

Advanced Circuits Topics Part 2

by Dr. Colton, Physics 441 (last updated: Fall 2019)

Part 2: Power, Transfer Functions, and Applications

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2. Power in AC problems
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4. Voltage Dividers, Real and Complex
5. Specific Applications
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1. Review of Advanced Circuits Topics Part 1

In *Advanced Circuits Topics Part 1*, we learned how to use complex numbers to represent:

- voltages and currents that oscillate sinusoidally with phase shifts
- impedances for circuit elements that create phase shifts

Summary: The way to represent sinusoidally oscillating voltages as complex numbers is via an implied time dependence of $e^{i\omega t}$. The angle of the complex number is the phase of the voltage or current.

$$\begin{aligned} V_0 \cos(\omega t + \phi_1) &\leftrightarrow V_0 e^{i(\omega t + \phi_1)} \leftrightarrow V_0 \angle \phi_1 \\ I_0 \cos(\omega t + \phi_2) &\leftrightarrow I_0 e^{i(\omega t + \phi_2)} \leftrightarrow I_0 \angle \phi_2 \end{aligned}$$

The phase shifts induced by capacitors and inductors between their voltages and currents can be incorporated into complex impedances, which allows for Ohm's law to be satisfied for each circuit element (as well as the circuit as a whole), and also allows the circuit elements to be added in series and in parallel using the standard resistor formulas.

$$\Delta V = IZ$$

$$Z_R = R$$

$$Z_C = \frac{1}{i\omega C} = -\frac{i}{\omega C}$$

$$Z_L = i\omega L$$

$$\text{Series: } Z_{total} = Z_1 + Z_2$$

$$\text{Parallel: } Z_{total} = (1/Z_1 + 1/Z_2)^{-1} = Z_1 // Z_2$$

The actual time dependence of any/all voltages and currents can be made explicit by adding back in the implied $e^{i\omega t}$ and taking the real part.

2. Power in AC problems

Instantaneous Power: The instantaneous power supplied by a power supply (or consumed by a circuit element) at time t is $P(t) = \text{Re}\{V(t)\}\text{Re}\{I(t)\}$ —you must multiply the *actual* voltage and *actual* currents together to get the *actual* power. That is, you must take the real parts of the complex voltage and complex current *before* multiplying them together.

For example, if the voltage supplied by a power supply is given by $1V \angle 0$ and the current is given by $0.2A \angle -0.927$, then the power as a function of time is:

$$\begin{aligned} P(t) &= \text{actual } V \times \text{actual } I \\ P(t) &= 1V \cos(\omega t) \cdot 0.2A \cos(\omega t - 0.927) \\ P(t) &= 0.2W \cos(\omega t) \cos(\omega t - 0.927) \end{aligned}$$

Caution: That is *not* the same thing you would get if you multiplied the two complex quantities together before converting to the real values:

$$\begin{aligned} P(t) &\neq \text{complex } V \times \text{complex } I \\ P(t) &\neq (1V \angle 0) (0.2A \angle -0.927) \\ P(t) &\neq 0.2W \angle -0.927 \\ P(t) &\neq 0.2W \cos(\omega t - 0.927) \end{aligned}$$

As you can see, $0.2W \cos(\omega t) \cos(\omega t - 0.927)$ (correct answer) is *not* the same as $0.2W \cos(\omega t - 0.927)$ (incorrect answer)! So taking the real parts before multiplying is critical here.

Average Power: Typically a more useful quantity than instantaneous power as a function of time is the *average* power that a battery provides or circuit element consumes. The power averaged over time is often written as $\langle P \rangle$, and is given by any of the following formulas, where ϕ is the complex phase angle of the circuit's complex impedance Z :

$$\begin{aligned} \boxed{\langle P \rangle = \frac{1}{2} V_0 I_0 \cos \phi_{tot} = \frac{1}{2} \frac{V_0^2}{|Z|} \cos \phi_{tot}} & \quad (\text{use the real amplitudes } V_0 \text{ and } I_0) \\ \boxed{\langle P \rangle = V_{rms} I_{rms} \cos \phi_{tot}} & \quad (\text{use the real RMS amplitudes, where e.g. } V_{rms} = \frac{V_0}{\sqrt{2}}) \\ \boxed{\langle P \rangle = \text{Re} \left\{ \frac{1}{2} V I^* \right\}} & \quad (\text{use the complex } V \text{ and } I; I^* \text{ is complex conjugate of } I) \end{aligned}$$

The first equation can be proved like this:

$$\begin{aligned} P(t) &= \text{Re}\{V(t)\} \text{Re}\{I(t)\} \\ &= V_0 I_0 \cos(\omega t) \cos(\omega t - \phi_{tot}) \\ &= V_0 I_0 \cos(\omega t) (\cos(\omega t) \cos(\phi_{tot}) + \sin(\omega t) \sin(\phi_{tot})) \\ &= V_0 I_0 (\cos^2(\omega t) \cos(\phi_{tot}) + \sin(\omega t) \cos(\omega t) \sin(\phi_{tot})) \\ \langle P \rangle &= V_0 I_0 \left(\frac{1}{2} \cos(\phi_{tot}) + 0 \sin(\phi_{tot}) \right) \\ \langle P \rangle &= \frac{1}{2} V_0 I_0 \cos \phi_{tot} \end{aligned}$$

In the second-to-last step, the factor of $\frac{1}{2}$ arises from the time averaging of $\cos^2(\omega t)$ and the factor of 0 arises from the time averaging of $\sin(\omega t) \cos(\omega t)$.

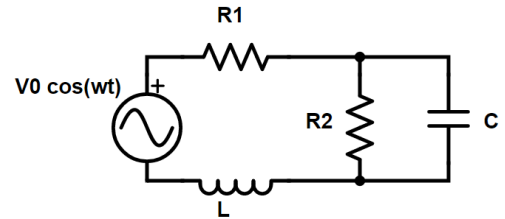
The derivation of the others is left as an exercise for the reader.

Due to the second boxed equation above:

- $\cos \phi_{tot}$ is often called the **power factor** of the circuit
- ϕ_{tot} (which you recall is the phase angle of the total complex impedance) is often called the **power angle** of the circuit

Worked Problem

In this circuit, suppose $V_0 = 1$ V, $R_1 = 10$ Ω , $R_2 = 20$ Ω , $C = 0.1$ μ F, and $L = 0.1$ mH. What is the average power supplied by the power supply as a function of ω ? Make a plot of $\langle P \rangle$ vs ω .



Solution: I'll use Mathematica to make my life easy.

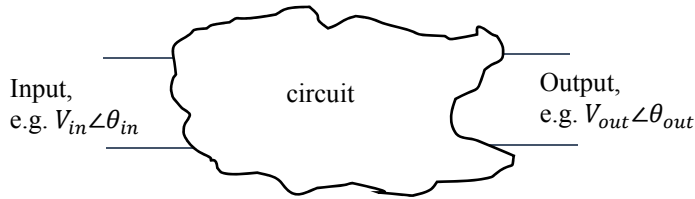
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In[1]:= Z[w_] =
  R1 + (1/R2 + I w C)^-1 + I w L /.
  {R1 -> 10, R2 -> 20, C -> 0.1*^-6, L -> 0.1*^-3} // ComplexExpand
Out[1]= 10. +  $\frac{0.05}{0.0025 + (0. + 1. \times 10^{-7} w)^2}$  + i  $\left( 0. + 0.0001 w - \frac{1. \times 10^{-7} w}{0.0025 + (0. + 1. \times 10^{-7} w)^2} \right)$ 
In[2]:= V0 = 1;
  I0[w_] = V0 / Abs[Z[w]];
  power[w_] = 1 / 2 V0 I0[w] Cos[Arg[Z[w]]]
  Plot[power[w], {w, 0, 1 000 000}, PlotRange -> All]
Out[4]=  $\frac{\text{Cos}\left[\text{Arg}\left[10. + \frac{0.05}{0.0025 + (0. + 1. \times 10^{-7} w)^2} + i \left( 0. + 0.0001 w - \frac{1. \times 10^{-7} w}{0.0025 + (0. + 1. \times 10^{-7} w)^2} \right)\right]\right]}{2 \text{Abs}\left[10. + \frac{0.05}{0.0025 + (0. + 1. \times 10^{-7} w)^2} + i \left( 0. + 0.0001 w - \frac{1. \times 10^{-7} w}{0.0025 + (0. + 1. \times 10^{-7} w)^2} \right)\right]}$ 
Out[5]=

```

3. Transfer Functions

All circuits with resistors, inductors, and capacitors will have frequency dependence. Often such circuits are used to enhance or reduce voltages at various frequencies. Phase shifts can also be induced, whether deliberately or not. The frequency dependence of a circuit is characterized by what is called its “transfer function”, sometimes given the symbol H or \tilde{H} . Here I’m using Griffith’s convention that you’ll see later in the book, of putting a tilde over functions that represent complex numbers if/when a reminder of their complex nature is helpful. Conceptually, the situation is like this:



The effect of the circuit is indicated by describing how much the amplitude of output gets increased or decreased relative to the amplitude of the input, and by describing how the phase of the output gets changed relative to the phase of the input. The former is V_{out}/V_{in} ; the latter is $\theta_{out} - \theta_{in}$.

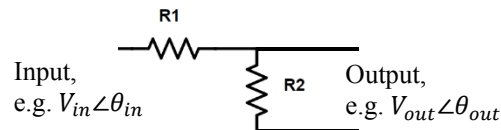
The two effects can be combined in one complex function indicating the ratio of output to input:

$$\tilde{H}(\omega) = \frac{V_{out} \angle \theta_{out}}{V_{in} \angle \theta_{in}} = \frac{V_{out}}{V_{in}} \angle (\theta_{out} - \theta_{in})$$

The transfer function \tilde{H} is a function of ω since the amplitude and phase of the output will depend on ω . For a given ω the transfer function \tilde{H} is a single complex number; its magnitude tells you the ratio of V_{out}/V_{in} and its phase tells you $\theta_{out} - \theta_{in}$ (for that ω).

4. Voltage Divider

Most of the important filter circuits can be represented as voltage dividers. The basic voltage divider circuit for resistors is this:



It is easily shown that as long as no current flows to the output wires, which is the case if e.g. the circuit the output wires are connecting to has a high input impedance, then

$$V_{out} = \frac{R_2}{R_1 + R_2} V_{in}$$

Or, written more generally for complex inputs, outputs, and impedances, we have:

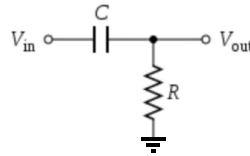
$$\tilde{V}_{out} = \frac{Z_2}{Z_1 + Z_2} \tilde{V}_{in}$$

This is the voltage divider equation, and the transfer function is easily read off as $\frac{Z_2}{Z_1 + Z_2}$ which is complex in general and a function of ω .

5. Specific Applications

(a) High Pass Filter

Using yet another notational shortcut, I will eliminate the lower “input” and “output” lines (which are connected), as that voltage is defined as ground. Here is the prototypical high pass filter circuit.



High pass filter circuit

The calculation of the transfer function is like this:

$$\tilde{H} = \frac{Z_2}{Z_1 + Z_2}$$

$$\tilde{H} = \frac{R}{R - \frac{i}{\omega C}} \times \frac{\omega C}{\omega C}$$

$$\tilde{H} = \frac{R\omega C}{R\omega C - i}$$

Making note that $1/(RC)$ has units of ω and giving it the special symbol $\omega_{3dB} = \frac{1}{RC}$ for reasons that I'll explain momentarily, we can rewrite the transfer function in this form:

$$\tilde{H} = \frac{\omega/\omega_{3dB}}{\omega/\omega_{3dB} - i}$$

High pass filter transfer function

Why is $1/(RC)$ called ω_{3dB} ? Well, at that particular frequency we have:

$$\tilde{H}(\omega = \omega_{3dB}) = \frac{1}{1 - i} = \frac{1}{\sqrt{2}} \angle -90^\circ$$

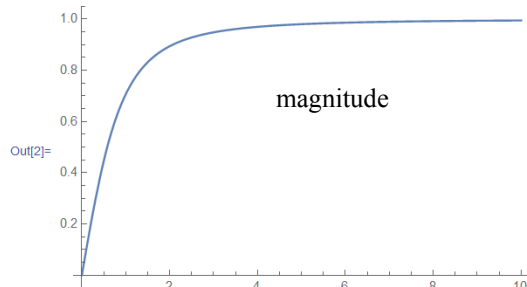
In other words, the amplitude at that particular frequency is equal to $1/\sqrt{2}$ of the initial amplitude, and since power goes as amplitude squared, the output power is decreased to 50% of the input power. "3dB" stands for "3 decibels", which by definition equals $\log_{10} 3 \approx 0.5$, so the frequency where the power decreases to 50% is universally called the 3dB point and given the symbol ω_{3dB} .

I will use Mathematica to plot the magnitude and phase of the transfer function for $\omega_{3dB} = 1$ rad/s. Or, you can think of the x-axis as representing ω/ω_{3dB} if you'd like. It's called a high pass filter circuit because the transfer function magnitude goes to 0 at low frequencies and 1 at high frequencies. The circuit prevents low frequencies from getting through, but passes high frequencies.

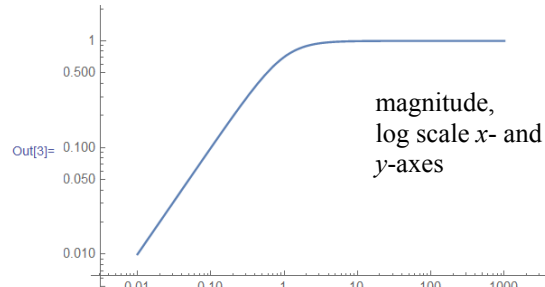
```
In[1]= H[w_] = w / (w - I) (* setting w3dB=1 for plotting *)
```

```
Out[1]=  $\frac{w}{-i + w}$ 
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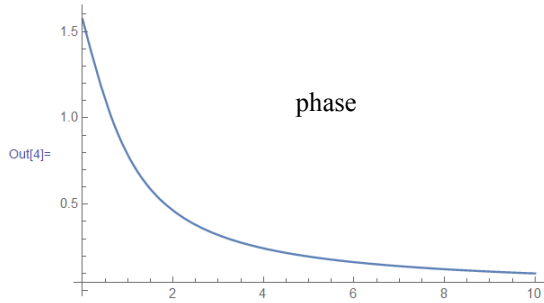
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In[2]= Plot[Abs[H[w]], {w, 0, 10}, PlotRange -> All]
```



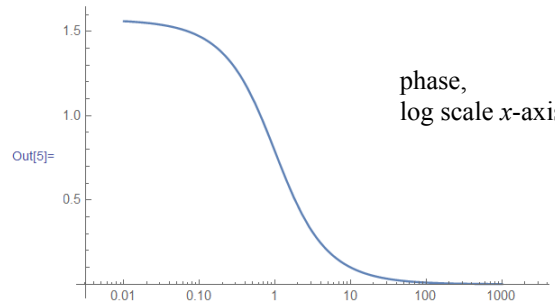
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In[3]= LogLogPlot[Abs[H[w]], {w, .01, 1000}, PlotRange -> All]
```



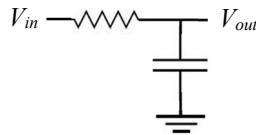
In[4]= Plot[Arg[H[w]], {w, 0, 10}, PlotRange -> All]



In[5]= LogLinearPlot[Arg[H[w]], {w, 0.01, 1000}, PlotRange -> All]



(b) Low Pass Filter



Low pass filter circuit

This is the prototypical low pass filter circuit. Here is a calculation of the transfer function, again using the symbol $\omega_{3dB} = \frac{1}{RC}$.

$$\tilde{H} = \frac{Z_2}{Z_1 + Z_2}$$

$$\tilde{H} = \frac{-\frac{i}{\omega C}}{R - \frac{i}{\omega C}} \times \frac{-\omega C}{-\omega C}$$

$$\tilde{H} = \frac{i}{-R\omega C + i}$$

$$\tilde{H} = \frac{i}{i - \omega/\omega_{3dB}}$$

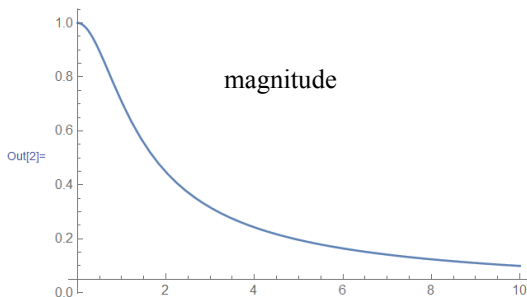
Low pass filter transfer function

Here are plots of the transfer function for the low pass filter with $\omega_{3dB} = 1$ rad/s for plotting purposes. It's called a low pass filter because the transfer function magnitude goes to 0 at high frequencies and 1 at low frequencies. The circuit prevents high frequencies from getting through, but passes low frequencies.

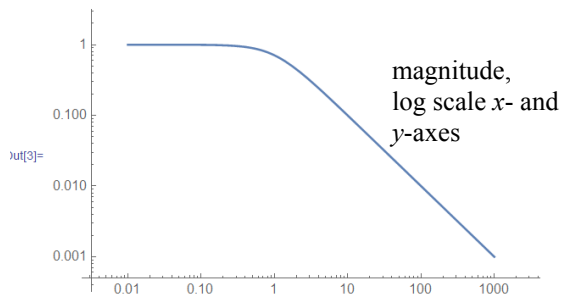
In[1]= H[w_] = I / (I - w) (* setting w3dB=1 for plotting *)

Out[1]= $\frac{i}{1 - w}$

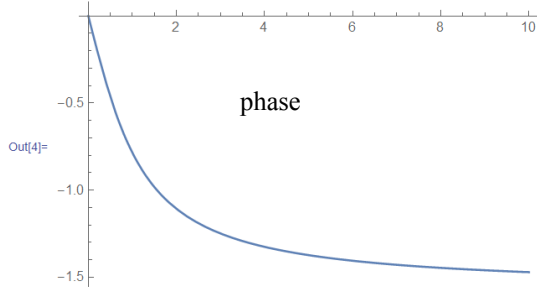
In[2]= Plot[Abs[H[w]], {w, 0, 10}, PlotRange -> All]



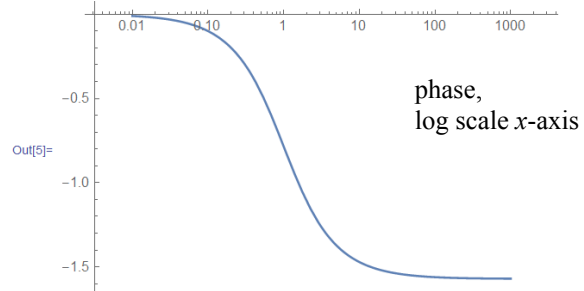
In[3]= LogLogPlot[Abs[H[w]], {w, .01, 1000}, PlotRange -> All]



In[4]= Plot[Arg[H[w]], {w, 0, 10}, PlotRange -> All]

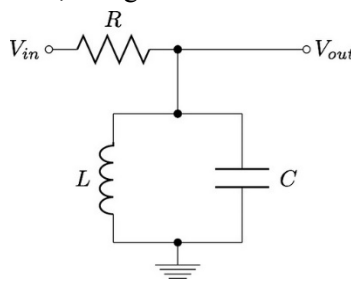


In[5]= LogLinearPlot[Arg[H[w]], {w, 0.01, 1000}, PlotRange -> All]



(c) Band Pass Filter

Here is the simplest band pass filter circuit, along with calculation of its transfer function.



Band pass filter circuit

In this case I've written the equations in terms of the special frequency $\omega_0 = \frac{1}{\sqrt{LC}}$. You'll see the term $Z_L // Z_C$ show up. It's left as an exercise for the reader to show that that $Z_L // Z_C = \frac{i\omega L}{1 - \omega^2 LC} = \frac{i\omega L}{1 - \omega^2/\omega_0^2}$.

$$\tilde{H} = \frac{Z_2}{Z_1 + Z_2}$$

$$\tilde{H} = \frac{Z_L // Z_C}{R + Z_L // Z_C}$$

$$\tilde{H} = \frac{\frac{i\omega L}{1 - \omega^2/\omega_0^2}}{R + \frac{i\omega L}{1 - \omega^2/\omega_0^2}} \times \frac{1 - \omega^2/\omega_0^2}{1 - \omega^2/\omega_0^2}$$

$$\tilde{H} = \frac{i\omega L}{R \left(1 - \frac{\omega^2}{\omega_0^2}\right) + i\omega L}$$

Band pass filter transfer function

Before I plot this, let's look at what happens when $\omega = \omega_0$:

$$\tilde{H}(\omega = \omega_0) = \frac{i\omega_0/(R/L)}{1 - 1 + i\omega_0/(R/L)}$$

$$\tilde{H}(\omega = \omega_0) = 1$$

At that frequency, called the resonant frequency, the output voltage is equal to the input voltage. As you go away from that frequency the output voltage is suppressed. This circuit is called a band pass filter because it allows a particular band of frequencies to pass through, namely those close to ω_0 , but blocks all other frequencies.

Here are some plots for $R = 100 \text{ k}\Omega$, $C = 10 \text{ nF}$, and $L = 10 \text{ mH}$ (which gives $\omega_0 = 100,000 \text{ rad/s}$).

```

In[1]= ZC = -I / (w C);
      ZL = I w L;
      Ztot[w_] = R + (1 / ZC + 1 / ZL) ^ -1;
      Z2[w_] = (1 / ZC + 1 / ZL) ^ -1;
      H[w_] = Z2[w] / Ztot[w] // Simplify

```

$$\text{Out[5]= } \frac{i L w}{R + i L w - C L R w^2}$$

```

In[6]= 1 / Sqrt[ L C ] /. {C -> 10*^-9, L -> 10*^-3}

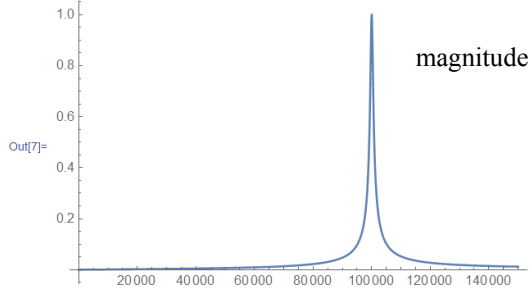
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Out[6]= 100 000

```

In[7]= Plot[Abs[ H[w] ] /. {R -> 100*^3, C -> 10*^-9, L -> 10*^-3}],
      {w, 0, 150 000}, PlotRange -> All]

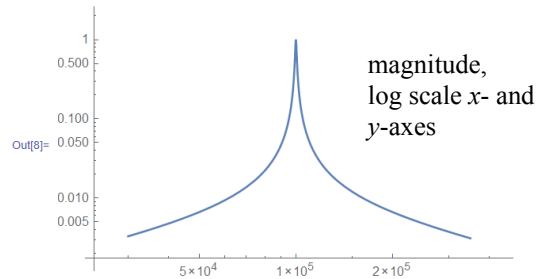
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In[8]= LogLogPlot[Abs[ H[w] ] /. {R -> 100*^3, C -> 10*^-9, L -> 10*^-3}],
      {w, 30 000, 350 000}, PlotRange -> All]

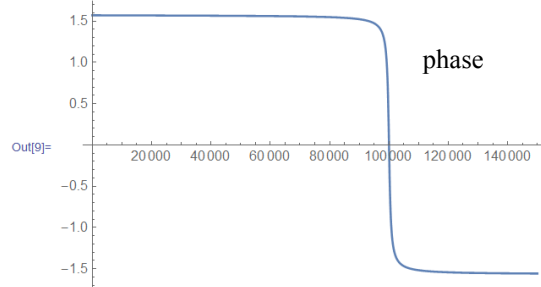
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```

In[9]= Plot[Arg[ H[w] ] /. {R -> 100*^3, C -> 10*^-9, L -> 10*^-3}],
      {w, 0, 150 000}, PlotRange -> All]

```



```

In[10]= LogLinearPlot[Arg[ H[w] ] /. {R -> 100*^3, C -> 10*^-9, L -> 10*^-3}],
      {w, 95 000, 105 000}, PlotRange -> All]

```

