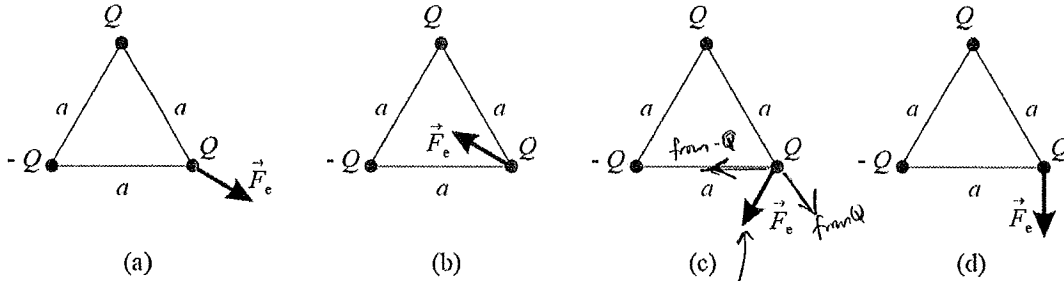


Phus 441 Exam 1 Solns

(20 pts) **Problem 1:** Multiple choice, 2 pts each. Circle the correct answer.

1.1. Three small charged bodies are placed at the vertices of an equilateral triangle. The electric force \vec{F}_e on the lower right charge is



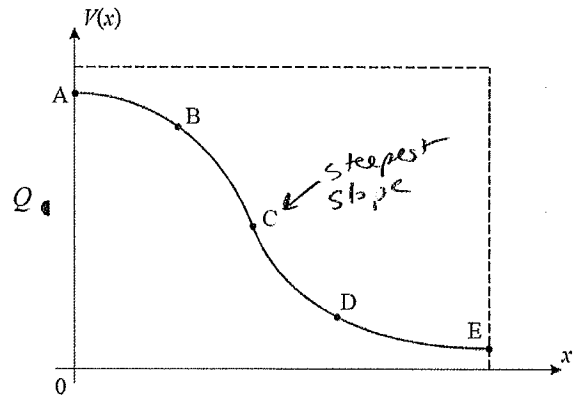
- (a) as in Figure (a).
- (b) as in Figure (b).
- (c) as in Figure (c).
- (d) as in Figure (d).
- (e) zero.

1.2. The electrostatic potential V in a region of space is only a function of x , and $V(x)$ is shown in the figure. Consider the electric field magnitudes at points A, B, C, D, and E. The largest magnitude is at point

- (a) A.
- (b) B.
- (c) C.
- (d) D.
- (e) E.

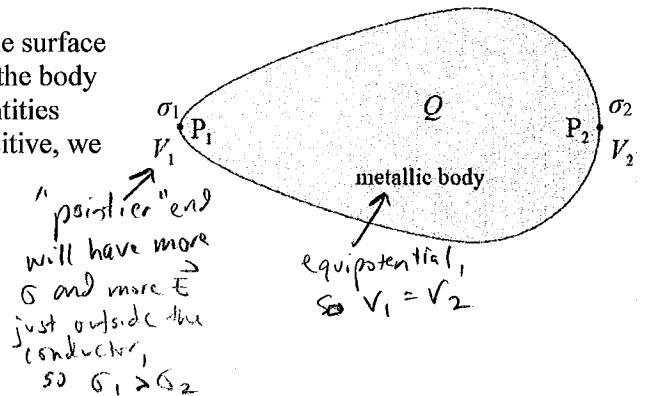
$$\vec{E} = -\vec{\nabla} V$$

\hookrightarrow relates to slope



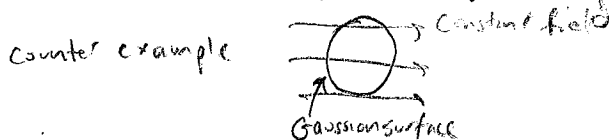
1.3. A charged metallic body is situated in free space. The surface charge density and electric potential at a point P_1 of the body are σ_1 and V_1 , respectively. At a point P_2 , these quantities equal σ_2 and V_2 . If the total charge of the body is positive, we have that

- (a) $\sigma_1 = \sigma_2$ and $V_1 = V_2$.
- (b) $\sigma_1 < \sigma_2$ and $V_1 = V_2$.
- (c) $\sigma_1 > \sigma_2$ and $V_1 = V_2$.
- (d) $\sigma_1 < \sigma_2$ and $V_1 < V_2$.
- (e) $\sigma_1 > \sigma_2$ and $V_1 > V_2$.



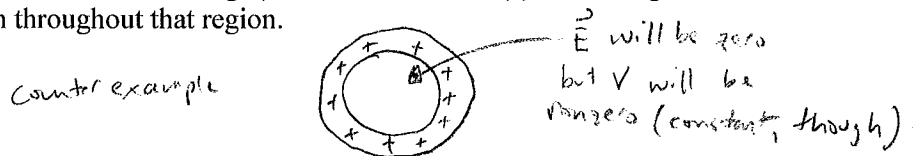
1.4. If the total charge enclosed inside a Gaussian surface is zero, then \vec{E} everywhere on the Gaussian surface must be zero.

- (a) True
- (b) False



1.5. A spherical region (radius R , centered on the origin) has electric field $\vec{E}(\mathbf{r}) = 0$ throughout. The voltage $V(\mathbf{r})$ must also vanish throughout that region.

- (a) True
- (b) False



1.6. A spherical region (radius R , centered on the origin) has voltage $V(\mathbf{r}) = 0$ throughout. The electric field $\mathbf{E}(\mathbf{r})$ must also vanish throughout that region.

- (a) True
(b) False

$$\vec{E} = -\nabla V$$

if V is a constant zero (or any constant value), then \vec{E} will be zero.

1.7. If you have a point charge at rest at the point (x_0, y_0, z_0) , and no other charges anywhere, then $\nabla \cdot \mathbf{E} = 0$ everywhere in all space.

- (a) True
(b) False

The divergence will be nonzero at the location of the charge. $\nabla \cdot \vec{E} = \rho/\epsilon_0$

ρ will be $q \times \delta(\vec{r} - \vec{r}_0)$, nonzero at $\vec{r}_0 = (x_0, y_0, z_0)$

1.8. If you have a point charge at rest at the point (x_0, y_0, z_0) , and no other charges anywhere, then $\nabla \times \mathbf{E} = 0$ everywhere in all space.

- (a) True
(b) False

For time-independent cases, $\nabla \times \vec{E} = 0$ everywhere (electric field has no curl)

1.9. A surface in the x - y plane (i.e. $z = 0$) has a constant surface charge density σ on it. Just above the surface, the electric field is $+3\hat{z}$ N/C, just below the surface the field is $-3\hat{z}$ N/C. What is the surface charge density σ ?

- (a) $3 \text{ N/C} \times \epsilon_0$
(b) $3 \text{ N/C} \div \epsilon_0$
(c) $6 \text{ N/C} \times \epsilon_0$
(d) $6 \text{ N/C} \div \epsilon_0$
(e) Zero

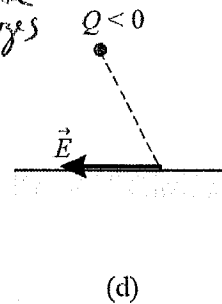
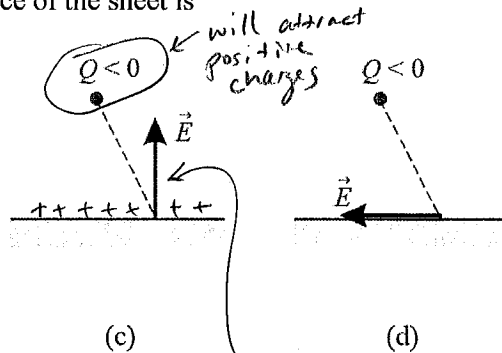
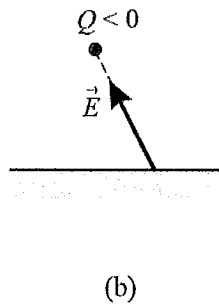
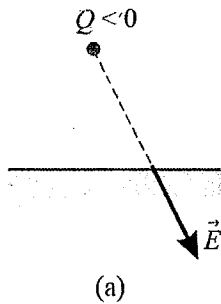
Discontinuity: $E_{\perp \text{ above}} - E_{\perp \text{ below}} = \sigma/\epsilon_0$

$$\sigma = \epsilon_0 \Delta E_{\perp}$$

$$= \epsilon_0 (+3\hat{z} - -3\hat{z})$$

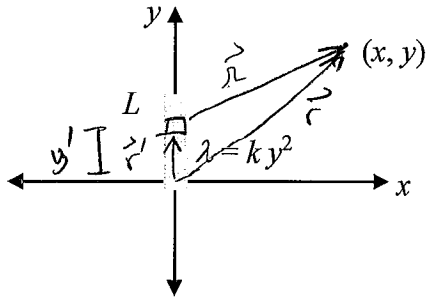
$$= \boxed{12\epsilon_0}$$

1.10. A negative point charge Q is located in air above a large, flat conducting sheet. The electric field intensity vector in air immediately above the surface of the sheet is



- (a) as in Figure (a).
(b) as in Figure (b).
(c) as in Figure (c).
(d) as in Figure (d).
(e) zero.

(14 pts) **Problem 2:** You have a short line of charge on the y -axis from $y=0$ to $y=L$. The linear charge density is $\lambda = ky^2$. Set up the integral that you would need to do in order to directly calculate the electric field $\mathbf{E}(x,y)$ for an arbitrary point in the x - y plane. You don't need to do the integral, just **get it into a form that e.g. you could type into Mathematica to get the answer.**



$$\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} \\ \vec{r}' &= y'\hat{y} \\ \vec{r} &= \vec{r} - \vec{r}' = x\hat{x} + y\hat{y} - y'\hat{y} \\ &= x\hat{x} + (y-y')\hat{y} \\ r \text{ (magnitude)} &= \sqrt{x^2 + (y-y')^2} \end{aligned}$$

direct integration: $\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\text{line}} \lambda \frac{d\vec{r}}{r^3}$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_0^L (ky'^2) dy' \frac{(x\hat{x} + (y-y')\hat{y})}{(x^2 + (y-y')^2)^{3/2}}$$

could go a little further...

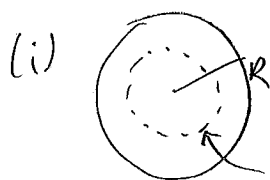
$$= \frac{k}{4\pi\epsilon_0} (x\hat{x} + y\hat{y}) \int_0^L \frac{y'^2 dy'}{(x^2 + (y-y')^2)^{3/2}}$$

$$+ \frac{-k}{4\pi\epsilon_0} \hat{y} \int_0^L \frac{-y'^3 dy'}{(x^2 + (y-y')^2)^{3/2}}$$

(16 pts) **Problem 3.** A sphere of radius R has a spherically-symmetric charge density given by:

$$\rho(r) = \frac{\rho_0 e^{-r/R}}{(r/R)^2}$$

Determine the electric field for the two regions: (i) $r < R$, (ii) $r > R$.



Gaussian Surface,
 $E = \text{constant}$
 $\vec{E} = \text{parallel } d\vec{a}$

so this
 $= \vec{E} \cdot A$
 $= E \cdot 4\pi r^2$

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{\text{enc}}}{\epsilon_0}$$

$$q_{\text{enc}} = \int \rho \, d\tau$$

$$= \int_0^r \left(\frac{\rho_0 e^{-r'/R}}{(r'/R)^2} \right) 4\pi r'^2 \, dr'$$

$$= \rho_0 R^2 \cdot 4\pi \int_0^r e^{-r'/R} \, dr'$$

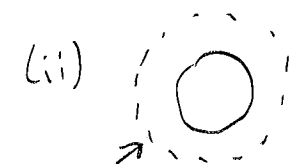
$$= \frac{e^{-r'/R}}{-1/R} \Big|_0^r$$

$$= R(1 - e^{-r/R})$$

$$q_{\text{enc}} = \rho_0 R^2 4\pi R (1 - e^{-r/R})$$

$$E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} \rho_0 R^3 4\pi (1 - e^{-r/R})$$

$$\boxed{\vec{E} = \frac{\rho_0 R^3}{\epsilon_0} \frac{1 - e^{-r/R}}{r^2} \hat{r}}$$



Gaussian Surface

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{\text{enc}}}{\epsilon_0}$$

$$q_{\text{enc}} = \rho_0 R^3 4\pi (1 - e^{-1})$$

$$E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} \rho_0 R^3 4\pi (1 - e^{-1})$$

$$\boxed{\vec{E} = \frac{\rho_0 R^3 (1 - e^{-1})}{\epsilon_0 r^2} \hat{r}}$$

same integral as before
 but upper limit is R

(18 pts) **Problem 4.** Suppose the electric field in the previous problem came out to be:

$$r < R: \vec{E} = \frac{\rho_0 R e^{-r/R}}{\epsilon_0} \hat{r}$$

$$r > R: \vec{E} = \frac{\rho_0 R^3 e^{-r/R}}{\epsilon_0 r^2} \hat{r}$$

(Note: those are *not* the correct answers!)

Find the potential $V(r)$ for the two regions, using the typical convention that $V(r = \infty) = 0$.

$$V = - \int \vec{E} \cdot d\vec{l}$$

$$r > R: V = - \int_{\infty}^r \vec{E}_{\text{outside}} \cdot d\vec{l}$$

$$= - \int_{\infty}^r \frac{\rho_0 R^3 e^{-r'/R}}{\epsilon_0 r'^2} dr'$$

$$= - \frac{\rho_0 R^3 e^{-1}}{\epsilon_0} \int_{\infty}^r r'^{-2} dr'$$

$$= - r'^{-1} \Big|_{\infty}^r$$

$$= - \frac{1}{r}$$

$$\boxed{V(r) = \frac{\rho_0 R^3 e^{-1}}{\epsilon_0} \frac{1}{r} \quad r > R}$$

$$r < R: V = - \int_{\infty}^R \vec{E}_{\text{outside}} \cdot d\vec{l} - \int_R^r \vec{E}_{\text{inside}} \cdot d\vec{l}$$

$$= \frac{\rho_0 R^3 e^{-1}}{\epsilon_0} \frac{1}{R}$$

from previous

$$= - \int_R^r \frac{\rho_0 R}{\epsilon_0} e^{-r'/R} dr'$$

$$= - \frac{\rho_0 R}{\epsilon_0} \left. e^{-r'/R} \right|_R^r$$

$$= + \frac{\rho_0 R^2}{\epsilon_0} (e^{-r/R} - e^{-1})$$

$$V = \frac{\rho_0 R^2}{\epsilon_0} e^{-1} + \frac{\rho_0 R^2}{\epsilon_0} e^{-r/R} - \frac{\rho_0 R^2}{\epsilon_0} e^{-1}$$

$$\boxed{V(r) = \frac{\rho_0 R^2}{\epsilon_0} e^{-r/R} \quad r < R}$$

(18 pts) **Problem 5.** Using the same electric field as the previous problem, namely:

$$r < R: \vec{E} = \frac{\rho_0 R e^{-r/R}}{\epsilon_0} \hat{r}$$

$$r > R: \vec{E} = \frac{\rho_0 R^3 e^{-1}}{\epsilon_0 r^2} \hat{r}$$

(a) What is the volume charge density ρ which gives rise to this field?

(b) Is there a surface charge density at $r = R$? Justify your answer.

(a) Gauss's Law: $\nabla \cdot \vec{E} = \rho / \epsilon_0 \rightarrow \rho = \epsilon_0 \nabla \cdot \vec{E}$

in spherical coords, $\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta E_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} E_\phi$
 (only E_r exists)

for $r < R$: $\rho = \epsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\rho_0 R e^{-r/R}}{\epsilon_0} \right)$
 $= \rho_0 R \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 e^{-r/R} \right)$
 $r^2 \left(-\frac{1}{R} \right) e^{-r/R} + 2r e^{-r/R}$

$$\rho = -\rho_0 e^{-r/R} + 2\rho_0 \frac{R}{r} e^{-r/R} \quad r < R$$

for $r > R$: $\rho = \epsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{\rho_0 R^3 e^{-1}}{\epsilon_0} \right)$
 constant, so $\partial/\partial r = 0$

$$\rho = 0 \quad r > R$$

(b) Boundary condition $E_{\perp \text{ above}} - E_{\perp \text{ below}} = \sigma / \epsilon_0$

$$\frac{\rho_0 R^3 e^{-1}}{\epsilon_0 R^2} - \frac{\rho_0 R e^{-1}}{\epsilon_0} = \sigma / \epsilon_0$$

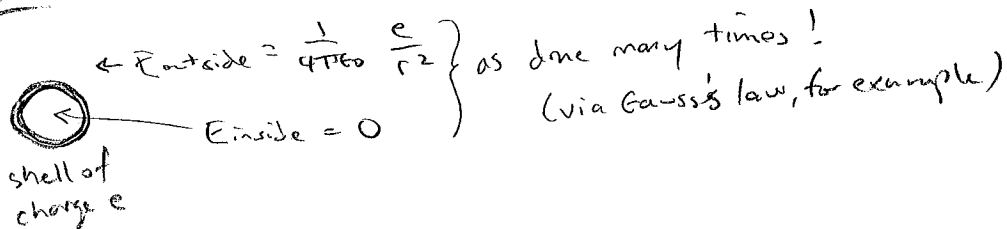
$$0 = \sigma / \epsilon_0$$

$$\sigma = 0$$

No discontinuity in E_{\perp} ,
 so no surface charge

(14 pts) **Problem 6.** As mentioned in class, the electromagnetic energy of a point charge diverges (is infinite) due to the $r = 0$ limit of integration. Because of this result it has been suggested that physical particles (like the electron) are not point particles. One possible model for the electron is that it is a spherical shell of charge with small but nonzero radius, having a constant surface charge density everywhere on the shell. Calculate the radius you get for such an electron by assuming that the total energy stored in the electron's electric field is equal to Einstein's equation for the inherent energy of a particle: $Energy = mc^2$. (Give a numerical as well as symbolic answer.)

electron



$$U = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

$$= \frac{\epsilon_0}{2} \left(\int_0^R \underbrace{E^2}_{=0} 4\pi r^2 dr + \int_R^\infty E_{\text{outside}}^2 4\pi r^2 dr \right)$$

$$= \frac{\epsilon_0}{2} \int_R^\infty \frac{1}{(4\pi\epsilon_0)^2} \frac{e^2}{r^4} 4\pi r^2 dr$$

$$= \frac{e^2}{2} \frac{1}{4\pi\epsilon_0} \int_R^\infty \frac{1}{r^2} dr$$

$$= -\frac{1}{r} \Big|_R^\infty$$

$$= \frac{1}{R}$$

$$U = \frac{e^2}{8\pi\epsilon_0} \cdot \frac{1}{R}$$

$$U_{\text{also}} = mc^2$$

$$mc^2 = \frac{e^2}{8\pi\epsilon_0} \frac{1}{R}$$

$$R = \frac{e^2}{8\pi\epsilon_0 mc^2}$$

$$= \frac{(1.6 \cdot 10^{-19})^2}{8\pi (8.85 \cdot 10^{-12}) (9.11 \cdot 10^{-31}) (3 \cdot 10^8)^2}$$

$$= 1.40 \cdot 10^{-15} \text{ m}$$