Fall 2016
Physics 441
Exam 2
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40104917

No time limit. Student calculators are allowed. One page of <u>handwritten</u> notes allowed (front & back). Books not allowed. A HANDOUT WITH FRONT AND BACK INSIDE COVERS OF GRIFFITHS TEXTBOOK SHOULD BE PROVIDED. If not, please ask the Testing Center for it and/or have them call me.

Name Solutions

Instructions: Please label & circle/box your answers. **Show your work**, where appropriate! And remember: **in any problems involving Gauss's Law, you should explicitly show your Gaussian surface**. For all problems, unless otherwise specified you may assume that you are dealing with **electrostatics**, i.e. the charges are not moving and the fields have come to equilibrium.

Integral/derivative table: One or more of the following integrals or derivatives may or may not be helpful on the exam. If you find yourself needing anything more complicated than this, then you have likely made an error.

(∂_x = Mathematica for $\frac{d}{dx}$)

In[1]:= $\int e^{a x}\, dx$

Out[1]= $\frac{e^{a x}}{a}$

In[2]:= $\int x\, e^{a x}\, dx$ // Expand

Out[2]= $-\frac{e^{a x}}{a^2} + \frac{e^{a x} x}{a}$

In[3]:= $\int x^2\, e^{a x}\, dx$ // Expand

Out[3]= $\frac{2 e^{a x}}{a^3} - \frac{2 e^{a x} x}{a^2} + \frac{e^{a x} x^2}{a}$

In[4]:= $\partial_x e^{a x}$

Out[4]= $a\, e^{a x}$

In[5]:= $\partial_x (x\, e^{a x})$

Out[5]= $e^{a x} + a\, e^{a x} x$

In[6]:= $\partial_x (x^2\, e^{a x})$

Out[6]= $2\, e^{a x} x + a\, e^{a x} x^2$

Some Legendre polynomials:
$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = 3/2\, x^2 - 1/2$
$P_3(x) = 5/2\, x^3 - 3/2\, x$

Orthogonality of the Legendre polynomials:

$$\int_{-1}^{1} P_\ell(x)\, P_m(x) dx = \begin{cases} 0 & \text{if } \ell \neq m \\ \dfrac{2}{2\ell+1} & \text{if } \ell = m \end{cases}$$

$$\int_{0}^{\pi} P_\ell(\cos\theta)\, P_m(\cos\theta) \sin\theta\, d\theta = \begin{cases} 0 & \text{if } \ell \neq m \\ \dfrac{2}{2\ell+1} & \text{if } \ell = m \end{cases}$$

Orthogonality of the sine functions:

$$\int_{0}^{1} \sin(n\pi x) \sin(m\pi x)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ 1/2 & \text{if } n = m \end{cases}$$

$$\int_{0}^{2\pi} \sin(nx) \sin(mx)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ 1/2 & \text{if } n = m \end{cases}$$

Phys 441 Exam 2 Solutions

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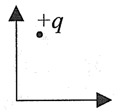
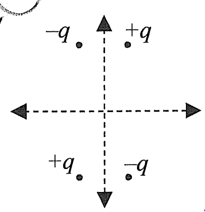
(~~20~~ pts) **Problem 1:** Multiple choice, 2 pts each. Circle the correct answer.

1.1. A charge q is near two infinite grounded planes that run along the positive x and y axes as depicted below.

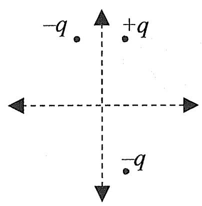


What collection of image charges would map the same potential?

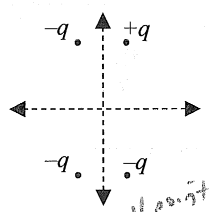
(a)



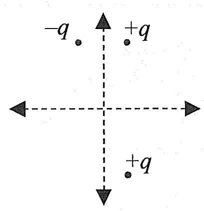
(b)



(c)

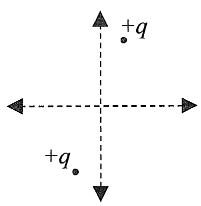


(d)

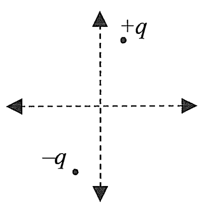


→ for this choice, only, symmetry guarantees that all points on both axes are at 0V

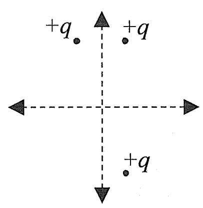
(e)



(f)



(g)



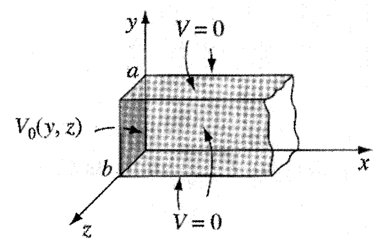
(h) None of them.

1.2. In class we solved Laplace's equation in 2D, obtaining $V(x, y)$. You have also worked one or more 2D problems for homework. Which of the following is true about situations like that where we can write V is a function of x and y only?

(a) The third dimension must be infinitely thin.
(b) The third dimension must be infinitely long.
(c) Such a potential is just a mathematical construct; it has no physical basis in reality.

→ No z-dependence means object is infinitely long in z-direction

1.3. An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$. See the figure.



The potential can be described through the general equation:

$$V(x, y, z) = \left(Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x}\right)(C \sin ky + D \cos ky)(E \sin lz + F \cos lz)$$

Which of the following is true about the constants in the equation?

✓ I. A = 0 otherwise $V \to \infty$ at $x \to \infty$

✓ II. D = 0 bndy conds for y mean we only have sines

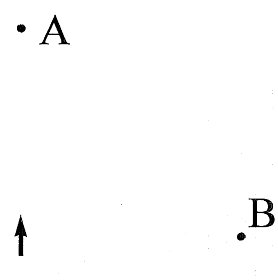
✓ III. F = 0 " " " z " " " " "

✓ IV. $k = \frac{n\pi}{a}$ (where n = an integer)

bndy cond for y, $V(y=a)=0$ is satisfied only if $k = \frac{n\pi}{a}$

(a) I only
(b) I and II only
(c) I and III only
(d) I and IV only
(e) II and III only
(f) II and IV only
(g) III and IV only
(h) I, II, and III only
(i) I, II, and IV only
(j) I, III, and IV only
(k) II, III, and IV only
(l) I, II, III, and IV

1.4. Points A and B are the same large distance from an electric dipole, but in different directions as per the figure. The dipole is depicted by the arrow (dipole moment in the direction of the arrow). What is true of the magnitude of the **E** field at point A compared to point B?



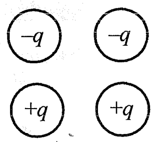
(a) $|\vec{E}_a| = 0$
(b) $|\vec{E}_b| = 0$
(c) $|\vec{E}_a|$ and $|\vec{E}_b|$ both $= 0$
(d) $|\vec{E}_a| > |\vec{E}_b|$ (and $|\vec{E}_b| \neq 0$)
(e) $|\vec{E}_a| < |\vec{E}_b|$ (and $|\vec{E}_a| \neq 0$)
(f) $|\vec{E}_a| = |\vec{E}_b|$ (and neither one is zero)

$$\vec{E}_{dip} = \frac{p}{4\pi\epsilon_0}\frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

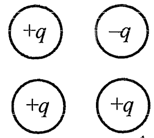
$$A: \theta = 0° \rightarrow \vec{E}_A = \frac{p}{4\pi\epsilon_0}\frac{1}{r^3}(2\hat{r})$$

$$B: \theta = 90° \rightarrow \vec{E}_B = \frac{p}{4\pi\epsilon_0}\frac{1}{r^3}(1\hat{\theta})$$

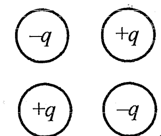
1.5. Which of the following configurations would have the fastest "fall off" of the electric field with r, for regions far from the charges? (The spheres are not conducting.)

(a) 

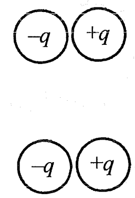
dipole

(b) 

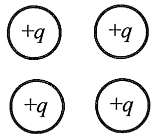
monopole

(c) 

quadrupole $(E \sim \frac{1}{r^4})$

(d) 

dipole

(e) 

monopole

(f) It's a tie between more than one of them.

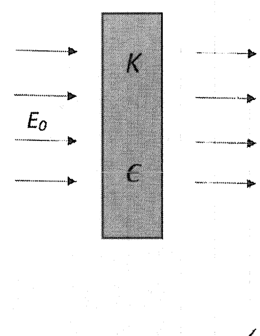
1.6. What does the "polarization" of a material describe?

(a) The net amount of charge in a unit volume, changing throughout the region of space
(b) A conductor's ability to fluctuate its net charge from positive to negative continuously
(c) Leakage of excess charge in a densely charged area or "pole" of a material
(d) The vector sum of all dipole moments per small volume element at various points within a material
(e) The displacement from equilibrium position due to free charges in a material

1.7. Which of the following is true of the displacement field, **D**?

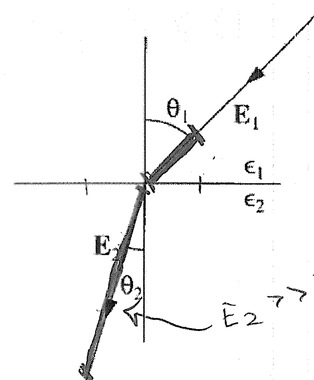
(a) It is determined solely by the free volume charge density, ρ_{free}. $\nabla\cdot\vec{D} = \rho_{free}$ ✓
(b) It represents the electric field created by the bound charges. No → free charges, not bound charges
(c) **D** field lines must begin and end on surface charges No, could have volume or point charges
(d) The **D** field is only non-zero in regions where there is polarization of charges. No, consider q this region zero net flux, but non zero $\vec{D}$
(e) More than one of the above.

1.8. An infinite slab of insulating material with dielectric constant K and permittivity $\epsilon = \epsilon_0 K$ is placed in a uniform electric field of magnitude E_0. The field is perpendicular to the surface of the material, as shown in the figure. What is the magnitude of the electric field inside the material?



(a) E_0/K As discussed in class, E field is lowered by dielectric constant $(= \epsilon_r)$
(b) $E_0/(K\epsilon_0)$
(c) E_0
(d) $\epsilon_0 K E_0$
(e) KE_0

1.9. The following diagram depicts how the **E**-field changes across the boundary of two linear dielectric mediums. What can be said about ϵ_1 and ϵ_2? Assume there are no free charges at the boundary and $\theta_1 = 45°$. (Careful: even though the figure looks a bit like optical refraction, these are not light rays.)



(a) $\epsilon_1 > \epsilon_2$
(b) $\epsilon_1 < \epsilon_2$
(c) $\epsilon_1 = \epsilon_2$
(d) $\epsilon_1 = 0$
(e) $\epsilon_2 = 0$

BC: $E_{\parallel 1} = E_{\parallel 2}$ means $\vec{E}_1$ and $\vec{E}_2$ have relative magnitudes as shown

BC $\epsilon_1 E_{\perp 1} - \epsilon_2 E_{\perp 2} = 0$ (because $\sigma_{free} = 0$)

means $\frac{E_{\perp 2}}{E_{\perp 1}} = \frac{\epsilon_1}{\epsilon_2}$, so ϵ_1 must be 3-4 x larger than ϵ_2

(14 pts) **Problem 2**: A thin insulating spherical shell (radius R) has a uniform surface charge density σ_0 coating it. The center of the sphere is fixed at a distance d above the center of an infinite, grounded conducting plate.

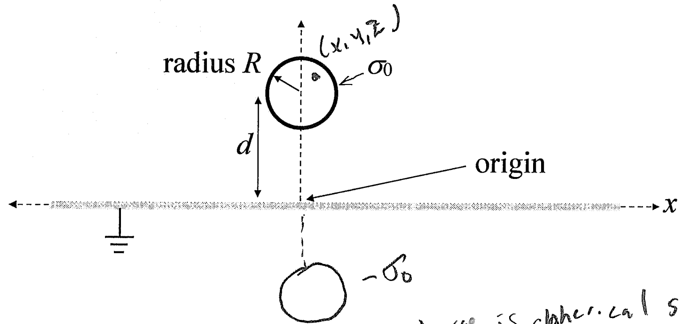


image charge is spherical shell with $-\sigma_0$

(a) Find the potential V for an arbitrary point (x, y, z) that lies *inside* the spherical shell. Use the specified origin of coordinates.

From knowledge of Gauss's Law, V from $+\sigma_0$ = constant (since $\vec{E}$ must be zero)

V from $-\sigma_0$ = same as from a pt charge at $(0,0,-d)$ having charge $q = \sigma A = -\sigma_0 4\pi R^2$

The constant value is same as from point charge, $V = -\int_\infty^R \left(\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\right)dr = \frac{q}{4\pi\epsilon_0}\frac{1}{R}$

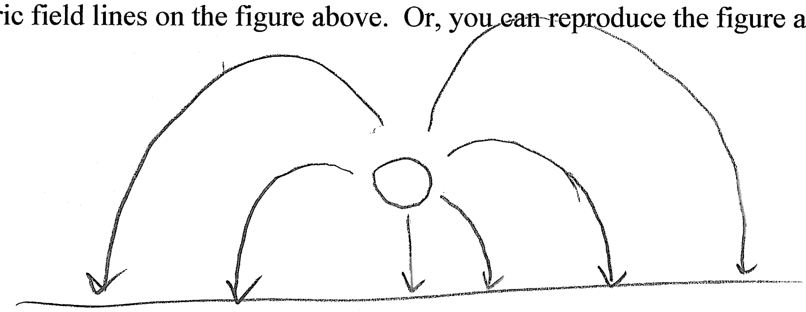
answer by superposition of actual charge and image charge is...

$$\boxed{V(x,y,z) = \frac{\sigma_0 4\pi R^2}{4\pi\epsilon_0}\frac{1}{R} - \frac{\sigma_0 \cdot 4\pi R^2}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+(z+d)^2}}}$$

for points inside the spherical shell

could simplify if desired, $V = \frac{\sigma_0 R}{\epsilon_0}\left(1 - \frac{R}{\sqrt{x^2+y^2+(z+d)^2}}\right)$

(b) Sketch the electric field lines on the figure above. Or, you can reproduce the figure and sketch here.



(like the upper half of a dipole's field lines)

(16 pts) **Problem 3**: A sphere of radius R has potential $V = V_0 \cos^2 \theta$ on its surface. Find the potential as a function of r and θ, for points *outside* the sphere. Hint: first find $\cos^2 \theta$ in terms of the Legendre polynomials in θ. See the first page of the exam.

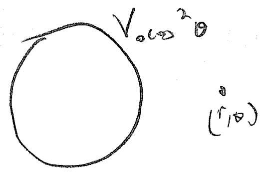
First: $P_2 = \frac{3}{2}\cos^2\theta - \frac{1}{2} \rightarrow$

$$2P_2 = 3\cos^2\theta - 1$$

$$2P_2 = 3\cos^2\theta - P_0$$

so $\underline{\cos^2\theta = \frac{2}{3}P_2 + \frac{1}{3}P_0}$

Now



$\nabla^2 V = 0$, spherical coords with no ϕ

$\Rightarrow V = R(r)\,\Theta(\theta)$

BC 1 $V = V_0\cos^2\theta$ at $r=R$

BC 2 $V =$ finite at $r = \infty$

and from book / class notes

$$R = \begin{cases} r^{\ell} & \rightarrow \text{BC 2 means we throw this one out} \\ \frac{1}{r^{\ell+1}} \end{cases}$$

$$\Theta = P_\ell(\cos\theta)$$

Sum over ℓ ... $\underline{V = \sum_\ell C_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)}$ general formula

Impose BC 1:

$$V_0\cos^2\theta = \sum_\ell C_\ell \frac{1}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$V_0 \frac{2}{3}P_2 + V_0\frac{1}{3}P_0 = \sum_\ell C_\ell \frac{1}{R^{\ell+1}} P_\ell(\cos\theta)$$

equate coefficients $\rightarrow$ only $\ell=2$ and $\ell=0$ terms survive

$\ell=0$: $\frac{V_0}{3} = C_0 \frac{1}{R} \rightarrow C_0 = \frac{V_0 R}{3}$

$\ell=2$: $\frac{2V_0}{3} = C_2\frac{1}{R^3} \rightarrow C_2 = \frac{2}{3}V_0 R^3$

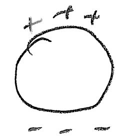
put back in general formula:

$$\boxed{V = \frac{V_0 R}{3}\frac{1}{r} + \frac{2}{3}V_0 R^3 \frac{1}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)}$$

(16 pts) **Problem 4**: A sphere of radius R centered on the origin carries a surface charge density of $\sigma_0\cos\theta$. Find the approximate potential and the field for this charge distribution for points along the y-axis ~~(0, 0, y)~~, where $|y| \gg R$.

$(0, y, 0)$

This will have a dipole moment in the $\hat{z}$ direction



First find dipole moment $\vec{p}$:

$$\vec{p} = \int \sigma(\vec{r}\,')\,\vec{r}\,'\,da'$$

I'll just keep the z-component of this, $= r'\cos\theta'\,\hat{z}$

$$\vec{p} = \int (\sigma_0\cos\theta')(\underbrace{r'}_{=R \text{ on surface}}\cos\theta'\,\hat{z})\,(R^2\sin\theta'\,d\theta'\,\underbrace{d\phi'}_{\text{integrates to } 2\pi})$$

$$= 2\pi\sigma_0 R^3\,\hat{z}\,\underbrace{\int_0^\pi \cos^2\theta'\sin\theta'\,d\theta'}_{-\frac{1}{3}\cos^3\theta'\big|_0^\pi}$$

$$= \frac{1}{3}\cos^3 0 - \frac{1}{3}\cos^3\pi = \frac{2}{3}$$

$$\underline{\vec{p} = \frac{4\pi}{3}\sigma_0 R^3\,\hat{z}}$$

$$V_{\text{dipole}} = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

For points on the y-axis $\theta = \pi/2$, $\hat{\theta} = -\hat{z}$, $r = |y|$, $\hat{r} = \hat{y}$

$$\theta = \pi/2 \rightarrow \boxed{V = 0}$$

By the way, this is exact. Due to symmetry, $V = 0$ everywhere in the x-y plane.

$$\vec{E}_{\text{dipole}} = \frac{p}{4\pi\epsilon_0}\frac{1}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

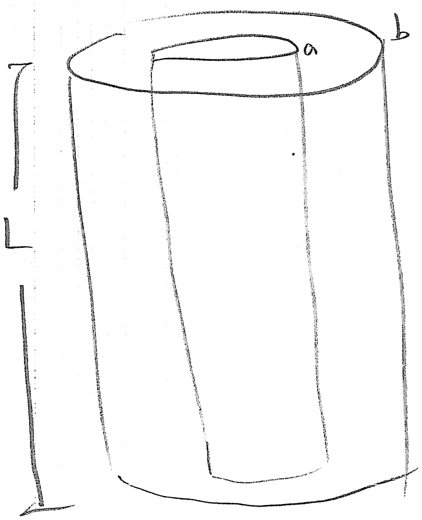
$$\vec{E} = \frac{\frac{\cancel{4\pi}}{3}\sigma_0 R^3}{\cancel{4\pi}\epsilon_0}\frac{1}{|y|^3}\left(\cancel{2\cos 90^\circ\,\hat{y}}^{\,0} + \sin 90^\circ(-\hat{z})\right)$$

$$\boxed{\vec{E} = \frac{\sigma_0}{3\epsilon_0}\frac{R^3}{|y|^3}(-\hat{z})}$$

for points on the y axis, with $y \gg R$

(I used $|y|$ because for the negative y-axis, the $\vec{E}$ field is still in the $-\hat{z}$ direction)

(11 pts) **Problem 5**. A finite cylindrical shell of length L, inner radius a, and outer radius b carries a polarization $\mathbf{P} = ks\hat{\mathbf{s}}$, where k is a constant and s is the usual cylindrical coordinate. Calculate the bound charges σ_b (on all four surfaces) and ρ_b.



$$\sigma_b = \vec{P} \cdot \hat{n}$$

Top: $\hat{n} = \hat{z}$
Bottom: $\hat{n} = -\hat{z}$
so $\boxed{\sigma_b = 0.}$ for top + bottom
since $\vec{P} \perp \hat{n}$

inner surface: $\hat{n} = -\hat{s}$

$$\sigma_b = \vec{P} \cdot \hat{n}\Big|_{s=a} = ka\,\hat{s} \cdot (-\hat{s})$$

$$\boxed{\sigma_b = -ka} \quad s = a$$

outer surface: $\hat{n} = +\hat{s}$

$$\sigma_b = \vec{P} \cdot \hat{n}\Big|_{s=b} = kb\,\hat{s} \cdot (+\hat{s})$$

$$\boxed{\sigma_b = kb} \quad s = b$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}(s P_s) + \cancel{\frac{1}{s}\frac{\partial P_\phi}{\partial \phi}}^{\,0} + \cancel{\frac{\partial P_z}{\partial z}}^{\,0}$$ from Griffiths cover

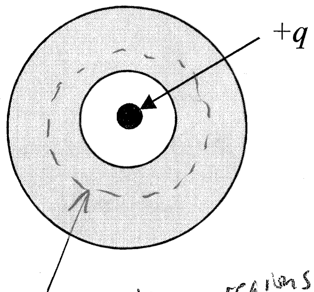
b

$$= \frac{1}{s}\frac{\partial}{\partial s}(ks^2)$$

$$= \frac{k}{s}\,2s$$

$$\boxed{\rho_b = 2k}$$

(12 pts) **Problem 6**. A point charge ($+q$) is at the origin. It is surrounded by a spherical dielectric shell (dielectric constant ε_r) with inner radius a and outer radius b. (a) Determine $\boldsymbol{D}$ in the three regions: (i) $r < a$, (ii) $a < r < b$, and (iii) $r > b$. (b) Determine $\boldsymbol{E}$ in the same three regions.



(a) For all three regions can use Gaussian surface like this, and $q_{free\,enc} = +q$ for all three!

$$\oint \vec{D} \cdot d\vec{a} = q_{free\,enc}$$

$$D \cdot 4\pi r^2 = q$$

$$\boxed{\vec{D} = \frac{q}{4\pi r^2}\,\hat{r}}$$ everywhere!

(b) $\vec{D} = \epsilon_0 \epsilon_r \vec{E} \rightarrow \vec{E} = \dfrac{\vec{D}}{\epsilon_0 \epsilon_r}$ where ϵ_r is the dielectric constant for the particular region

Regions (i) and (iii): $\epsilon_r = 1$

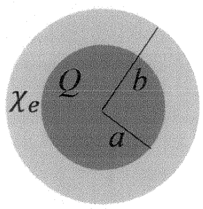
so $\vec{E} = \dfrac{\vec{D}}{\epsilon_0} \rightarrow \boxed{\vec{E} = \dfrac{q}{4\pi\epsilon_0 r^2}\,\hat{r}}$ $r < a$ and $r > b$

Region (ii): ϵ_r = given ϵ_r

so $\vec{E} = \dfrac{\vec{D}}{\epsilon_0 \epsilon_r} \rightsquigarrow \boxed{\vec{E} = \dfrac{q}{4\pi\epsilon_0\epsilon_r r^2}\,\hat{r}}$ $a < r < b$

(13 pts) **Problem 7**. A spherical conductor of radius a, carries a charge Q as shown below. It is surrounded by a linear dielectric material of susceptibility χ_e, which extends out to radius b. Find the energy stored in the electric field for this configuration in terms of the variables given.

(Note $\vec{E}$ and $\vec{D} = 0$ for $r < a$)



Similar to last problem

$a < r < b \rightarrow \vec{D} = \frac{Q}{4\pi}\frac{1}{r^2}\hat{r}$

and $\vec{E} = \frac{Q}{4\pi\epsilon_0\epsilon_r}\frac{1}{r^2}\hat{r}$

$r > b \rightarrow \vec{D} = \frac{Q}{4\pi}\frac{1}{r^2}\hat{r}$

and $\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}$.

$$U = \frac{1}{2}\int_{\text{allspace}} \vec{D}\cdot\vec{E}\, d\tau \qquad d\tau = 4\pi r^2 dr$$

$$= \frac{1}{2}\int_0^a 0\, d\tau + \frac{1}{2}\int_a^b \left(\frac{Q}{4\pi r^2}\right)\left(\frac{Q}{4\pi\epsilon_0\epsilon_r r^2}\right)(4\pi r^2 dr) + \frac{1}{2}\int_b^\infty \left(\frac{Q}{4\pi r^2}\right)\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)(4\pi r^2 dr)$$

$$= \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0\epsilon_r}\underbrace{\int_a^b \frac{dr}{r^2}}_{-\frac{1}{r}\Big|_a^b = \frac{1}{a}-\frac{1}{b}} + \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\underbrace{\int_b^\infty \frac{dr}{r^2}}_{-\frac{1}{r}\Big|_b^\infty = \frac{1}{b}}$$

$$\boxed{U = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\left[\frac{1}{\epsilon_r}\left(\frac{1}{a}-\frac{1}{b}\right) + \frac{1}{b}\right]}$$

where $\epsilon_r = 1 + \chi_e$