Phys 441 Exam 1 Solns

(20 pts) **Problem 1:** Multiple choice, 2 pts each. Circle the correct answer.

1.1. The electric field lines at a certain location:

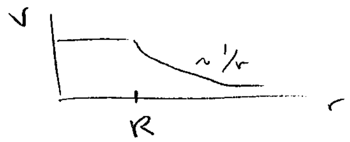
- ✓ I. Never cross
- ~~II.~~ Are always parallel to equipotential lines They are ⊥
- ✓ III. Are always parallel to the electric force on a test charge placed at that location

(a) I
(b) I and II
(c) II and III
(d) I and III
(e) I, II, and III

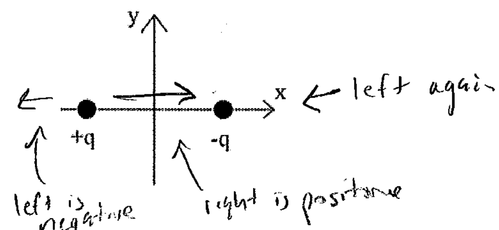
1.2. A solid, conducting sphere of radius R is positively charged. Of the following distances from the center of the sphere, which location will have the greatest electric potential? (Take $V = 0$ at $r \to \infty$)

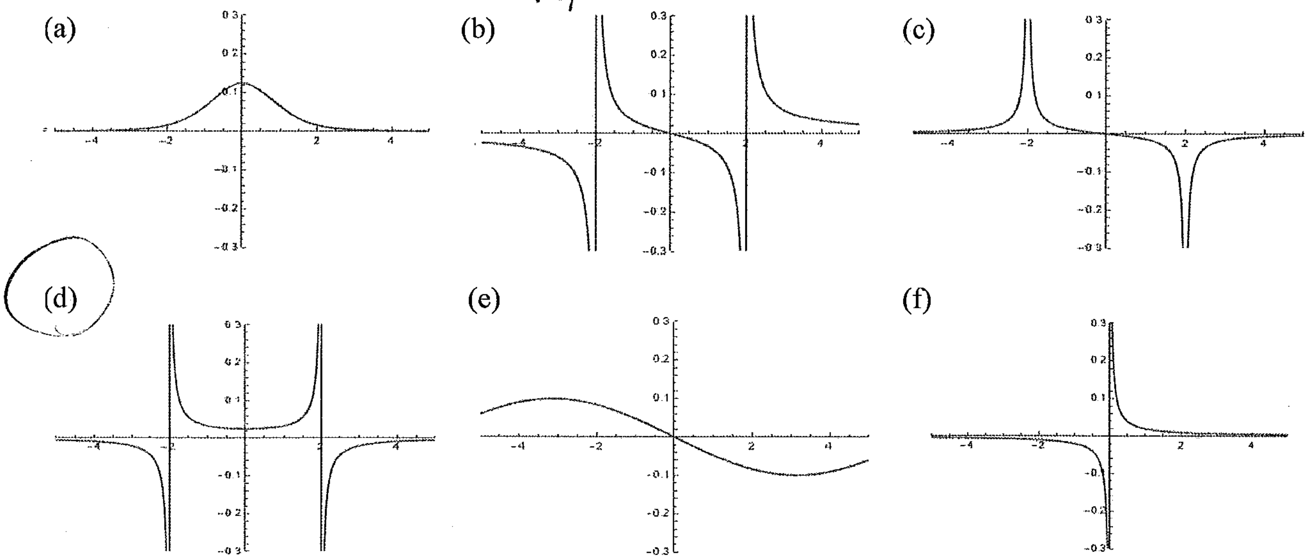
(a) 0 (center of the sphere).
(b) 1.1 R
(c) 1.25 R
(d) 2 R
(e) None of the above because the potential is constant.

potential will be like this:

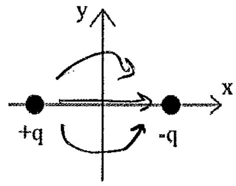


1.3. Two charges are assembled as shown below. Which graph correctly depicts values of E_x for points along the x-axis? (Don't worry about the numbers, just the shape of the graphs.)



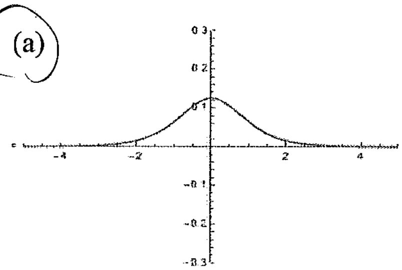


1.4. Same situation. Which graph correctly depicts values of E_x for points along the y-axis?

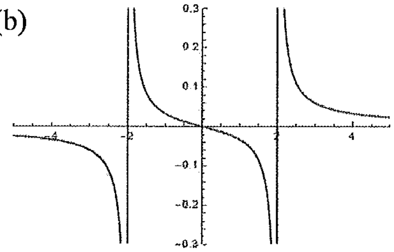


Always right, but strongest right at $y = 0$

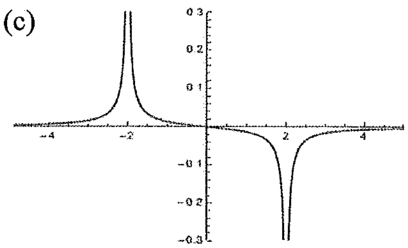
(a)



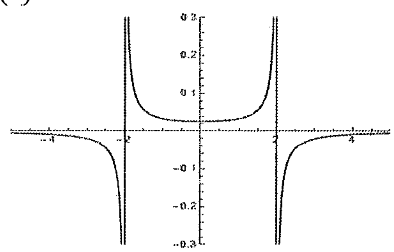
(b)



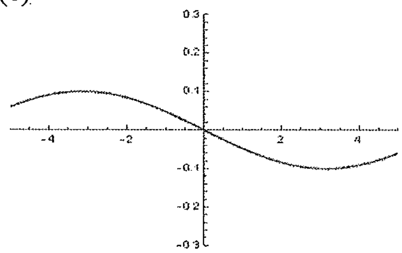
(c)



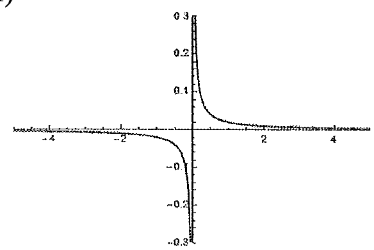
(d)



(e)

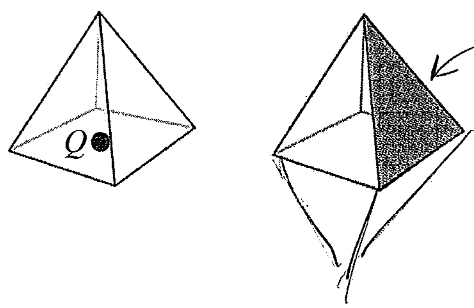


(f)



1.5. A charge Q is placed at the center of the base of a triangular prism (pyramid with five sides as shown). What is the flux in terms of Q through the shaded side of the prism?

(a) 0
(b) $\frac{Q}{\varepsilon_0}$
(c) $\frac{Q}{4\varepsilon_0}$
(d) $\frac{Q}{5\varepsilon_0}$
(e) $\frac{Q}{8\varepsilon_0}$

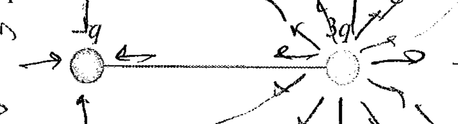


$\frac{1}{8}$ of total flux goes through here (and total flux is Q/ε_0)

1.6. Which of the following does not have symmetry sufficient to use a Gaussian surface to relatively easily find **E**?
(a) Sphere of constant charge density
(b) Infinite plane of constant charge density
(c) Infinite cylinder of constant charge density
(d) Cube of constant charge density   As discussed in class
(e) Infinite line of constant charge density
(f) None of the above (ALL can be solved with a Gaussian surface)

field lines are like this

1.7. Two opposite charges are placed on a line as show in the figure below.



Spot where $E = 0$

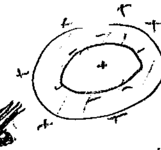
The charge on the right is three times the magnitude of the charge on the left. Besides infinity, where else can the electric field ~~possibly be zero?~~

(a) Between the two charges
(b) To the right of the charge on the right
(c) To the left of the charge on the left
(d) Nowhere else can it be zero

1.8. Which of the following conditions allows the electric field to be written as $\mathbf{E} = -\nabla V$, where V is the electrostatic potential?

(a) $\nabla \cdot E = 0$
(b) $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$
(c) $\nabla \times E = 0$ gradient field $\Leftrightarrow$ curl-less field
(d) $F = QE$
(e) $\phi_E = \int_S E \cdot da$

1.9. A conducting shell is placed around the origin. Which of the following is true?



induced charges will result in field outside the shell

(this is a "Faraday cage")

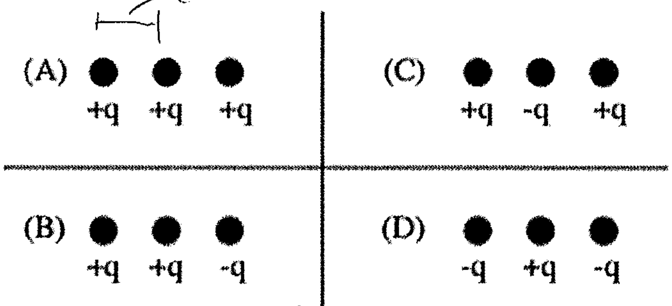
~~I.~~ Charges inside the shell will not create electric fields outside the shell.
II. Charges outside the shell will not create electric fields inside the shell.

(a) I only
(b) II only
(c) Both I and II
(d) Neither I nor II

electric field in conductor will be zero, so inner surface and whole inside will be an equipotential, and $\vec{E}\,(=-\nabla V)$ will be zero

1.10. Rank the work needed to assemble the charge distributions below (i.e. moving the charges in from infinity to be next to each other), from least to greatest. In all cases, the separation between the middle charge and the outer charges are the same. call distance one unit

(a) A < B < C = D
(b) C = D < B < A
(c) A < B = C = D
(d) B = C = D < A
(e) D < C = B < A



$$PE = \frac{1}{4\pi\varepsilon_0} \sum_{\text{all pairs}} \frac{q_i q_j}{r_{ij}}$$

A) $PE \sim \frac{q^2}{1} + \frac{q^2}{1} + \frac{q^2}{2} = q^2(1+1+\frac{1}{2}) = 2\frac{1}{2} q^2$

B) $PE \sim \frac{q^2}{1} + \frac{-q^2}{1} + \frac{-q^2}{2} = q^2(1-1-\frac{1}{2}) = -\frac{1}{2} q^2$

c) $PE \sim -\frac{q^2}{1} + \frac{-q^2}{1} + \frac{q^2}{2} = q^2(-1-1+\frac{1}{2}) = -1\frac{1}{2} q^2$

D) $PE \sim -\frac{q^2}{1} - \frac{q^2}{1} + \frac{q^2}{2} = q^2(-1-1+\frac{1}{2}) = -1\frac{1}{2} q^2$

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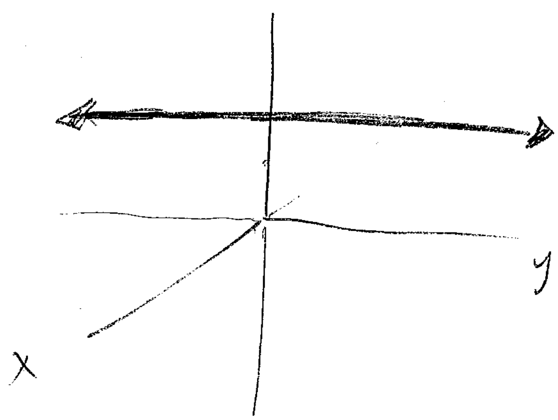
(~~5~~ pts) **Problem 2**: A volume charge density is given by $\rho(x, y, z) = c\,\delta(x)\,\delta(z-2)$

x must be 0

z must be 2

(y can be anything)

(a) Sketch the charge distribution and also describe it in a few words.



This is a line of charge extending from $y = -\infty$ to $+\infty$, fixed at $x = 0$ and $z = 2$.

dx has units of m

$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$ means $\delta(x)$ has units of m^{-1}

(b) What are the units of c? (Hint: the delta function has units. If you don't know them offhand, you'll need to figure them out.)

$\rho = c\,\delta(x)\,\delta(z-2)$

units: $\frac{C}{m^3} = [c]\,\frac{1}{m}\cdot\frac{1}{m}$

$\boxed{\text{units of } c \text{ are } \frac{\text{Coulomb}}{m}}$

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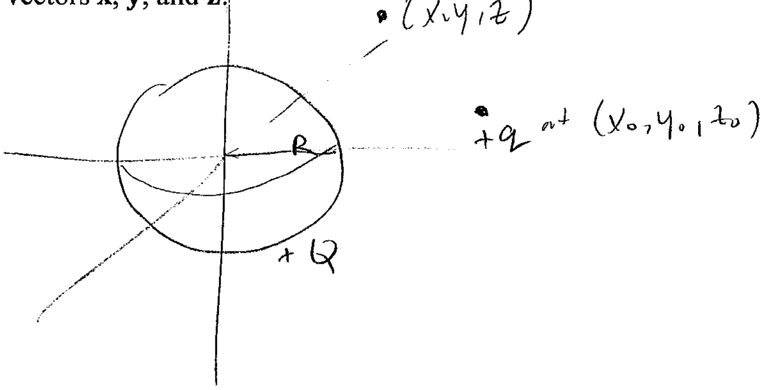
(~~10~~ pts) **Problem 3**: A spherical shell of charge (radius R, charge Q uniformly distributed on the shell) is at the origin. A point charge (charge q) placed outside the sphere is at the coordinates (x_0, y_0, z_0). What is the electric field $\mathbf{E}$ at an arbitrary point (x, y, z)? Indicate directions via the regular rectangular unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.



Field from sphere at origin is $\vec{E} = \dfrac{Q}{4\pi\epsilon_0}\dfrac{\vec{r}}{r^3}$ because it acts like pt charge at the origin

$$= \frac{Q}{4\pi\epsilon_0}\frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

Field from pt charge at (x_0, y_0, z_0) is $\vec{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\vec{\mathscr{r}}}{\mathscr{r}^3}$

where $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{r}\,' = x_0\hat{x} + y_0\hat{y} + z_0\hat{z}$

$\vec{\mathscr{r}} = (x-x_0)\hat{x} + (y-y_0)\hat{y} + (z-z_0)\hat{z}$

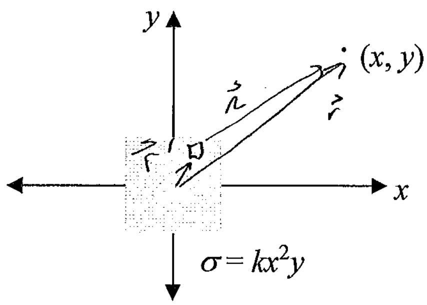
$\mathscr{r} = \left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}$

So $\vec{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{(x-x_0)\hat{x} + (y-y_0)\hat{y} + (z-z_0)\hat{z}}{(\text{stuff})^{3/2}}$

Superposition: add them together

$$\boxed{\vec{E}_{tot} = \frac{Q}{4\pi\epsilon_0}\frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}} + \frac{q}{4\pi\epsilon_0}\frac{(x-x_0)\hat{x} + (y-y_0)\hat{y} + (z-z_0)\hat{z}}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}}}$$

(10 pts) **Problem 4:** A square of charge of side length L is centered on the origin as shown. The surface charge density is a function of x and y as per this equation: $\sigma = kx^2y$. Set up the integral that you would need to do in order to directly calculate the electric potential for an arbitrary point in the x-y plane, $V(x,y)$. You don't need to do the integral, just **get it into a form that e.g. you could type into Mathematica to get the answer.**



$$V(x,y) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r}\,')\,da'}{\mathscr{r}}$$

$$\vec{r} = x\hat{x} + y\hat{y}$$

$$\vec{r}\,' = x'\hat{x} + y'\hat{y}$$

$$\vec{\mathscr{r}} = (x-x')\hat{x} + (y-y')\hat{y}$$

$$\mathscr{r} = \left((x-x')^2 + (y-y')^2\right)^{1/2}$$

$$da' = dx'dy'$$

$$\boxed{V(x,y) = \frac{1}{4\pi\epsilon_0} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{k x'^2 y' \, dx' dy'}{\left((x-x')^2 + (y-y')^2\right)^{1/2}}}$$

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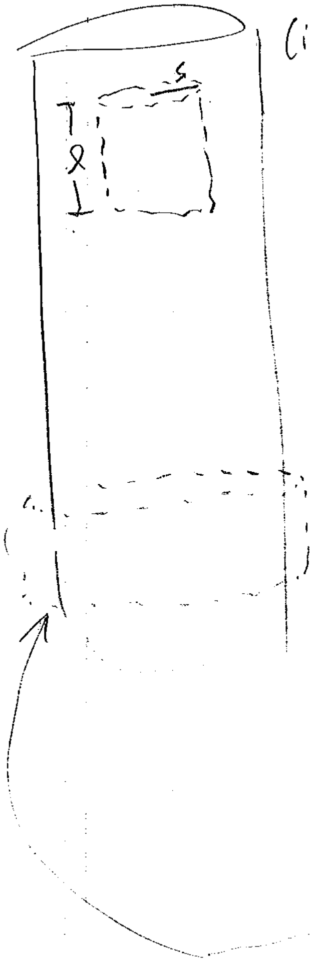
(~~[illegible]~~ pts) **Problem 5.** An infinite cylinder of radius R has a charge density given by:

$$\rho(s) = \frac{\rho_0 e^{-s/R}}{(s/R)}$$

by symmetry, $\vec{E}$ will be in $\hat{s}$ direction, and only a function of s

where ρ_0 has units of charge density (C/m^3) and s is the usual cylindrical coordinate.

Determine the electric field for the two regions: (i) $s < R$, (ii) $s > R$.



(i) $s < R$

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

$\cancel{\int_{top} + \int_{bottom}} + \int_{wrapper}$

(no flux)

$$\oint \vec{E} \cdot d\vec{a} = EA$$

from proper symmetry

$$q_{enc} = \int \rho \, d\tau$$

$$= \int \left(\frac{\rho_0 e^{-s/R}}{s/R}\right)(s\,ds\,d\phi\,dz)$$

integrates to 2π — integrates to l

$$= \rho_0 R \cdot 2\pi l \underbrace{\int_0^s e^{-s/R}\,ds}_{\left.\frac{e^{-s/R}}{-1/R}\right|_0^s \;=\; R(1-e^{-s/R})}$$

$$= 2\pi \rho_0 R^2 l (1 - e^{-s/R})$$

$$E \cdot (\cancel{2\pi} s \cancel{l}) = \frac{1}{\epsilon_0} \cdot \cancel{2\pi} \rho_0 R^2 \cancel{l} (1 - e^{-s/R})$$

$$\boxed{\vec{E} = \frac{\rho_0}{\epsilon_0} R^2 \frac{1}{s} (1 - e^{-s/R})\,\hat{s}}$$ for $s < R$

(ii) $s > R$

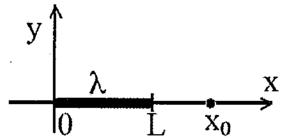
Only change is q_{enc}, now limit goes to R, not just s

$$q_{enc} = 2\pi \rho_0 R^2 l (1 - e^{-1})$$

$$E \cdot \cancel{2\pi} s \cancel{l} = \frac{1}{\epsilon_0} \cancel{2\pi} \rho_0 R^2 \cancel{l} (1 - e^{-1})$$

$$\boxed{\vec{E} = \frac{\rho_0}{\epsilon_0} R^2 (1 - e^{-1}) \frac{1}{s}\,\hat{s}}$$ for $s > R$

(~~10~~ 7 pts) **Problem 6**. A finite line of charge exists along the x-axis from 0 to L as shown. It has a constant linear charge density, λ. Without doing any calculations, what should the potential at the point x_0 be, in the limiting case that $x_0 \gg L$? Don't just say "it goes to zero", but rather write down an equation describing *how* it goes to zero in terms of the variables given.



the finite line will look like a pt charge

with $q = \int \lambda \, dx = \lambda L$

and distance away $= x_0$

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{x_0}}$$

(denominator of

$x_0 - L$

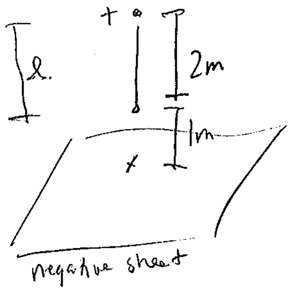
or $x_0 - \frac{L}{2}$

also OK, although since

$x_0 \gg L$, they both

reduce to merely x_0)

(10 pts) **Problem 7**. A small positive charge ($Q = 10^{-15}$ C) is located at rest 3 meters above an infinite sheet of negative charge ($\sigma = -10^2$ C/m^2). It is released and "falls" towards the sheet. How much kinetic energy will it have when it is 1 meter above the sheet? You may ignore gravity, air resistance, etc.



$$\vec{E} = \frac{\sigma}{2\epsilon_0}(-\hat{z})$$ (from Gauss's Law, if you don't remember it)

(= constant!)

$$\Delta V = -\int E \cdot dl$$

$$= -E \cdot l$$ since E is constant, $l = 2m$

$$W = Q\Delta V$$

$$KE_{\text{gained}} = W_{\text{done by field}}$$

$$= Q \cdot E \cdot l$$

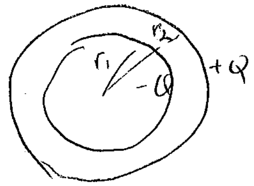
$$= Q \frac{\sigma}{2\epsilon_0} \cdot l$$

$$= (10^{-15})\left(\frac{10^2}{2 \cdot 8.85 \cdot 10^{-12}}\right) \cdot (2)$$

$$= \boxed{.0113 \text{ J}}$$

(17 pts) **Problem 8**. The ionosphere of the Earth is a layer in the atmosphere about 100 km above the surface of the Earth that is made conducting due to the atoms there having been ionized by UV rays from the sun. The ground/ionosphere system for the Earth can be modeled as two concentric conducting spheres at $r_1 = 6400$ km (the surface of the Earth) and $r_2 = 6500$ km (the ionosphere), that have a potential difference of 300 kV. The Earth's surface is negatively charged and the ionosphere is positively charged; you may assume that the two charges are equal. Give numerical answers to the following (and be sure to show your work):

(a) What is the capacitance of the system?
(b) What is the total charge Q, and the surface charge density σ, of the Earth's surface?
(c) How much electrical energy is stored in the electric field between the concentric spheres?



(a) in between a and b, field acts like pt charge at origin,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}(-\hat{r})$$

$$\Delta V = -\int \vec{E}\cdot d\vec{\ell}$$

$$= -\int_{r_2}^{r_1} \frac{-Q}{4\pi\epsilon_0}\frac{1}{r^2}\,dr$$

$$|\Delta V| = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$C = \frac{Q}{|\Delta V|} = 4\pi\epsilon_0\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^{-1}$$

$$= 4\pi\cdot 8.85\cdot 10^{-12}\left(\frac{1}{6400\cdot 10^3}-\frac{1}{6500\cdot 10^3}\right)^{-1}$$

$$= \boxed{.0463\ F}$$

(b) $Q = C|\Delta V| = (.0463)(300\cdot 10^3)$

$$= \boxed{13879\ C}$$

$$\sigma = \frac{Q}{A} = \frac{13879}{4\pi(6400\cdot 10^3)^2} = \boxed{2.70\cdot 10^{-11}\ \frac{C}{m^2}}$$

(c) could do $\frac{\epsilon_0}{2}\int E^2\,d\tau$ but much easier to do

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \text{ for capacitor}$$

$$U = \frac{1}{2}(.0463)(300\cdot 10^3)^2 = \boxed{2.08\cdot 10^9\ J}$$