No time limit. Student calculators are allowed. One page of handwritten notes allowed (front & back). Books not allowed.

Name: 

Instructions: Please label & circle/box your answers. Show your work, where appropriate! And remember: in any problems involving Gauss's Law, you should explicitly show your Gaussian surface. For all problems, unless otherwise specified you may assume that you are dealing with electrostatics, i.e. the charges are not moving and the fields have come to equilibrium, and that all dielectrics are linear and isotropic.

Griffiths front and back covers

<table>
<thead>
<tr>
<th>VECTOR DERIVATIVES</th>
<th>VECTOR IDENTITIES</th>
</tr>
</thead>
</table>

**Cartesian.** \( dV = dx \hat{i} + dy \hat{j} + dz \hat{k}, \quad dT = dx dy dz \)

**Gradient:** \( \nabla V = \frac{\partial V}{\partial x} \hat{i} + \frac{\partial V}{\partial y} \hat{j} + \frac{\partial V}{\partial z} \hat{k} \)

**Divergence:** \( \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \)

**Curl:** \( \nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{i} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{j} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{k} \)

**Laplacian:** \( \nabla^2 \mathbf{V} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \)

**Spherical.** \( dV = dr \hat{r} + r^2 \phi \hat{\theta} + rs \sin \theta \phi \hat{\phi}, \quad dT = dV \phi dr d\theta d\phi \)

**Gradient:** \( \nabla V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\phi} \)

**Divergence:** \( \nabla \cdot \mathbf{v} = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( r^2 \sin \theta v_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \)

**Curl:** \( \nabla \times \mathbf{v} = \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial \theta} (r^2 v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right) \hat{r} + \left( \frac{\partial v_r}{\partial \phi} - \frac{\partial v_\phi}{\partial r} \right) \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( r^2 v_\phi \right) \hat{\phi} \)

**Laplacian:** \( \nabla^2 \mathbf{V} = \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) \right) + \frac{\partial^2 V}{\partial \theta^2} \sin \theta + \frac{1}{r \sin \theta} \frac{\partial^2 V}{\partial \phi^2} \)

**Cylindrical.** \( dV = dr \hat{r} + s d\phi \hat{\phi} + dz \hat{z}, \quad dT = 2 \pi r dz dr d\phi \)

**Gradient:** \( \nabla V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \phi} \hat{\phi} + \frac{\partial V}{\partial z} \hat{z} \)

**Divergence:** \( \nabla \cdot \mathbf{v} = \left( \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \right) \hat{r} + \frac{1}{r} \left( \frac{\partial (rv_\phi)}{\partial \theta} - \frac{\partial v_r}{\partial \phi} \right) \hat{\phi} + \frac{\partial v_\phi}{\partial z} \hat{z} \)

**Curl:** \( \nabla \times \mathbf{v} = \frac{1}{r} \left( \frac{\partial}{\partial \phi} (v_z) - \frac{\partial v_\phi}{\partial z} \right) \hat{r} + \frac{1}{r} \left( \frac{\partial v_r}{\partial \phi} - \frac{\partial v_\phi}{\partial r} \right) \hat{\phi} + \frac{\partial}{\partial z} \left( rv_\phi - \frac{\partial v_z}{\partial \phi} \right) \hat{z} \)

**Laplacian:** \( \nabla^2 \mathbf{V} = \frac{1}{r} \left( \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2 V}{\partial z^2} \)

Special case derivatives:

(1) \( \nabla \cdot \vec{F} = 4\pi \delta(r) \)

(2) \( \nabla^2 \frac{1}{r} = -4\pi \delta(r) \)

Physics 441 Exam 1 – pg 1
### Maxwell's Equations

In general:
\[
\begin{align*}
\mathbf{V} \cdot \mathbf{E} &= \frac{\partial \rho}{\partial t} \\
\mathbf{V} \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} \\
\mathbf{V} \cdot \mathbf{B} &= 0 \\
\mathbf{V} \times \mathbf{B} &= \mu_0 \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\end{align*}
\]

In matter:
\[
\begin{align*}
\mathbf{V} \cdot \mathbf{D} &= \rho_f \\
\mathbf{V} \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} \\
\mathbf{V} \cdot \mathbf{B} &= 0 \\
\mathbf{V} \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}
\end{align*}
\]

### Fundamentals Constants

<table>
<thead>
<tr>
<th>Property</th>
<th>Value (SI unit)</th>
</tr>
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<tbody>
<tr>
<td>( \varepsilon_0 )</td>
<td>( 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2 )</td>
</tr>
<tr>
<td>( \mu_0 )</td>
<td>( 4\pi \times 10^{-7} \text{ N/A}^2 )</td>
</tr>
<tr>
<td>( c )</td>
<td>( 3.00 \times 10^8 \text{ m/s} )</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>( 1.60 \times 10^{-19} \text{ C} )</td>
</tr>
<tr>
<td>( m )</td>
<td>( 9.11 \times 10^{-31} \text{ kg} )</td>
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</tbody>
</table>

### Spherical and Cylindrical Coordinates

#### Spherical

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>( r \sin \theta \cos \phi )</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( r \sin \theta )</td>
</tr>
<tr>
<td>( \phi )</td>
<td>( r \cos \theta )</td>
</tr>
<tr>
<td>( x )</td>
<td>( \sin \theta \cos \phi )</td>
</tr>
<tr>
<td>( y )</td>
<td>( \sin \theta \sin \phi )</td>
</tr>
<tr>
<td>( z )</td>
<td>( \cos \theta )</td>
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</table>

#### Cylindrical

<table>
<thead>
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<th>Coordinate</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>( \cos \phi )</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( \sin \phi )</td>
</tr>
<tr>
<td>( z )</td>
<td>( z )</td>
</tr>
</tbody>
</table>

### Some Miscellaneous Mathematical Stuff

\[
\begin{align*}
\int_0^a \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi n x}{a} \right) \, dx &= \begin{cases} 
0, & \text{if } n \neq m \\
\frac{a}{2}, & \text{if } n = m
\end{cases} \\
\int_0^1 \rho(x) P_n(x) \, dx &= \begin{cases} 
0, & \text{if } \ell \neq m \\
\frac{1}{2\ell+1}, & \text{if } \ell = m
\end{cases}
\end{align*}
\]

\[
\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{3}{2} x^2 - \frac{1}{2}
\end{align*}
\]

### Legendre Polynomials

- \( P_0(x) = 1 \)
- \( P_1(x) = x \)
- \( P_2(x) = \frac{3}{2} x^2 - \frac{1}{2} \)

Some definite integrals:
\[
\begin{align*}
\int_0^\pi \sin^2 x \, dx &= \frac{\pi}{2} \\
\int_0^\pi \cos^2 x \, dx &= \frac{\pi}{2} \\
\int_0^{2\pi} \sin^2 x \, dx &= \pi \\
\int_0^{2\pi} \cos^2 x \, dx &= \pi \\
\int_0^\pi \sin^3 x \, dx &= 0 \\
\int_0^\pi \cos^3 x \, dx &= 0 \\
\int_0^{2\pi} \sin^3 x \, dx &= 0 \\
\int_0^{2\pi} \cos^3 x \, dx &= 0 \\
\int_0^\pi \sin^4 x \, dx &= \frac{3\pi}{8} \\
\int_0^\pi \cos^4 x \, dx &= \frac{3\pi}{8} \\
\int_0^{2\pi} \sin^4 x \, dx &= \frac{3\pi}{4} \\
\int_0^{2\pi} \cos^4 x \, dx &= \frac{3\pi}{4}
\end{align*}
\]

Physics 441 Exam 1 – pg 2
(16 pts) **Problem 1:** Multiple choice, 2 pts each. Circle the correct answer.

1.1. For an infinitesimal $dA$ on the top surface of a cylinder, what would be the appropriate formula to use in cylindrical coordinates?

(a) $s \, ds \, d\phi \, \hat{z}$
(b) $s \, ds \, d\phi \, \hat{r}$
(c) $s \, ds \, d\phi \, \hat{\phi}$
(d) $s \, d\phi \, dz \, \hat{z}$
(e) $s \, d\phi \, dz \, \hat{\phi}$
(f) $s \, d\phi \, dz \, \hat{r}$

1.2. A potential varies as $1/r^3$. Which of the following could it be?

(a) A monopole potential

(b) A dipole potential

(c) A quadrupole potential

(d) An octopole potential

(e) A higher order potential

1.3. A spherical surface of radius $R$ is somehow maintained at the following potential: $V(\theta) = V_0(\cos^4 \theta + 1)$. The formula for the potential inside the surface will involve:

(a) An infinite sum of Legendre polynomials in $\cos \theta$.

(b) A finite sum of Legendre polynomials in $\cos \theta$.

(c) An infinite sum of sines/cosines and/or exponentials.

(d) A finite sum of sines/cosines and/or exponentials.

1.4. A cube of side length $a$ is somehow maintained at the following potential: $V = 0$ for all sides except for one, and is a constant $V_0$ on that side. The formula for the potential inside the surface will involve:

(a) An infinite sum of Legendre polynomials in $\cos \theta$.

(b) A finite sum of Legendre polynomials in $\cos \theta$.

(c) An infinite sum of sines/cosines and/or exponentials.

(d) A finite sum of sines/cosines and/or exponentials.

1.5. A dielectric is placed in an external electric field whose magnitude was $E_0$ before the dielectric was put in the region. (Initially the region contained only a vacuum.) Which is true about the total electric field inside the dielectric, $E_{tot}$?

(a) $E_{tot} < E_0$, and they are in the same direction

(b) $E_{tot} = E_0$, and they are in the same direction

(c) $E_{tot} > E_0$, and they are in the same direction

(d) $E_{tot} < E_0$, and they are in the opposite direction

(e) $E_{tot} = E_0$, and they are in the opposite direction

(f) $E_{tot} > E_0$, and they are in the opposite direction
1.6. Same situation. $D_0$ is the magnitude of the $D$ field before the dielectric was put in the region. What is true about the total $D$ field inside the dielectric, $D_{\text{tot}}$? (Assume no edge effects, if it matters.)

(a) $D_{\text{tot}} < D_0$, and they are in the same direction  
(b) $D_{\text{tot}} = D_0$, and they are in the same direction  
(c) $D_{\text{tot}} > D_0$, and they are in the same direction  
(d) $D_{\text{tot}} < D_0$, and they are in the opposite direction  
(e) $D_{\text{tot}} = D_0$, and they are in the opposite direction  
(f) $D_{\text{tot}} > D_0$, and they are in the opposite direction

Given this, we can say that

$$E_{\text{tot}} = \frac{E_0}{\varepsilon}$$  (as discussed in class).

Also, $D = \varepsilon_0 E$ inside and $D = \varepsilon E$ outside

and

$$\frac{D_{\text{tot}}}{D_0} = \frac{\varepsilon_0 E_{\text{tot}}}{\varepsilon_0 E} = \frac{\varepsilon_0 \varepsilon}{\varepsilon_0 \varepsilon} = 1$$

$D_{\text{tot}} = D_0$

These last two aren’t multiple choice questions but they seemed to fit best here, and each one counts 2 pts like the multiple choice questions.

1.7. Given a dipole moment $p$ and point $X$ as shown in the figure, what is the direction of $\mathbf{f}$ as must be used in the dipole electric field equation, namely $E = \frac{1}{4\pi\varepsilon_0} \frac{p}{r^3} (2 \cos \theta \cdot \hat{\mathbf{r}} + \sin \theta \cdot \hat{\mathbf{\theta}})$? Draw the $\mathbf{f}$ direction on the figure just above.

1.8. Same situation... what is the direction of $\hat{\mathbf{\theta}}$? Draw the $\hat{\mathbf{\theta}}$ direction on the figure just above.
(9 pts) **Problem 2:** Short answers—use **words**, not equations. Or at most, perhaps a single equation in a supplementary role.

(a) Briefly explain how/why the method of relaxation works.

How: Numerically set each point to the average of surrounding pts, and iterate until no change in numerical values.

Why: Solution to Laplace's Eqn must be as "smooth as possible," which means each pt is average of surrounding points. Uniqueness theorem means that when you find a function which does this and also satisfies boundary conditions, it is the correct function.

(b) Briefly explain why you may generally stop at the first non-zero term when solving problems with the multipole expansion.

Multipole expansion terms have increasing powers of 1/r. If r is big compared to size of charge distribution then each successive term is much smaller than the preceding terms and can therefore be neglected as long as you have a non-zero term.

(c) Briefly explain what the polarization field, \( \vec{P} \), means.

\( \vec{P} \) is the density of dipole moments in the material, i.e. how much dipole moment per volume exists at a given \((x,y,z)\) coordinate. This is in the approximation that the dipole moment density is a smoothly varying function ignoring the ups and downs from individual atoms.
(11 pts) **Problem 3:** Two charges are above a grounded conducting plane: charge \( +q \) a distance \( d \) above the plane, and another charge \( +q \) a distance \( d \) directly above that (i.e. \( 2d \) from the plane). What is the net force on the upper charge?

By symmetry the plane will have \( V = 0 \) everywhere.

Force on the charge is superposition of forces from the other three charges:

\[
F = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{d^2} - \frac{1}{4\pi\varepsilon_0} \frac{q^2}{(3d)^2} - \frac{1}{4\pi\varepsilon_0} \frac{q^2}{(4d)^2}
\]

\[
= \frac{1}{4\pi\varepsilon_0} \frac{q^2}{d^2} \left( 1 - \frac{1}{9} - \frac{1}{16} \right)
\]

\[
\approx 0.826 \quad \text{(or } \frac{119}{144} \text{ if you want exact)}
\]

\[
F = 0.826 \frac{q^2}{4\pi\varepsilon_0} \frac{1}{d^2}
\]

\[
\left( \text{or } \frac{119}{144} \frac{q^2}{4\pi\varepsilon_0} \frac{1}{d^2} \right)
\]
(16 pts) Problem 4: An insulating spherical shell has potential on its surface (at \( r = R \)) given by \( V(R, \theta) = V_0 \cos^2 \theta \). Find \( V(r, \theta) \) for all points outside the shell. Hint: your first step could be to use page 2 of the exam to find \( \cos^2 \theta \) in terms of Legendre polynomials.

\[
P_0 = 1 \quad P_1 = x \quad P_2 = \frac{3}{2} x^2 - \frac{1}{2}
\]

Thus \( x = \cos \theta \), we want \( x^2 \) in terms of \( p_0, p_1, p_2 \)

Start with \( P_2 \):

\[
\frac{2}{3} P_2 = \frac{2}{3} \cdot \frac{3}{2} x^2 - \frac{3}{2} \cdot \frac{1}{2}
\]

\[
\frac{2}{3} P_2 = x^2 - \frac{1}{3}
\]

So \( x^2 = \frac{2}{3} P_2 + \frac{1}{3} P_0 \)

Separation of Variables

in spherical coords w/o \( \phi \)

\[ V = R(r) \Omega(\theta) \]

leads to \[ R = \frac{1}{r^{d+1}} \]

\[ \Omega = P_\ell(\cos \theta) \]

General form \[ V = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{C_{\ell m}}{r^{\ell+1}} P_\ell(\cos \theta) \]

Only \( \ell = 0 \)

\[ V(r, \theta) = V_0 \left( \frac{2}{3} P_2 + \frac{1}{3} P_0 \right) \]

\[
\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{C_{\ell m}}{r^{\ell+1}} P_\ell(\cos \theta) = V_0 \left( \frac{2}{3} P_2 + \frac{1}{3} P_0 \right)
\]

Equate coefficients, all terms \( = 0 \) except \( \ell = 0 \) and \( \ell = 2 \)

\[ \ell = 0 \quad C_0 \frac{1}{R} = \frac{V_0}{3} \quad \rightarrow \quad C_0 = \frac{V_0 R}{3} \]

\[ \ell = 2 \quad C_2 \frac{1}{R^2} = \frac{2V_0}{3} \quad \rightarrow \quad C_2 = \frac{2V_0 R^3}{3} \]

Final answer:

\[
V_{\text{outside}} = \frac{V_0}{3} \frac{R}{r} P_0(\cos \theta) + \frac{2V_0 R^3}{3} \frac{1}{r^3} P_2(\cos \theta)
\]

or

\[
V_{\text{outside}} = \frac{V_0}{3} \frac{R}{r} + \frac{2V_0 R^3}{3} \frac{1}{r^3} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)
\]

if you want to plug in for \( P_0 \) and \( P_2 \)
Problem 5: An insulating spherical shell (radius \( R \)) has a surface charge density of \( \sigma(R, \theta) = \sigma_0 \sin^2 \theta \cos \phi \). The sphere has no net charge, but it does have a dipole moment.

(a) Make a sketch of the shell, indicating where the charges will be positive, negative and zero, and use that to deduce the direction of the dipole moment. Or, if your sketching ability is not up to it, describe very carefully with words where the charges will be positive, negative, and zero, and what that means for the direction of the dipole moment.

From graphs above we see \( \sigma \) will go to zero at \( \theta = 0 \) and \( \theta = \pi \) and will be maximal at \( \theta = \frac{\pi}{2} \).

Also \( \sigma \) will be positive around \( \phi = 0 \) (positive x-axis), negative around \( \phi = \frac{\pi}{2} \) and \( \phi = \frac{3\pi}{2} \) (positive z-axis)

From picture \( \hat{p} = +\hat{x} \) direction

(points from negatives to positives)

(b) Calculate the dipole moment, hopefully using your answer to part (a) to help you avoid some work. Hint: there are some definite integrals given on page 2 of the exam; one or more should be helpful.

\[
\hat{p} = \int \hat{r}' \sigma(\hat{r}') \, d\hat{r}' \quad \text{in general, or} \quad \hat{p} = \int \hat{r}' \sigma(\hat{r}') \, d\hat{a}'
\]

for surface charge densities

\[
\hat{r}' = R \hat{r} = R (\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k})
\]

\( d\hat{a}' = R^2 \sin \theta \, d\theta \, d\phi \hat{r} \)

\( \text{unimportant due to symmetry from part a} \)

\[
\hat{p} = \int (R \sin \theta \cos \phi \hat{x}) (\sigma_0 \sin^2 \theta \cos \phi) (R^2 \sin \theta \, d\theta \, d\phi)
\]

I'm going to drop the primes for convenience

\[
\hat{p} = R^3 \sigma_0 \hat{x} \int_0^\pi \sin^3 \theta \, d\theta \int_0^{2\pi} \cos^2 \phi \, d\phi
\]

\( \text{from page 2} \)

\[
\hat{p} = \frac{3\pi^2 R^3 \sigma_0 \hat{x}}{8}
\]

Physics 441 Exam 1 – pg 8
Problem 6. An infinitely long cylinder (radius $R$) has a built-in polarization given by $\mathbf{P}(s) = P_0 \left(\frac{s}{R}\right) \hat{s}$. There are no free charges present.

(a) Find the bound volume and surface charge densities.

\[
\begin{align*}
\text{volume:} & \quad \rho_b = -\nabla \cdot \mathbf{P} \\
& = \left( \frac{1}{s} \frac{\partial}{\partial s} (s \mathbf{P}_s) \right) \\
& = -\frac{1}{5} \frac{\partial}{\partial s} \left( \frac{P_0}{R} s^2 \right) \\
& = -\frac{1}{5} \frac{P_0}{R} 2s
\end{align*}
\]

\[
\rho_b = \begin{cases} 
  -\frac{2P_0}{R} & s < R \\
  \frac{P_0}{R} & s = R 
\end{cases}
\]

Note both charge densities are constant so no integrals need be done to calculate charge enclosed in next step.

(b) Determine $\mathbf{E}(s)$ inside the cylinder via Gauss’s Law for $\mathbf{E}$.

Surface shown above, $\mathbf{E}$ is constant and $\parallel$ to $\hat{s}$, (wedge)

\[
\begin{align*}
\oint \mathbf{E} \cdot d\mathbf{a} &= \rho_{\text{enc}} \varepsilon_0 \\
\text{Gauss Law only if } \mathbf{E} &\text{ contains the } s = R \text{ surface} \\
\mathbf{E} &\cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \left( \frac{2P_0}{R} \right) s^2 \hat{s} \cdot \hat{s}
\end{align*}
\]

\[
\mathbf{E} = -\frac{P_0}{\varepsilon_0 R s} \hat{s}, \quad s \leq R
\]

(problem continues on next page)
(c) Determine $\mathbf{E}(s)$ inside the cylinder via Gauss’s Law for $\mathbf{D}$, and the equation $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$. If you do things correctly your answers to (b) and (c) will agree.

\[
\int \mathbf{D} \cdot \mathbf{d}a = \varepsilon_0 \text{ free encl.}
\]

\[
\nabla \cdot \mathbf{D} = 0
\]

\[
0 \cdot 2\pi s l = 0
\]

\[
\nabla \cdot \mathbf{D} = 0
\]

Since $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, we have

\[
0 = \varepsilon_0 \mathbf{E} + \mathbf{P}
\]

\[
\mathbf{E} = -\frac{1}{\varepsilon_0} \mathbf{P}
\]

\[
\mathbf{E} = -\frac{1}{\varepsilon_0} \mathbf{P}_0 \oint R \mathbf{S}
\]

Yay! It matches part b! 😊
(16 pts) Problem 7. A spherical capacitor is made by putting dielectric (relative permittivity $\epsilon_r$) between two concentric spherical conductors as shown.

(a) Find the capacitance of the system.

\[ \rho_{\text{enc}} = \frac{Q}{V} \rightarrow \frac{Q}{V} \rightarrow \text{use} \quad C = \frac{Q}{V} \]

\[ \oint \vec{D} \cdot d\vec{s} = \Phi_{\text{magnetic}} \]

\[ \Phi_{\text{magnetic}} = \oint \vec{B} \cdot d\vec{l} = \Phi \]

\[ \vec{D} = \frac{\Phi}{4\pi r^2} \hat{r} \]

Use \( \vec{D} = \frac{Q}{4\pi \epsilon_0} \hat{r} \rightarrow \vec{E} = \frac{Q}{4\pi \epsilon_0 \epsilon_r} \frac{1}{r} \hat{r} \]

\[ \Delta V = -\int_{b}^{a} \vec{E} \cdot d\vec{r} = -\int_{b}^{a} \frac{Q}{4\pi \epsilon_0 \epsilon_r} \frac{dr}{r} \]

\[ \Delta V = \frac{Q}{4\pi \epsilon_0 \epsilon_r} \ln \left( \frac{a}{b} \right) \]

\[ |\Delta V| = \frac{Q}{4\pi \epsilon_0 \epsilon_r} \left( \frac{1}{a} - \frac{1}{b} \right) \]

(b) Find the polarization function \( P(\vec{r}) \) for the dielectric in terms of the given quantities.

\[ \text{CM use} \quad \vec{P} = \epsilon_0 \chi_e \vec{E} \quad \text{and} \quad \chi_e = \frac{Q}{V} \]

\[ \vec{P} = \epsilon_0 (\epsilon_r - 1) \vec{E} \quad \text{(use \( \vec{E} \) from above)} \]

\[ \vec{P} = \frac{\epsilon_0 (\epsilon_r - 1)}{4\pi} \frac{Q}{\epsilon_0} \frac{1}{r^2} \hat{r} \]

\[ \vec{P} = \frac{Q}{4\pi} \frac{\epsilon_r - 1}{\epsilon_r} \frac{1}{r^2} \hat{r} \]

(pro_problem continues on next page)
(c) The bound volume charge density is zero (you don’t have to verify this). Find the bound surface charge densities at the inner and outer surfaces of the dielectric.

\[
\sigma_b = \mathbf{P} \cdot \mathbf{n}
\]

inner surface \( \mathbf{n} = -\hat{r}, \ r = a \)
\[
\sigma_b = -\mathbf{P} \cdot \hat{r} \bigg|_{r=a}
\]

inner \( \sigma_b = -\frac{Q}{4\pi} \frac{\varepsilon_r-1}{\varepsilon_r a^2} \)

outer surface \( \mathbf{n} = \hat{r}, \ r = b \)
\[
\sigma_b = +\mathbf{P} \cdot \hat{r} \bigg|_{r=b}
\]

outer \( \sigma_b = +\frac{Q}{4\pi} \frac{\varepsilon_r-1}{\varepsilon_r b^2} \)

(d) Verify that the total bound charge is zero.

inner \( q_b = \sigma_b \times \text{area} \)
\[
= \left(-\frac{Q}{4\pi} \frac{\varepsilon_r-1}{\varepsilon_r a^2}\right) (4\pi a^2)
\]

inner \( q_b = -Q \left(\frac{\varepsilon_r-1}{\varepsilon_r}\right) \)

outer \( q_b = \sigma_b \times \text{area} \)
\[
= \left(+\frac{Q}{4\pi} \frac{\varepsilon_r-1}{\varepsilon_r} \frac{1}{b^2}\right) (4\pi b^2)
\]

outer \( q_b = +Q \frac{\varepsilon_r-1}{\varepsilon_r} \)

\[ q_b \text{ total} = q_b \text{ inner} + q_b \text{ outer} = 0 \]