}}{\left(0.0025+\left(0.-\frac{1.\times10^{7}}{w}\right)^{2}\right)w}+0.0001\,w\right)\right]}$$

Out[5]=



2. Transfer Functions

All circuits with resistors, inductors, and capacitors will have frequency dependence. Often such circuits are used to enhance or reduce voltages at various frequencies. Phase shifts can also be induced, whether deliberately or not. The frequency dependence of a circuit is characterized by what is called its "transfer function", sometimes given the symbol H or $\tilde{H}$. Here I'm using Griffith's convention that you'll see later in the book, of putting a tilde over functions that represent complex numbers if/when a reminder of their complex nature is helpful. Conceptually, the situation is like this:



The effect of the circuit is indicated by describing how much the amplitude of output gets increased or decreased relative to the amplitude of the input, and by describing how the phase of the output gets changed relative to the phase of the input. The former is $\frac{V_{out}}{V_{in}}$; the latter is $\theta_{out} - \theta_{in}$.

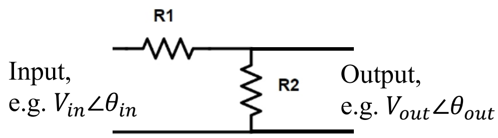
The two effects can be combined in one complex function indicating the ratio of output to input:

$$\boxed{\tilde{H}(\omega) = \frac{V_{out}\angle\theta_{out}}{V_{in}\angle\theta_{in}} = \frac{V_{out}}{V_{in}}\angle(\theta_{out} - \theta_{in})}$$

The transfer function $\widetilde{H}$ is a function of ω since the amplitude and phase of the output will depend on ω. For a given ω it's just a regular complex number; the magnitude and phase of $\widetilde{H}$ give you $\frac{V_{out}}{V_{in}}$ and $\theta_{out} - \theta_{in}$, respectively.

3. Voltage Divider

Most important filter circuits can be represented as voltage dividers. The voltage divider circuit is this:



It is easily shown that as long as no current flows to the output wires, which is the case if e.g. the circuit the output wires are connecting to has a high input impedance, then

$$V_{out} = \frac{R_2}{R_1 + R_2} V_{in}$$

Or, written more generally for complex inputs, outputs, and impedances, we have:

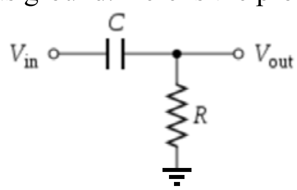
$$\tilde{V}_{out} = \frac{Z_2}{Z_1 + Z_2} \tilde{V}_{in}$$

This is the voltage divider equation, and the transfer function is easily read off as $\frac{Z_2}{Z_1+Z_2}$ which is complex in general and a function of ω.

4. Specific Applications (aka Worked Problems)

4a. <u>High Pass Filter</u>

Using yet another notational shortcut, I will eliminate the lower "input" and "output" lines (which are connected), as that voltage is defined as ground. Here is the prototypical high pass filter circuit.



The calculation of the transfer function is like this:

$$\widetilde{H} = \frac{Z_2}{Z_1 + Z_2}$$

$$\widetilde{H} = \frac{R}{R - \frac{i}{\omega C}} \times \frac{\omega C}{\omega C}$$

$$\widetilde{H} = \frac{R\omega C}{R\omega C - i}$$

Making note that $1/(RC)$ has units of ω and giving it the special symbol $\omega_{3dB} = \frac{1}{RC}$ for reasons that I'll explain momentarily, we can rewrite the transfer function in this form:

$$\boxed{\widetilde{H} = \frac{\omega/\omega_{3dB}}{\omega/\omega_{3dB} - i}}$$

Why is $1/(RC)$ called ω_{3dB}? Well, at that particular frequency we have:

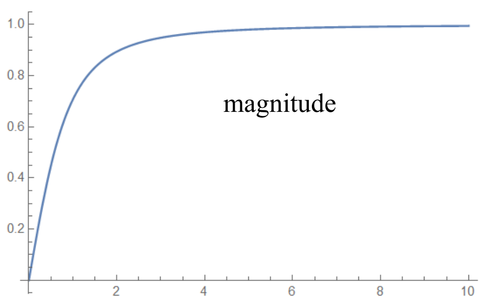
$$\widetilde{H}(\omega = \omega_{3dB}) = \frac{1}{1-i} = \frac{1}{\sqrt{2}}\angle{-90°}$$

In other words, at that particular frequency the amplitude is equal to $\frac{1}{\sqrt{2}}$ of the initial amplitude, and since power goes as amplitude squared that means the power has decreased to 50% of the initial power. "3dB" stands for "3 decibels", which by definition equals $\log_{10} 3 \approx 0.5$, so the frequency where the power decreases to 50% is universally called the 3dB point and given the symbol ω_{3dB}.

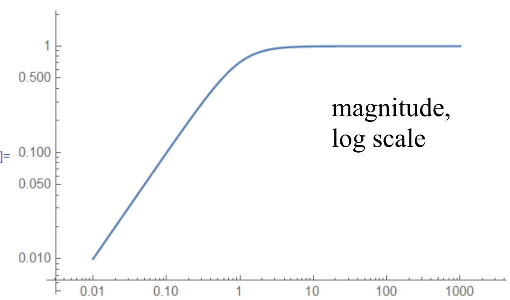
I will use Mathematica to plot the magnitude and phase of the transfer function for $\omega_{3dB} = 1$. Or, you can think of the x-axis as representing ω/ω_{3dB} if you'd like. The plots on the right have log scales for one or both axes. It's a high pass filter because the transfer function magnitude goes to 0 at low frequencies and 1 at high frequencies. The circuit prevents low frequencies from passing through, but passes high frequencies.

```
In[1]:= H[w_] = w / (w - I)  (* setting w3dB=1 for plotting *)
Out[1]= w/(-i + w)
```

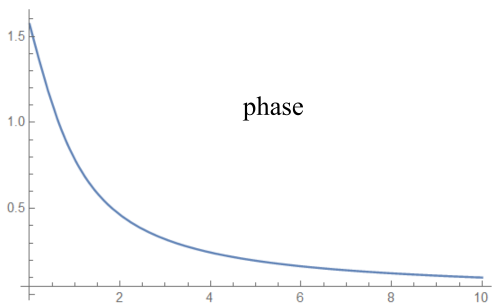
```
In[2]:= Plot[Abs[H[w]], {w, 0, 10}, PlotRange → All]
Out[2]=
```



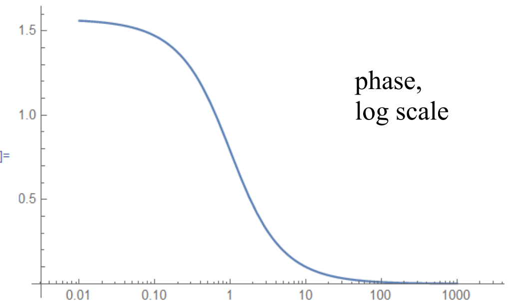

```
In[3]:= LogLogPlot[Abs[H[w]], {w, .01, 1000}, PlotRange → All]
Out[3]=
```



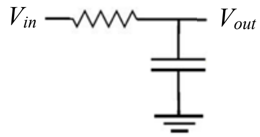

```
In[4]:= Plot[Arg[H[w]], {w, 0, 10}, PlotRange → All]
Out[4]=
```




```
In[5]:= LogLinearPlot[Arg[H[w]], {w, 0.01, 1000}, PlotRange → All]
Out[5]=
```



4b. Low Pass Filter



There is the prototypical low pass filter circuit, and a calculation of the transfer function using $\omega_{3dB} = \frac{1}{RC}$ again.

$$\widetilde{H} = \frac{Z_2}{Z_1 + Z_2}$$

$$\widetilde{H} = \frac{-\frac{i}{\omega C}}{R - \frac{i}{\omega C}} \times \frac{-\omega C}{-\omega C}$$

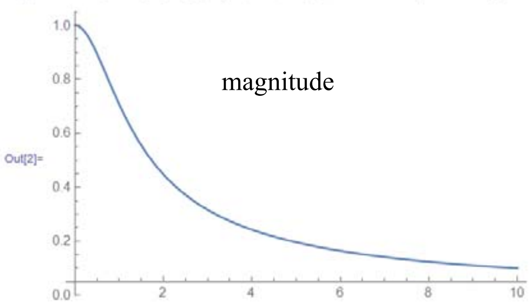
$$\widetilde{H} = \frac{i}{-R\omega C + i}$$

$$\widetilde{H} = \frac{i}{i - \omega/\omega_{3dB}}$$

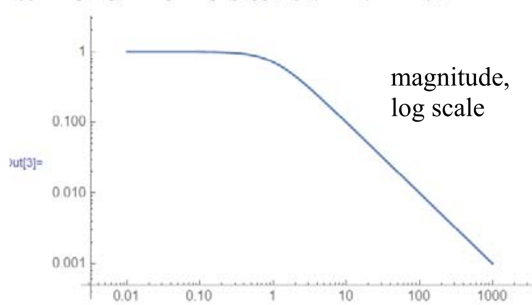
Here are plots of the transfer function for the low pass filter, again setting $\omega_{3dB} = 1$ for plotting purposes.

```
In[1]:= H[w_] = I / (I - w)  (* setting w3dB=1 for plotting *)
Out[1]= i / (i - w)
```

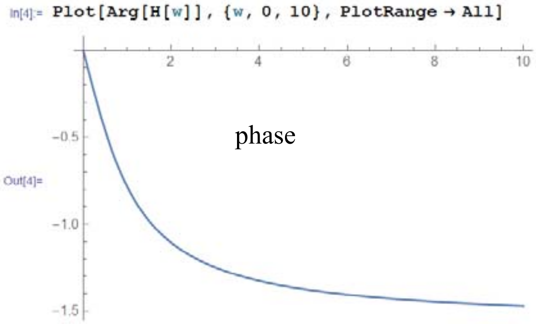
```
In[2]:= Plot[Abs[H[w]], {w, 0, 10}, PlotRange → All]
```



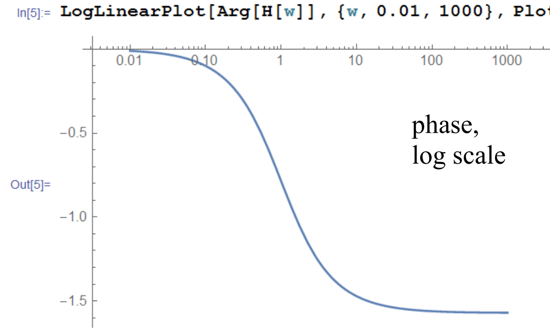

```
In[3]:= LogLogPlot[Abs[H[w]], {w, .01, 1000}, PlotRange → All]
```




```
In[4]:= Plot[Arg[H[w]], {w, 0, 10}, PlotRange → All]
```



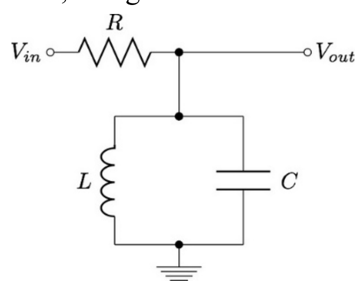

```
In[5]:= LogLinearPlot[Arg[H[w]], {w, 0.01, 1000}, PlotRange → All]
```



It's a low pass filter because the transfer function magnitude goes to 0 at high frequencies and 1 at low frequencies. The circuit prevents high frequencies from passing through, but passes low frequencies.

4c. Band Pass Filter

Here is the simplest band pass filter circuit, along with calculation of the transfer function.



In this case I've written the equations in terms of the special frequency $\omega_0 = \frac{1}{\sqrt{LC}}$. You'll see the term $Z_L//Z_C$ show up. It's left as an exercise for the reader (easily done with Mathematica) to show that that term is equal to $\frac{i\omega L}{1-\omega^2 LC} = \frac{i\omega L}{1-\omega^2/\omega_0^2}$.

$$\widetilde{H} = \frac{Z_2}{Z_1 + Z_2}$$

$$\widetilde{H} = \frac{Z_L//Z_C}{R + Z_L//Z_C}$$

$$\widetilde{H} = \frac{\frac{i\omega L}{1-\omega^2/\omega_0^2}}{R + \frac{i\omega L}{1-\omega^2/\omega_0^2}} \times \frac{1-\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}$$

$$\boxed{\widetilde{H} = \frac{i\omega L}{R\left(1 - \frac{\omega^2}{\omega_0^2}\right) + i\omega L}}$$

Before I plot this, let's look at what happens when $\omega = \omega_0$:

$$\widetilde{H}(\omega = \omega_0) = \frac{i\omega_0/(R/L)}{1 - 1 + i\omega_0/(R/L)} = 1$$

At that frequency, called the resonant frequency, the output voltage is equal to the input voltage. As you can see in the plots below, as you go away from that frequency the output voltage gets suppressed. This circuit allows a particular band of frequencies to pass through but blocks all other frequencies.

Here are some plots for *R* = 100 kΩ, *C* = 10 nF, and *L* = 10 mH (which gives $\omega_0 = 100{,}000$ rad/s).

In[1]:=
```
ZC = -I / (w C);
ZL = I w L;
Ztot[w_] = R + (1 / ZC + 1 / ZL) ^ -1;
Z2[w_] = (1 / ZC + 1 / ZL) ^ -1;
H[w_] = Z2[w] / Ztot[w] // Simplify
```

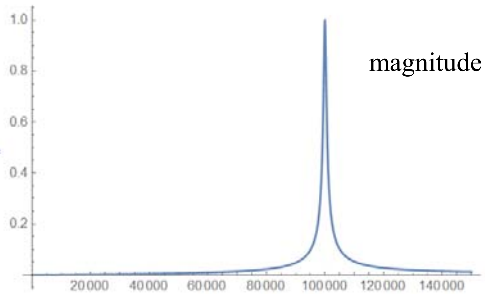
Out[5]= $\frac{i L w}{R + i L w - C L R w^2}$

In[6]:=
```
1 / Sqrt[L C] /. {C → 10*^-9, L → 10*^-3}
```

Out[6]= 100 000

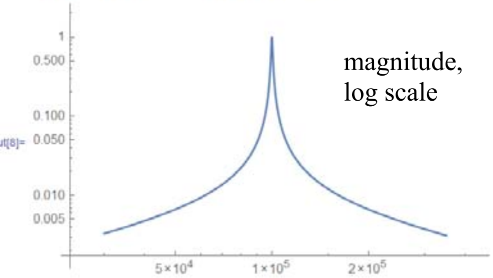
In[7]:=
```
Plot[Abs[H[w] /. {R → 100*^3, C → 10*^-9, L → 10*^-3}],
 {w, 0, 150 000}, PlotRange → All]
```

Out[7]=



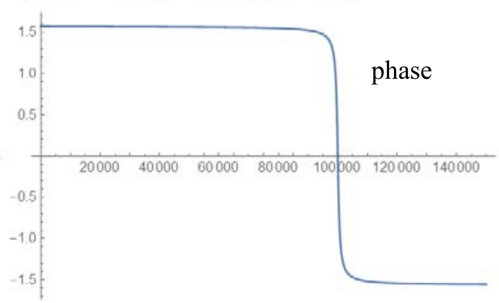
In[8]:=
```
LogLogPlot[Abs[H[w] /. {R → 100*^3, C → 10*^-9, L → 10*^-3}],
 {w, 30 000, 350 000}, PlotRange → All]
```

Out[8]=



In[9]:=
```
Plot[Arg[H[w] /. {R → 100*^3, C → 10*^-9, L → 10*^-3}],
 {w, 0, 150 000}, PlotRange → All]
```

Out[9]=



In[10]:=
```
LogLinearPlot[Arg[H[w] /. {R → 100*^3, C → 10*^-9, L → 10*^-3}],
 {w, 95 000, 105 000}, PlotRange → All]
```

Out[10]=