Fall 2017
Physics 441
Exam 2
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barcode here

No time limit. Student calculators are allowed. One page of handwritten notes allowed (front & back). Books not allowed.

Name Solutions

Instructions: Please label & circle/box your answers. **Show your work**, where appropriate! And remember: **in any problems involving Gauss's Law, you should explicitly show your Gaussian surface**. For all problems, unless otherwise specified you may assume that you are dealing with **electrostatics**, i.e. the charges are not moving and the fields have come to equilibrium, and that all **dielectrics are linear and isotropic**.

Griffiths front and back covers

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl:
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

Special case derivatives: $\quad \nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta(\mathbf{r}) \qquad \nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r})$
(similar things true for $\boldsymbol{\mathscr{r}}$)

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^8\ \mathrm{m/s}$	(speed of light)
$e = 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Some miscellaneous mathematical stuff:

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{a}{2}, & \text{if } n = m \end{cases}$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$
$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\int_{-1}^{1} P_\ell(x)\,P_m(x)dx = \begin{cases} 0, & \text{if } \ell \neq m \\ \frac{2}{2\ell+1}, & \text{if } \ell = m \end{cases}$$

$$\int_0^\pi P_\ell(\cos\theta)\,P_m(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0, & \text{if } \ell \neq m \\ \frac{2}{2\ell+1}, & \text{if } \ell = m \end{cases}$$

$P_\ell(x)$ are the Legendre polynomials; the first few are these:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

Some definite integrals:

$\int_0^\pi \sin^2 x\,dx = \frac{\pi}{2}$	$\int_0^\pi \cos^2 x\,dx = \frac{\pi}{2}$	$\int_0^{2\pi} \sin^2 x\,dx = \pi$	$\int_0^{2\pi} \cos^2 x\,dx = \pi$
$\int_0^\pi \sin^3 x\,dx = \frac{4}{3}$	$\int_0^\pi \cos^3 x\,dx = 0$	$\int_0^{2\pi} \sin^3 x\,dx = 0$	$\int_0^{2\pi} \cos^3 x\,dx = 0$
$\int_0^\pi \sin^4 x\,dx = \frac{3\pi}{8}$	$\int_0^\pi \cos^4 x\,dx = \frac{3\pi}{8}$	$\int_0^{2\pi} \sin^4 x\,dx = \frac{3\pi}{4}$	$\int_0^{2\pi} \cos^4 x\,dx = \frac{3\pi}{4}$

(16 pts) **Problem 1:** Multiple choice, 2 pts each. Circle the correct answer.

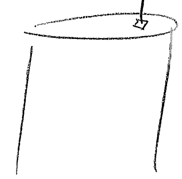
1.1. For an infinitesimal d**A** on the top surface of a cylinder, what would be the appropriate formula to use in cylindrical coordinates?

(a) $s\,ds\,d\phi\,\hat{\mathbf{s}}$
(b) $s\,ds\,d\phi\,\hat{\boldsymbol{\phi}}$
(c) $s\,ds\,d\phi\,\hat{\mathbf{z}}$
(d) $s\,d\phi dz\,\hat{\mathbf{s}}$
(e) $s\,d\phi dz\,\hat{\boldsymbol{\phi}}$
(f) $s\,d\phi dz\,\hat{\mathbf{z}}$

$d\vec{a}$ direction is $\hat{z}$

infinitesimal area

magnitude is $s\,ds\,d\phi$



1.2. A potential varies as $1/r^3$. Which of the following could it be?

(a) A monopole potential $\rightarrow V \sim \frac{1}{r},\ E \sim \frac{1}{r^2}$
(b) A dipole potential $\rightarrow V \sim \frac{1}{r^2},\ E \sim \frac{1}{r^3}$
(c) A quadrupole potential $\rightarrow V \sim \frac{1}{r^3},\ E \sim \frac{1}{r^4}$
(d) An octopole potential
(e) A higher order potential

1.3. A spherical surface of radius *R* is somehow maintained at the following potential: $V(\theta) = V_0(\cos^4\theta + 1)$. The formula for the potential inside the surface will involve:

(a) An infinite sum of Legendre polynomials in $\cos\theta$.
(b) A finite sum of Legendre polynomials in $\cos\theta$.
(c) An infinite sum of sines/cosines and/or exponentials.
(d) A finite sum of sines/cosines and/or exponentials.

$\cos^4\theta + 1$ can be written as finite sum of Legendre polynomials, therefore SoV technique will yield finite sum for V

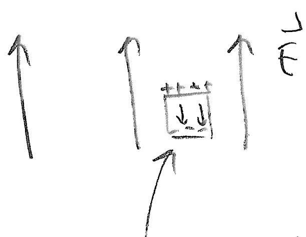
1.4. A cube of side length *a* is somehow maintained at the following potential: V = 0 for all sides except for one, and is a constant V_0 on that side. The formula for the potential inside the surface will involve:

(a) An infinite sum of Legendre polynomials in $\cos\theta$.
(b) A finite sum of Legendre polynomials in $\cos\theta$.
(c) An infinite sum of sines/cosines and/or exponentials.
(d) A finite sum of sines/cosines and/or exponentials.

We did this problem in class, and had to use an infinite sum to match the $V = V_0$ boundary condition.

1.5. A dielectric is placed in an external electric field whose magnitude was E_0 before the dielectric was put in the region. (Initially the region contained only a vacuum.) Which is true about the total electric field inside the dielectric, $\mathbf{E}_{tot}$?

(a) $E_{tot} < E_0$, and they are in the same direction
(b) $E_{tot} = E_0$, and they are in the same direction
(c) $E_{tot} > E_0$, and they are in the same direction
(d) $E_{tot} < E_0$, and they are in the opposite direction
(e) $E_{tot} = E_0$, and they are in the opposite direction
(f) $E_{tot} > E_0$, and they are in the opposite direction



as discussed in class, the material will polarize and set up counter field to oppose $\vec{E}_0$. It will be smaller than $\vec{E}_0$, though, so total field is reduced but still in $\vec{E}_0$ direction.

1.6. Same situation. D_0 is the magnitude of the **D** field before the dielectric was put in the region. What is true about the total **D** field inside the dielectric, $\mathbf{D_{tot}}$? (Assume no edge effects, if it matters.)

(a) $D_{tot} < D_0$, and they are in the same direction
(b) $D_{tot} = D_0$, and they are in the same direction
(c) $D_{tot} > D_0$, and they are in the same direction
(d) $D_{tot} < D_0$, and they are in the opposite direction
(e) $D_{tot} = D_0$, and they are in the opposite direction
(f) $D_{tot} > D_0$, and they are in the opposite direction

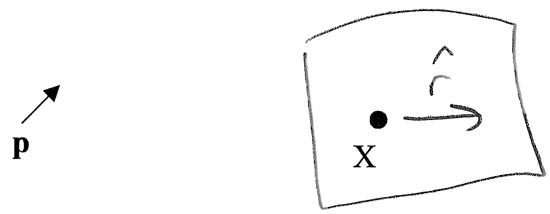
Given this, we can say that $E_{tot} = \frac{E_0}{\epsilon_r}$ (as discussed in class).

Also $D = \epsilon_0 \epsilon_r E$ inside and $D = \epsilon_0 E$ outside

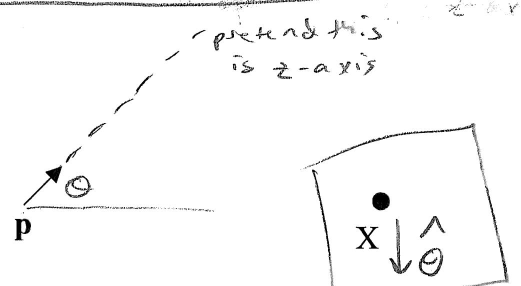
So $\frac{D_{tot}}{D_0} = \frac{\cancel{\epsilon_0}\epsilon_r E_{tot}}{\cancel{\epsilon_0} E_0} = \frac{\epsilon_r (E_0/\epsilon_r)}{E_0} = 1$

$D_{tot} = D_0$

These last two aren't multiple choice questions but they seemed to fit best here, and each one counts 2 pts like the multiple choice questions.



1.7. Given a dipole moment **p** and point X as shown in the figure, what is the direction of $\hat{\mathbf{r}}$ as must be used in the dipole electric field equation, namely $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\ \hat{\mathbf{r}} + \sin\theta\ \hat{\boldsymbol{\theta}}\right)$? Draw the $\hat{\mathbf{r}}$ direction on the figure just above.



1.8. Same situation… what is the direction of $\hat{\boldsymbol{\theta}}$? Draw the $\hat{\boldsymbol{\theta}}$ direction on the figure just above.

(9 pts) **Problem 2**: Short answers—use *words*, not equations. Or at most, perhaps a single equation in a supplementary role.

(a) Briefly explain how/why the method of relaxation works.

How: numerically set each point to the average of surrounding pts, and iterate until no change in numerical values.

Why: Solns to Laplace's Eqn must be as "smooth as possible", which means each pt is average of surrounding points. Uniqueness theorem means that when you find a function which does this and also satisfies boundary conditions, it is the correct function.

(b) Briefly explain why you may generally stop at the first non-zero term when solving problems with the multipole expansion.

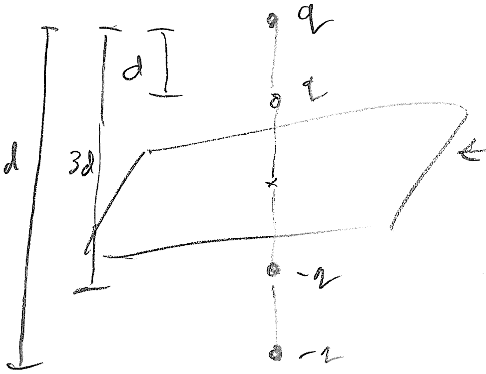
Multipole expansion terms have increasing powers of $\frac{1}{r}$.
If r is big compared to size of charge distribution then each successive term is much smaller than the preceding terms and can therefore be neglected as long as you have a non-zero term.

(c) Briefly explain what the polarization field, **P**, means.

$\vec{P}$ is the density of dipole moments in the material, i.e. how much dipole moment per volume exists at a given (x, y, z) coordinate. This is in the approximation that the dipole moment density is a smoothly varying function ignoring the ups and downs from individual atoms.

(11 pts) **Problem 3:** Two charges are above a grounded conducting plane: charge $+q$ a distance d above the plane, and another charge $+q$ a distance d directly above that (i.e. $2d$ from the plane). What is the net force on the upper charge?

image problem



By symmetry the plane will have $V=0$ everywhere.

Force on top charge is superposition of forces from the other three charges, $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{z}$ for each one.

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qq}{d^2}\hat{z} - \frac{1}{4\pi\epsilon_0}\frac{qq\,\hat{z}}{(3d)^2} - \frac{1}{4\pi\epsilon_0}\frac{qq\,\hat{z}}{(4d)^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{z}\underbrace{\left(1 - \frac{1}{9} - \frac{1}{16}\right)}_{\approx .826 \text{ (or } \frac{119}{144} \text{ if you want exact)}}$$

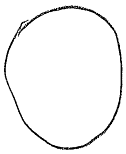
$$\boxed{\vec{F} = .826\,\frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{z}}$$

$$\left(\text{or } \frac{119}{144}\frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{z}\right)$$

$P_0 = 1 \qquad P_2 = \frac{3}{2}x^2 - \frac{1}{2}$

$P_1 = x$ $\Big\}$ from pg 2

(16 pts) **Problem 4**: An insulating spherical shell has potential on its surface (at $r = R$) given by $V(R,\theta) = V_0 \cos^2\theta$. Find $V(r,\theta)$ for all points *outside* the shell. Hint: your first step could be to use page 2 of the exam to find $\cos^2\theta$ in terms of Legendre polynomials.



$\to x = \cos\theta$, we want x^2 in terms of P_0, P_1, P_2

Start with P_2: $\frac{2}{3}P_2 = \frac{2}{3}\cdot\frac{3}{2}x^2 - \frac{2}{3}\cdot\frac{1}{2}$

$\frac{2}{3}P_2 = x^2 - \frac{1}{3}$

So $x^2 = \frac{2}{3}P_2 + \frac{1}{3}$

$$\boxed{x^2 = \frac{2}{3}P_2 + \frac{1}{3}P_0}$$

Separation of Variables in spherical coords w/o ϕ

$\to V = R(r)\,\Theta(\theta)$

Leads to $R = \cancel{r^\ell}, \frac{1}{r^{\ell+1}}$ throw out because "outside the shell" goes infinite at $r=\infty$

$\Theta = P_\ell(\cos\theta)$

General form $V_{outside} = \sum_{\ell=0}^{\infty} C_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$

Bndy cond $V(r=R) = V_0\left(\frac{2}{3}P_2 + \frac{1}{3}P_0\right)$

$$\sum_{\ell=0}^{\infty} C_\ell \frac{1}{R^{\ell+1}} P_\ell = \frac{V_0}{3}P_0 + \frac{2V_0}{3}P_2$$

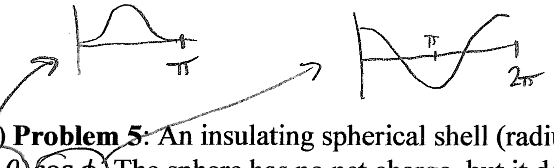
equate coefficients, all terms $=0$ except $\ell=0$ and $\ell=2$

$\ell=0$ $\quad C_0 \frac{1}{R} = \frac{V_0}{3} \to C_0 = \frac{V_0 R}{3}$

$\ell=2$ $\quad C_2 \frac{1}{R^3} = \frac{2V_0}{3} \to C_2 = \frac{2V_0}{3}R^3$

Final answer: $\boxed{V_{outside} = \frac{V_0}{3}\frac{R}{r}P_0(\cos\theta) + \frac{2V_0}{3}\frac{R^3}{r^3}P_2(\cos\theta)}$

or $\boxed{V_{outside} = \frac{V_0}{3}\frac{R}{r} + \frac{2V_0}{3}\frac{R^3}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)}$ if you want to plug in for P_0 and P_2

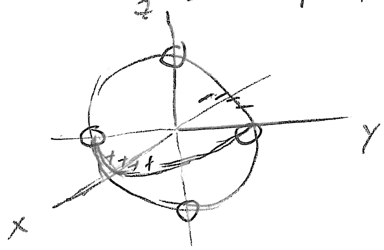


(16 pts) **Problem 5**: An insulating spherical shell (radius R) has a surface charge density of $\sigma(R,\theta) = \sigma_0 \sin^2\theta \cos\phi$. The sphere has no net charge, but it does have a dipole moment.

(a) Make a sketch of the shell, indicating where the charges will be positive, negative and zero, and use that to deduce the direction of the dipole moment. Or, if your sketching ability is not up to it, describe very carefully with words where the charges will be positive, negative, and zero, and what that means for the direction of the dipole moment.

From graphs above we see σ will go to zero at $\theta = 0$ and $\theta = \pi$ and will be maximal at $\theta = \pi/2$

Also σ will be positive around $\phi = 0$ (positive x-axis), ... negative around $\phi = \pi$ (negative x-axis), and zero around $\phi = \pi/2$ and $3\pi/2$ (pos + neg y-axis)



From picture $\boxed{\vec{p} = +\hat{x} \text{ direction}}$

(points from negatives to positives)

(b) Calculate the dipole moment, hopefully using your answer to part (a) to help you avoid some work. Hint: there are some definite integrals given on page 2 of the exam; one or more should be helpful.

$\vec{p} = \int \vec{r}\,' \rho(\vec{r}\,')\, d\tau'$ in general, or $\vec{p} = \int \vec{r}\,' \sigma(\vec{r}\,')\, da'$ for surface charge densities

$$\vec{r}\,' = R\hat{r} = R(\sin\theta'\cos\phi'\hat{x} + \underbrace{\ldots\hat{y} + \ldots\hat{z}}_{\text{unimportant due to symmetry from part a}})$$

$da' = R^2 \sin\theta'\, d\theta' d\phi'$

$$\vec{p} = \int (R\sin\theta'\cos\phi'\hat{x})(\sigma_0 \sin^2\theta'\cos\phi')(R^2\sin\theta'\, d\theta' d\phi')$$

I'm going to drop the primes for convenience

$$\vec{p} = R^3\sigma_0\hat{x} \underbrace{\int_0^{\pi} \sin^4\theta\, d\theta}_{3\pi/8} \underbrace{\int_0^{2\pi} \cos^2\phi\, d\phi}_{\pi}$$

(from page 2)

$$\boxed{\vec{p} = \frac{3\pi^2}{8} R^3 \sigma_0 \hat{x}}$$

$\vec{P} = P_0 \frac{s}{R} \hat{s}$

(16 pts) **Problem 6**. An infinitely long cylinder (radius R) has a built-in polarization given by $\mathbf{P}(s) = P_0 \left(\frac{s}{R}\right) \hat{\mathbf{s}}$. There are no free charges present.

(a) Find the bound volume and surface charge densities.

volume: $\rho_b = -\nabla \cdot \vec{P}$

$= -\left(\frac{1}{s}\frac{\partial}{\partial s}(s P_s) + \dots \right)$ unimportant terms

$= -\frac{1}{s}\frac{\partial}{\partial s}\left(\frac{P_0}{R} s^2\right)$

$= -\frac{1}{\cancel{s}}\frac{P_0}{R} 2\cancel{s}$

$$\boxed{\rho_b = -\frac{2P_0}{R}}$$

surface: $\sigma_b = \vec{P} \cdot \hat{n}$

$= \vec{P} \cdot \hat{s} \Big|_{s=R}$

$= P_0 \frac{\cancel{R}}{\cancel{R}}$

$$\boxed{\sigma_b = P_0 \text{ at } s = R}$$

Note both charge densities are constant so no integrals need be done to calculate $q_{enclosed}$ in next step

(b) Determine $\mathbf{E}(s)$ inside the cylinder via Gauss's Law for $\mathbf{E}$. Surface shown above, $\vec{E}$ = constant and // to $d\vec{a}$ (wrapper)

$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$

$\cancel{\int_{top}}^{0} + \cancel{\int_{bottom}}^{0} + \int_{wrapper}$ (no flux)

q_{enc} = only from ρ, because Gaussian surface doesn't contain the $s = R$ surface

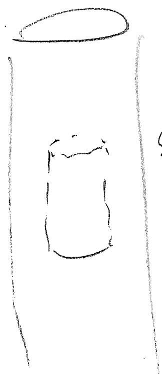
$= \rho V = \left(\frac{-2P_0}{R}\right)(\pi s^2 l)$

$E \cdot \cancel{2\pi s \, l} = \frac{1}{\epsilon_0}\left(-\cancel{2}\frac{P_0}{R}\right)\cancel{\pi} s^{\cancel{2}} \cancel{l}$

$$\boxed{\vec{E} = -\frac{P_0}{\epsilon_0 R} s \hat{s}}$$

(problem continues on next page)

(c) Determine **E**(s) <u>inside</u> the cylinder via Gauss's Law for **D**, and the equation $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$. If you do things correctly your answers to (b) and (c) will agree.



Same Gaussian surface

$\vec{D}$ constant, $\parallel d\vec{a}$ for wrapper

$$\oint \vec{D} \cdot d\vec{a} = \underbrace{q_{free\ enc}}_{=0}$$

$$D \cdot 2\pi s \ell = 0$$

$$\boxed{\vec{D} = 0}$$

Since $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$, we have

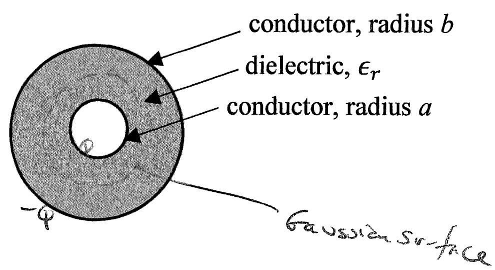
$$0 = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{E} = -\frac{1}{\epsilon_0}\vec{P}$$

$$\boxed{\vec{E} = -\frac{1}{\epsilon_0} P_0 \frac{s}{R} \hat{s}}$$

Yay! It matches part b! :)

(16 pts) **Problem 7**. A spherical capacitor is made by putting dielectric (relative permittivity ϵ_r) between two concentric spherical conductors as shown.



(a) Find the capacitance of the system.

Plan: find $\vec{E}$ → find ΔV → use $C = \frac{Q}{|\Delta V|}$

$$\oint \vec{D} \cdot d\vec{a} = q_{\text{free enc}}$$

$$D \cdot 4\pi r^2 = Q$$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

Use $\vec{D} = \epsilon_0 \epsilon_r \vec{E}$ ⟶ $\underline{\vec{E} = \frac{Q}{4\pi\epsilon_0\epsilon_r r^2}\hat{r}}$

$$\Delta V = -\int_b^a \vec{E} \cdot d\vec{l} = -\int_b^a \frac{Q}{4\pi\epsilon_0\epsilon_r}\frac{dr}{r^2}$$

$d\vec{l} = dr\,\hat{r}$

$$\Delta V = \frac{Q}{4\pi\epsilon_0\epsilon_r}\underbrace{\int_a^b \frac{1}{r^2}\,dr}_{-\frac{1}{r}\Big|_a^b}$$

$$|\Delta V| = \frac{Q}{4\pi\epsilon_0\epsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{Q}{|\Delta V|}$$

$$\boxed{C = 4\pi\epsilon_0\epsilon_r\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}}$$

(b) Find the polarization function **P(r)** for the dielectric in terms of the given quantities.

Can use $\vec{P} = \epsilon_0 \chi_e \vec{E}$ and $\chi_e = \epsilon_r - 1$

$$\vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E} \quad \text{(use } \vec{E} \text{ from above)}$$

$$\vec{P} = \cancel{\epsilon_0}(\epsilon_r - 1)\frac{Q}{4\pi\cancel{\epsilon_0}\epsilon_r}\frac{1}{r^2}\hat{r}$$

$$\boxed{\vec{P} = \frac{Q}{4\pi}\frac{\epsilon_r - 1}{\epsilon_r}\frac{1}{r^2}\hat{r}}$$

(problem continues on next page)

(c) The bound volume charge density is zero (you don't have to verify this). Find the bound surface charge densities at the inner and outer surfaces of the dielectric.

$$\sigma_b = \vec{P} \cdot \hat{n}$$

inner surface $\hat{n} = -\hat{r}$, $r = a$

$$\sigma_b = -\vec{P} \cdot \hat{r} \Big|_{r=a}$$

$$\boxed{\sigma_{b\,\text{inner}} = -\frac{Q}{4\pi} \frac{\epsilon_r - 1}{\epsilon_r} \frac{1}{a^2}}$$

outer surface $\hat{n} = +\hat{r}$, $r = b$

$$\sigma_b = +\vec{P} \cdot \hat{r} \Big|_{r=b}$$

$$\boxed{\sigma_{b\,\text{outer}} = +\frac{Q}{4\pi} \frac{\epsilon_r - 1}{\epsilon_r} \frac{1}{b^2}}$$

(d) Verify that the total bound charge is zero.

inner $q_b = \sigma_b \times \text{area}$

$$= \left(-\frac{Q}{\cancel{4\pi}} \frac{\epsilon_r - 1}{\epsilon_r} \frac{1}{\cancel{a^2}}\right)(\cancel{4\pi a^2})$$

$$\underline{\text{inner } q_b = -Q\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)}$$

outer $q_b = \sigma_b \times \text{area}$

$$= \left(+\frac{Q}{\cancel{4\pi}} \frac{\epsilon_r - 1}{\epsilon_r} \frac{1}{\cancel{b^2}}\right)(\cancel{4\pi b^2})$$

$$\underline{\text{outer } q_b = +Q\,\frac{\epsilon_r - 1}{\epsilon_r}}$$

$$\underline{q_{b\,\text{total}} = q_{b\,\text{inner}} + q_{b\,\text{outer}} = 0} \quad \checkmark$$