Fall 2018
Physics 441
Exam 1
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No time limit. Student calculators are allowed. One page of handwritten notes allowed (front & back). Books not allowed.

Name Solutions

Instructions: Please label & circle/box your answers. **Show your work**, where appropriate! And remember: **in any problems involving Gauss's Law, you should explicitly show your Gaussian surface**. For all problems, unless otherwise specified you may assume that you are dealing with **electrostatics**, i.e. the charges are not moving and the fields have come to equilibrium.

Griffiths front and back covers

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl: $\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\,d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Special case derivatives:
(similar things true for $\boldsymbol{\mathscr{r}}$)

$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta(\mathbf{r}) \qquad \nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^{8}\ \mathrm{m/s}$	(speed of light)
$e = 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Some miscellaneous mathematical stuff:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \qquad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

Some indefinite integrals:

$$\int \sin Cx = \frac{-\cos Cx}{C} \qquad \int \sin^2 Cx = \frac{x}{2} - \frac{\sin 2Cx}{4C}$$

$$\int \cos Cx = \frac{\sin Cx}{C} \qquad \int \cos^2 Cx = \frac{x}{2} + \frac{\sin 2Cx}{4C}$$

Some definite integrals:

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \begin{cases} 0, \text{ if } n \neq m \\ \frac{a}{2}, \text{ if } n = m \end{cases}$$

$$\int_0^{\pi} \sin^2 x\, dx = \frac{\pi}{2} \qquad \int_0^{\pi} \cos^2 x\, dx = \frac{\pi}{2} \qquad \int_0^{2\pi} \sin^2 x\, dx = \pi \qquad \int_0^{2\pi} \cos^2 x\, dx = \pi$$

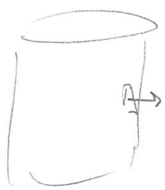
$$\int_0^{\pi} \sin^3 x\, dx = \frac{4}{3} \qquad \int_0^{\pi} \cos^3 x\, dx = 0 \qquad \int_0^{2\pi} \sin^3 x\, dx = 0 \qquad \int_0^{2\pi} \cos^3 x\, dx = 0$$

$$\int_0^{\pi} \sin^4 x\, dx = \frac{3\pi}{8} \qquad \int_0^{\pi} \cos^4 x\, dx = \frac{3\pi}{8} \qquad \int_0^{2\pi} \sin^4 x\, dx = \frac{3\pi}{4} \qquad \int_0^{2\pi} \cos^4 x\, dx = \frac{3\pi}{4}$$

(27 pts) **Problem 1**. Multiple choice, 1.5 pts each. Circle the correct answers for the multiple choice questions.

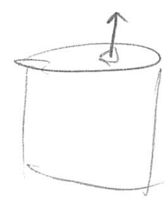
1.1. For an infinitesimal **dA** on the "wrapper" of a cylinder, what would be the appropriate formula to use in cylindrical coordinates?

(a) $s\,ds\,d\phi\,\hat{\mathbf{s}}$
(b) $s\,ds\,d\phi\,\hat{\boldsymbol{\phi}}$
(c) $s\,ds\,d\phi\,\hat{\mathbf{z}}$
(d) $s\,d\phi dz\,\hat{\mathbf{s}}$
(e) $s\,d\phi dz\,\hat{\boldsymbol{\phi}}$
(f) $s\,d\phi dz\,\hat{\mathbf{z}}$



1.2. For an infinitesimal **dA** on the <u>top</u> of a cylinder, what would be the appropriate formula to use in cylindrical coordinates?

(a) $s\,ds\,d\phi\,\hat{\mathbf{s}}$
(b) $s\,ds\,d\phi\,\hat{\boldsymbol{\phi}}$
(c) $s\,ds\,d\phi\,\hat{\mathbf{z}}$
(d) $s\,d\phi dz\,\hat{\mathbf{s}}$
(e) $s\,d\phi dz\,\hat{\boldsymbol{\phi}}$
(f) $s\,d\phi dz\,\hat{\mathbf{z}}$



1.3. What are the units of δ(x) if x is measured in meters?

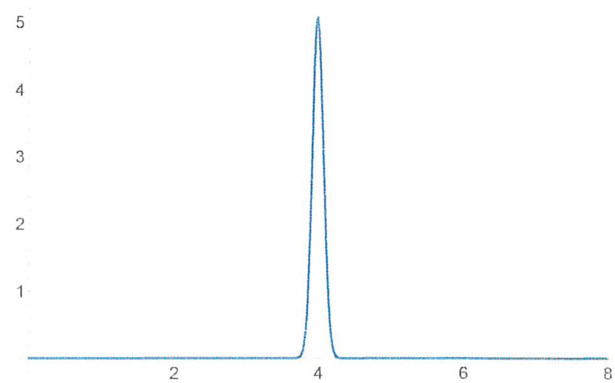
(a) δ is dimensionless (no units)
(b) [m]: Unit of length
(c) [m^2]: Unit of length squared
(d) [m^{-1}]: One over unit of length
(e) [m^{-2}]: One over unit of length squared

$\int \delta(x)\,dx = $ [no units]

length

therefore $\frac{1}{\text{length}}$

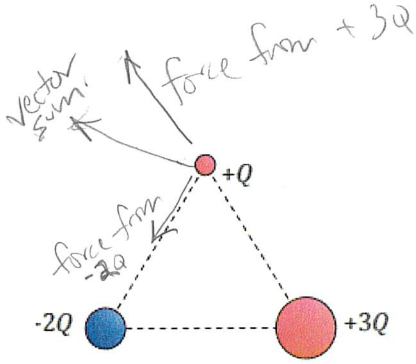
1.4. The plot to the side is of the function $f(x)$... it's a "spiky" function centered on 4 that has an area of 1. Indeed, it looks a little like a delta function although it doesn't spike all the way to infinity. Which of the following will be closest to $\int_0^8 f(x)\,e^{-x}dx$?



This is like $\delta(x-4)$

$\int_0^8 \delta(x-4)e^{-x}\,dx = \boxed{e^{-4}}$

(a) 0
(b) $\frac{1}{2}e^{-1}$
(c) e^{-1}
(d) $\frac{1}{2}e^{-2}$
(e) e^{-2}
(f) $\frac{1}{3}e^{-3}$
(g) $\frac{1}{2}e^{-3}$
(h) e^{-3}
(i) $\frac{1}{4}e^{-4}$
(j) $\frac{1}{2}e^{-4}$
(k) e^{-4}



1.5. Three point charges, of charge $+Q$, $-2Q$, and $+3Q$, are placed equidistant as shown. Which vector best describes the net direction of the electric field acting on the $+Q$ charge?

(a) 

(b) 

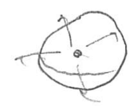
(c) 

(d) 

(e) 

1.6. A point charge is placed at the center of a spherical Gaussian surface. The net electric flux passing through the surface will changed for which of the following situations:

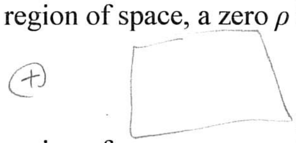
same for any surface containing the charge



(a) ~~The sphere is replaced by a cube of the same volume.~~
(b) ~~The sphere is replaced by a cube of one-tenth the volume.~~
(c) ~~The point charge is moved to a location within the sphere but close to the surface.~~
(d) The point charge is moved to a location just outside the sphere. Then no net flux
(e) ~~A second point charge is placed just outside the sphere.~~

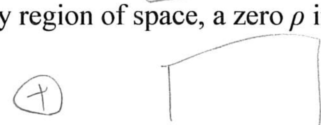
1.7. Which would be the better way to calculate the electric field produced by a solid cube of charge?
(a) Coulomb's law
(b) Gauss's law Can't use → not the correct symmetry

1.8. T/F: For an arbitrary region of space, a zero ρ inside the region implies a zero **E**.
(a) True
(b) False



region here still has $\vec{E}$

1.9. T/F: For an arbitrary region of space, a zero ρ inside the region implies a zero V.
(a) True
(b) False



region here still has V

1.10. T/F: For an arbitrary region of space, a zero **E** inside the region implies a zero ρ.
(a) True
(b) False

$\nabla \cdot \vec{E} = \rho/\epsilon_0 \rightarrow$ if $\vec{E}=0$, then $\nabla \cdot \vec{E} = 0$, then $\rho = 0$

1.11. T/F: For an arbitrary region of space, a zero **E** inside the region implies a zero V.
(a) True
(b) False

→ implies constant V, but doesn't have to be zero

zero $V \rightarrow$ zero $\vec{E}$ (since $\vec{E} = -\nabla V$), which then by same logic as above implies $\rho = 0$

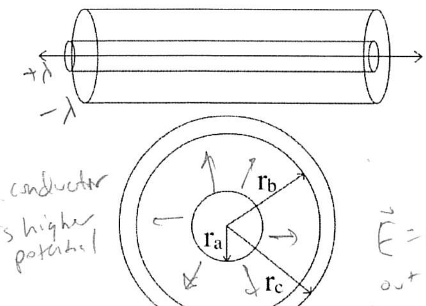
1.12. T/F: For an arbitrary region of space, a zero V inside the region implies a zero ρ.
(a) True
(b) False

1.13. T/F: For an arbitrary region of space, a zero V inside the region implies a zero **E**.
(a) True
(b) False

$\vec{E} = -\nabla V$ so if $V = 0$, $\vec{E} = 0$ also

1.14. Which of the following is correct?
(a) **E** and V are both always continuous across a boundary
(b) **E** is always continuous, but V can be discontinuous
(c) **E** can be discontinuous, but V is always continuous As discussed several times in class
(d) **E** and V can both be discontinuous

1.15. Consider a metal cylindrical shell of outer radius r_c and inner radius r_b which is concentric with a metal wire of radius r_a. The linear charge density of the wire is $+\lambda$ and the linear charge density of the cylinder is $-\lambda$. Which of the following statement(s) is (are) true?

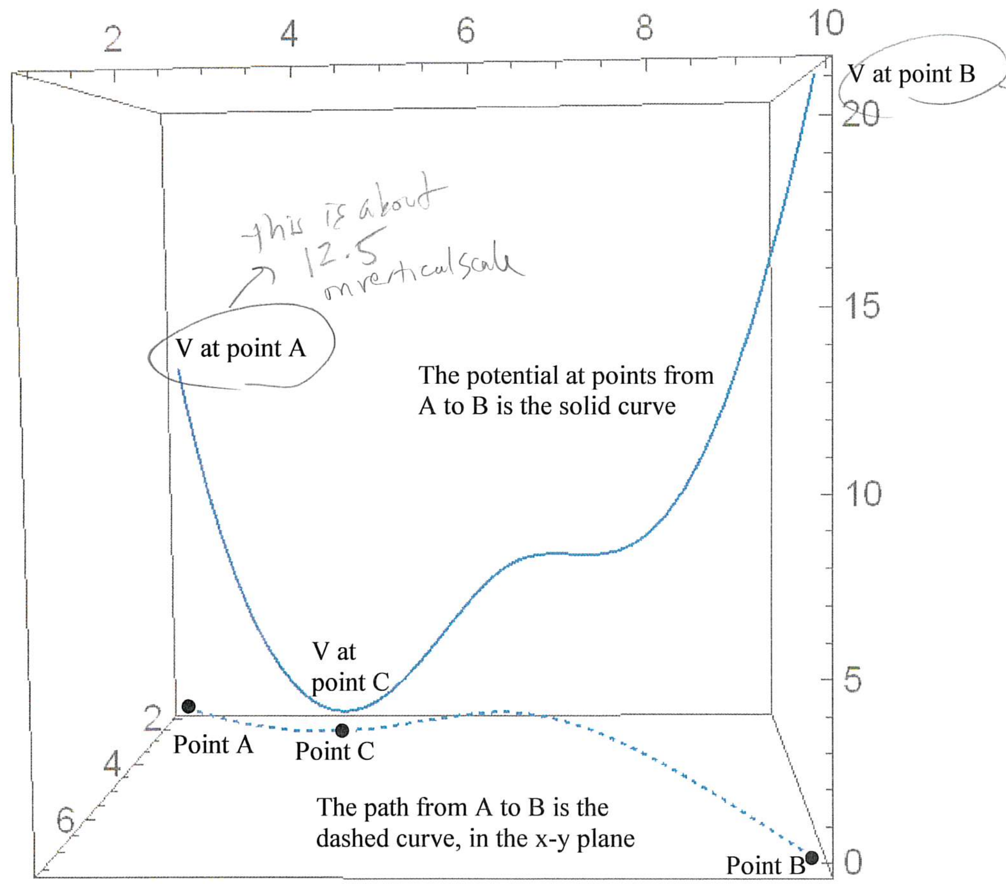


I. The potential difference between r_c and r_b is zero. True - inside conductor
II. The potential difference between r_b and r_a is zero. False - r_a is higher potential
III. The potential difference between a point outside the cylinder and r_c is zero. True, since $\vec{E} = 0$ outside
IV. The electric field between a point outside the cylinder and r_c is zero. True, see note

$\vec{E} = 0$ out here by symmetry (+ Gauss's law if you want)

(a) I and III
(b) I and IV
(c) II and III
(d) II and IV
(e) I, III and IV
(f) II, III, and IV
(g) I, II, III, and IV

1.16. Which of the following statements regarding conductors in electric fields (static situations) is false?
(a) The electric field inside any solid conductor is always zero. True
(b) The value of the electrostatic potential is the same at all points inside of, and on the surface of, a conductor of any shape. True
(c) Any excess charge placed on an isolated solid conductor will always move to the outside surface of that conductor. True
(d) The surface charge density (charge per unit area) on the surface of a conductor is always largest near sharp points. True
(e) The electric field at the surface of a conductor is always perpendicular to the surface of that conductor. True
(f) None (they are all true)



This is about 21 on vertical scale

1.17. A charged particle takes the dashed path from A to B in the *x-y* plane, and the potential V that it experiences along that path is plotted as a function of *x* and *y* as the solid curve. What is $\int_{pt\,A}^{pt\,B} \nabla V \cdot d\boldsymbol{\ell}$ for the particular path shown? (Choose the closest value.) Don't worry about units, just use the numbers from the plot if you need numbers.

(a) 0
(b) 8.5
(c) 12.5
(d) 21.0
(e) 22.5
(f) Can't tell; need more information

This is equal to $V_B - V_A$ (gradient theorem)
$= 21 - 12.5$
$= 8.5$

1.18. For the same plot, what direction is the electric field at point C? $\vec{E} = -\nabla V$

(a) $\hat{\mathbf{x}}$
(b) $-\hat{\mathbf{x}}$
(c) $\hat{\mathbf{y}}$
(d) $-\hat{\mathbf{y}}$
(e) $\hat{\mathbf{z}}$
(f) $-\hat{\mathbf{z}}$
(g) Can't tell; need more information

Need the whole potential surface to calculate the gradient. For example (in 2D) point C could look like this C which would imply no field.

Or it could be like this C which would have $\vec{E}$ in the x-direction.

(6 pts) **Problem 2**. Short answers.

(a) In terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, what is $\hat{\mathbf{r}}$ for the point (1, 2, 3)?

$$\hat{r} = \frac{\vec{r}}{r} = \frac{\hat{x} + 2\hat{y} + 3\hat{z}}{\sqrt{1+4+9}} = \boxed{\frac{1}{\sqrt{14}}\left(\hat{x} + 2\hat{y} + 3\hat{z}\right)}$$

(b) In terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, what is $\hat{\mathbf{s}}$ for the point (1, 2, 3)?

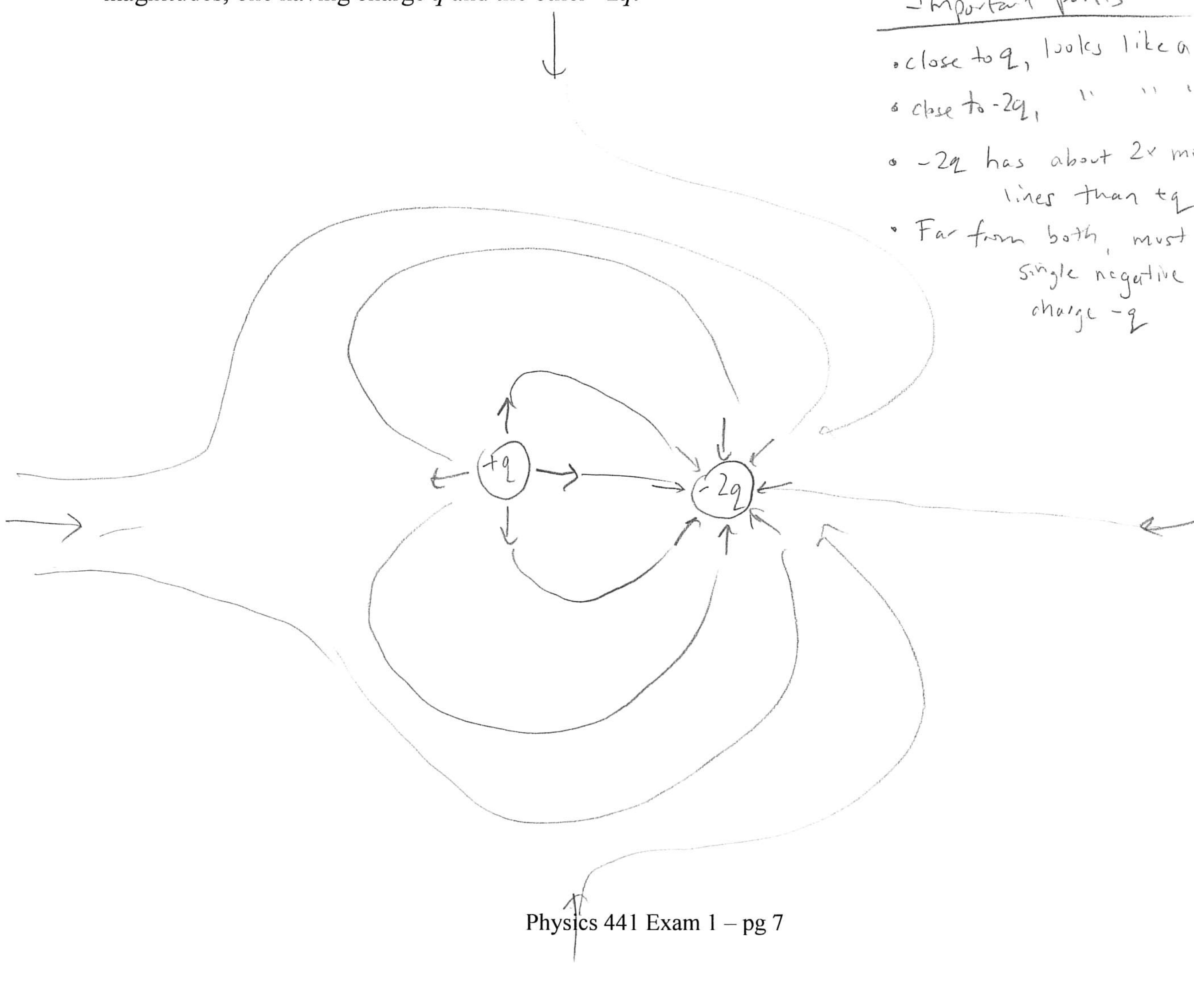
like part a but without the z-component

$$\hat{s} = \frac{\vec{s}}{s} = \boxed{\frac{\hat{x} + 2\hat{y}}{\sqrt{5}}}$$

(c) Draw the electric field lines in the region between and around two opposite charges of unequal magnitudes, one having charge q and the other $-2q$.

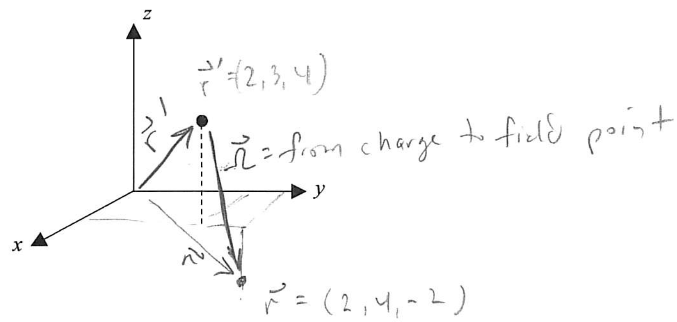
Important points

- close to q, looks like a single +charge
- close to -2q, " " " " - "
- -2q has about 2x more field lines than +q
- Far from both, must look like a single negative charge with charge -q



(12 pts) **Problem 3.** A charge q is at the location (2, 3, 4) as shown by the dot. Consider a “test point” at (2, 4, –2) (not shown).

Part 1: What are the following? Put your vector answers in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.



(a) $\mathbf{r}$ (vector) $\boxed{\vec{r} = 2\hat{x} + 4\hat{y} - 2\hat{z}}$

(b) $\mathbf{r}'$ (vector) $\boxed{\vec{r}\,' = 2\hat{x} + 3\hat{y} + 4\hat{z}}$

(c) $\boldsymbol{\mathscr{r}}$ (vector) $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = \boxed{\hat{y} - 6\hat{z}}$

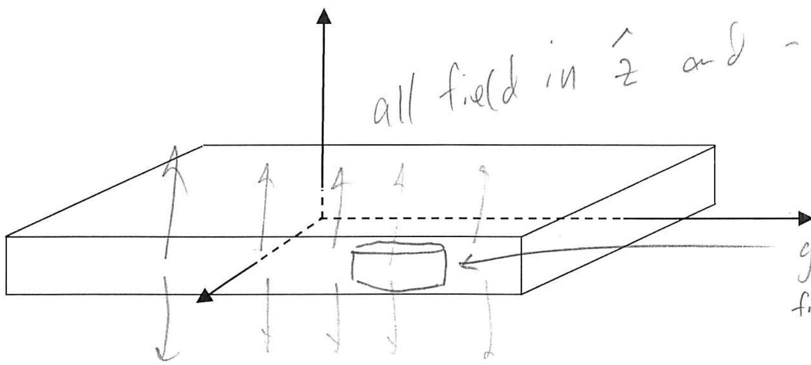
(d) $\mathscr{r}$ (scalar) $\mathscr{r} = |\vec{\mathscr{r}}| = \sqrt{1 + 36} = \boxed{\sqrt{37}}$

Part 2: Draw the test point on the graph above, and also draw in the three vectors from Part 1.

correct symmetry for Gauss's law since $\cos\left(\frac{\pi z}{2d}\right)$ is symmetric about $z=0$

(14 pts) **Problem 4**: An infinite slab with thickness 2*d* is centered on the *x*-*y* plane. A section is shown. It has charge density $\rho = \rho_0 \cos\left(\frac{\pi z}{2d}\right)$. Determine the electric field *inside* the slab, both magnitude and direction, as a function of *z*, for $0 < z < d$.

all field in $\hat{z}$ and $-\hat{z}$ directions. For $0<z<d$ it's in $+\hat{z}$ (assuming $\rho_0>0$)



gaussian surface is "pill box" from $-z$ to $+z$

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

$$\int_{top} + \int_{bottom} + \cancel{\int_{wrapper}} = \frac{q_{enc}}{\epsilon_0}$$

(no flux out the sides)

side note

$$q_{enc} = \int \rho \, d\tau'$$

$$= \rho_0 \int_{-z}^{z} \cos\frac{\pi z'}{2d} A \, dz'$$

$$= \rho_0 A \left. \frac{\sin\frac{\pi z'}{2d}}{\frac{\pi}{2d}} \right|_{-z}^{z}$$

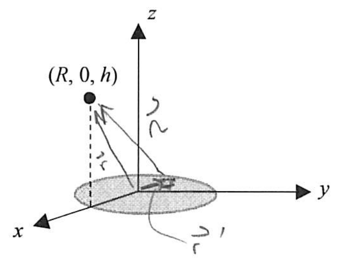
$$= \rho_0 A \cdot \frac{2d}{\pi} \cdot 2 \times \sin\frac{\pi z}{2d}$$

$$EA + EA = \frac{\rho_0 A}{\epsilon_0} \frac{4d}{\pi} \sin\frac{\pi z}{2d}$$

$$2EA =$$

$$\boxed{\vec{E} = \frac{2\rho_0 d}{\epsilon_0 \pi} \sin\frac{\pi z}{2d} \, \hat{z}}$$

(15 pts) **Problem 5**. A disc of charge with radius R lies in the x-y plane, centered on the z-axis as shown. Its charge density is: $\sigma = \sigma_0 \left(\frac{s}{R}\right)\cos(3\phi)$. Set up an integral that you could use to calculate the electric potential V at the point indicated, $(R, 0, h)$. Please don't do the integral, just set it up. Make sure all quantities in your integral are explicitly written in terms of constants, the given variables, or variables of integration.



$\vec{r} = R\hat{x} + h\hat{z}$

$\vec{r}\,' = s'\hat{s} = s'\cos\phi'\hat{x} + s'\sin\phi'\hat{y}$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = (R - s'\cos\phi')\hat{x} - s'\sin\phi'\hat{y} + h\hat{z}$

$\mathscr{r} = \sqrt{(R - s'\cos\phi')^2 + s'^2\sin^2\phi' + h^2}$

$V = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma\, da'}{\mathscr{r}}$

σ at $r' = \sigma_0\left(\frac{s'}{R}\right)\cos(3\phi')$

$da' = s'\,ds'\,d\phi'$

$\mathscr{r}$ = as just done

$$\boxed{V = \frac{\sigma_0}{4\pi\epsilon_0 R}\int_0^{2\pi}\int_0^{R} \frac{s'^2\cos 3\phi'\, ds'\, d\phi'}{\sqrt{(R - s'\cos\phi')^2 + s'^2\sin^2\phi' + h^2}}}$$

(12 pts) **Problem 6**. Don't worry about units in this problem. The electric field in a given region of space is given in spherical coordinates by $\mathbf{E} = (C_1 r^2 + C_2)\hat{\mathbf{r}}$, where C_1 and C_2 are positive numbers. Find the potential difference between $r = 5$ and $r = 3$, and state which point is at the higher potential.

$r=3$ is higher potential since $\vec{E}$ points outward (like a positive point charge)

$$\Delta V = -\int_5^3 \vec{E} \cdot d\vec{\ell}$$

$d\vec{\ell} = dr\,\hat{r}$

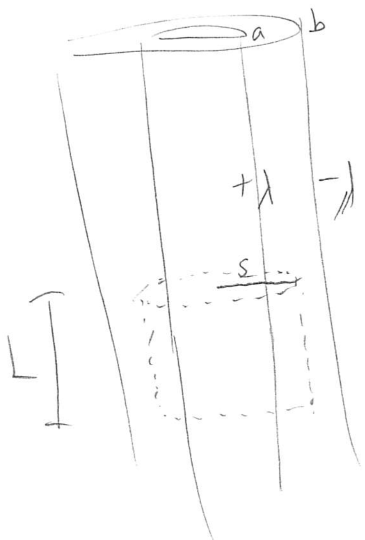
$\vec{E} = (C_1 r^2 + C_2)\,\hat{r}$

$$= \underbrace{-\int_5^3 C_1 r^2\,dr}_{-\frac{1}{3} C_1 r^3 \Big|_5^3} + \underbrace{-\int_5^3 C_2\,dr}_{-C_2 r \Big|_5^3}$$

$\frac{1}{3} C_1 (125-27)$ $\qquad$ $C_2(5-3)$

$$= \boxed{\frac{98}{3} C_1 + 2C_2}$$

(14 pts) **Problem 7**. Find the capacitance per length of a pair of concentric conducting cylinders, infinite in length, the inner one having radius a and the outer one having radius b. (Yes, this is exactly like a HW problem.)



1) Assume $+\lambda$ on inner and $-\lambda$ on outer conductors

2) Use Gauss's law to get field between conductors ($\vec{E}$ in $\hat{s}$ direction by symmetry)

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

All flux through wrapper

$$E(2\pi s \cancel{L}) = \frac{\lambda \cancel{L}}{\epsilon_0}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0} \frac{1}{s} \hat{s}$$

3) Integrate to get ΔV between a and b

$$\Delta V = -\int \vec{E} \cdot d\vec{\ell}$$

$$= -\int_b^a \frac{\lambda}{2\pi\epsilon_0} \frac{1}{s} \, ds$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln b/a$$

4) divide $\frac{Q}{\Delta V}$ to get $C \rightarrow$ divide $\frac{Q/L \; (\rightarrow = \lambda)}{\Delta V}$ to get $\frac{C}{L}$

$$\frac{C}{L} = \frac{\lambda}{\frac{\lambda}{2\pi\epsilon_0} \ln b/a}$$

$$\boxed{\frac{C}{L} = 2\pi\epsilon_0 \frac{1}{\ln b/a}}$$