Fall 2018
Physics 441
Exam 2
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40115526

No time limit. Student calculators are allowed. One page of handwritten notes allowed (front & back). Books not allowed.

Name Solutions

Instructions: Please label & circle/box your answers. **Show your work**, where appropriate! And remember: **in any problems involving Gauss's Law, you should explicitly show your Gaussian surface**. For all problems, unless otherwise specified you may assume that you are dealing with **electrostatics**, i.e. the charges are not moving and the fields have come to equilibrium.

Griffiths front and back covers

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl: $\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

Special case derivatives:
(similar things true for $\mathbf{\mathscr{r}}$)

$\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta(\mathbf{r})$ $\nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r})$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^8\ \mathrm{m/s}$	(speed of light)
$e = 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Some miscellaneous mathematical stuff:

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \qquad \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x \qquad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \qquad \cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

The first few Legendre polynomials:

$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \qquad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x \qquad P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

Some indefinite integrals:

$\int \sin Cx = \frac{-\cos Cx}{C} \qquad \int \sin^2 Cx = \frac{x}{2} - \frac{\sin 2Cx}{4C} \qquad \int \cos Cx = \frac{\sin Cx}{C} \qquad \int \cos^2 Cx = \frac{x}{2} + \frac{\sin 2Cx}{4C}$

Some definite integrals:

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{a}{2}, & \text{if } n = m \end{cases}$$

$\int_0^\pi \sin^2 x\,dx = \frac{\pi}{2} \qquad \int_0^\pi \cos^2 x\,dx = \frac{\pi}{2} \qquad \int_0^{2\pi} \sin^2 x\,dx = \pi \qquad \int_0^{2\pi} \cos^2 x\,dx = \pi$

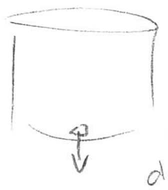
$\int_0^\pi \sin^3 x\,dx = \frac{4}{3} \qquad \int_0^\pi \cos^3 x\,dx = 0 \qquad \int_0^{2\pi} \sin^3 x\,dx = 0 \qquad \int_0^{2\pi} \cos^3 x\,dx = 0$

$\int_0^\pi \sin^4 x\,dx = \frac{3\pi}{8} \qquad \int_0^\pi \cos^4 x\,dx = \frac{3\pi}{8} \qquad \int_0^{2\pi} \sin^4 x\,dx = \frac{3\pi}{4} \qquad \int_0^{2\pi} \cos^4 x\,dx = \frac{3\pi}{4}$

(20 pts) **Problem 1**. Multiple choice, 1.5 pts each. (Plus you get a free 0.5 point. ☺) Circle the correct answers for the multiple choice questions.

1.1. For an infinitesimal d**A** on the bottom surface of a cylinder, what would be the appropriate formula to use in cylindrical coordinates?

(a) $s\,ds\,d\phi\,\hat{\mathbf{s}}$
(b) $s\,ds\,d\phi\,\hat{\boldsymbol{\phi}}$
(c) $s\,ds\,d\phi\,\hat{\mathbf{z}}$
(d) $s\,d\phi dz\,\hat{\mathbf{s}}$
(e) $s\,d\phi dz\,\hat{\boldsymbol{\phi}}$
(f) $s\,d\phi dz\,\hat{\mathbf{z}}$



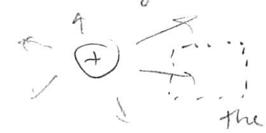
area $= s\,ds\,d\phi$
direction $= -\hat{z}$

(g) $-s\,ds\,d\phi\,\hat{\mathbf{s}}$
(h) $-s\,ds\,d\phi\,\hat{\boldsymbol{\phi}}$
(i) $-s\,ds\,d\phi\,\hat{\mathbf{z}}$
(j) $-s\,d\phi dz\,\hat{\mathbf{s}}$
(k) $-s\,d\phi dz\,\hat{\boldsymbol{\phi}}$
(l) $-s\,d\phi dz\,\hat{\mathbf{z}}$

1.2. Poisson's equation tells us that $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$. If the charge density throughout some volume is zero, what else must be true throughout that volume?

(a) $V = 0$
(b) $\mathbf{E} = 0$
(c) Both V and $\mathbf{E}$ must be zero
(d) None of the above is necessarily true.

could have charge outside the volume



the volume has nonzero V and $\vec{E}$

1.3. Which of the following is true of Laplace's equation?

(a) It can have more than one solution for a given set of boundary conditions.
(b) The solutions in one dimension must be sines and cosines (or a linear combination). → In 1D solutions are lines, actually: $\frac{d^2V}{dx^2} = 0$ leads to $V = Ax + B$ by integrating twice
(c) It is only valid in regions of space that contain no charges.
(d) It requires that nowhere within the region of interest can the potential be zero
(e) More than one of the above.

1.4. In class we solved Laplace's equation in 2D, obtaining $V(x, y)$. You have also worked one or more 2D problems for homework. Which of the following is true about situations like that where we can write V is a function of x and y only?

(a) The third dimension must be infinitely thin, or close enough that it's a good approximation.
(b) The third dimension must be finite.
(c) The third dimension must be infinitely long, or close enough that it's a good approximation.

If no z-dependence, V doesn't vary with z, so situation must be the same for all z

1.5. A cube of side length a is somehow maintained at the following potential: $V = 0$ for all sides except for one, and is a constant V_0 on that side. The formula for the potential inside the surface will involve:

(a) An infinite sum of sines/cosines and/or exponentials. we did this problem in class!
(b) A finite sum of sines/cosines and/or exponentials.
(c) An infinite sum of Legendre polynomials in $\cos\theta$.
(d) A finite sum of Legendre polynomials in $\cos\theta$.
(e) An infinite sum of Bessel functions.
(f) A finite sum of Bessel functions.

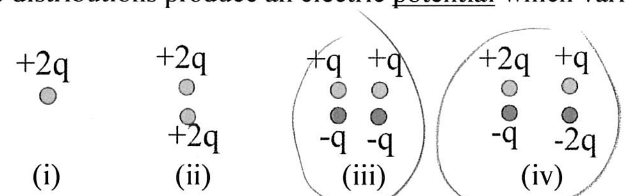
mono $E \sim \frac{1}{r^2}$ dipole $E \sim \frac{1}{r^3}$ quad $E \sim \frac{1}{r^4}$

1.6. A localized charge distribution has no net charge, zero electric dipole moment, but a nonzero electric quadrupole moment. At a large distance r from the distribution, the electric field will fall off like:

(a) No fall off; it's constant
(b) $1/r$
(c) $1/r^2$
(d) $1/r^3$
(e) $1/r^4$
(f) $1/r^5$

dipole potential

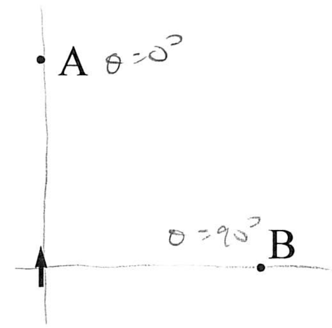
1.7. Which of these charge distributions produce an electric potential which varies as $1/r^2$ when you get far away?



both of these will have non zero dipole moments because + and − charges are separated

(a) (i) only
(b) (ii) only
(c) (iii) only
(d) (iv) only
(e) (i) and (ii)
(f) (i) and (iii)
(g) (i) and (iv)
(h) (ii) and (iii)
(i) (ii) and (iv)
(j) (iii) and (iv)

1.8. Points A and B are the same large distance from an electric dipole, but in different directions as per the figure. The dipole is depicted by the arrow (dipole moment in the direction of the arrow). What is true of the magnitude of the **E** field at point A compared to point B?



(a) $|\vec{E}_a|$ and $|\vec{E}_b|$ are both $= 0$
(b) $|\vec{E}_a| = 0$, $|\vec{E}_b| \neq 0$
(c) $|\vec{E}_b| = 0$, $|\vec{E}_a| \neq 0$
(d) $|\vec{E}_a| > |\vec{E}_b|$ (and both $\neq 0$)
(e) $|\vec{E}_a| < |\vec{E}_b|$ (and both $\neq 0$)
(f) $|\vec{E}_a| = |\vec{E}_b|$ (and both $\neq 0$)

$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$

this part zero for point B — this part zero for point A

so E_a will be $2\times E_b$ in magnitude

1.9. A dielectric is placed in an external electric field whose magnitude was E_0 before the dielectric was put in the region. (Initially the region contained only a vacuum.) Which is true about the total electric field inside the dielectric, $\mathbf{E_{tot}}$?

(a) $E_{tot} < E_0$, and they are in the same direction
(b) $E_{tot} = E_0$, and they are in the same direction
(c) $E_{tot} > E_0$, and they are in the same direction
(d) $E_{tot} < E_0$, and they are in the opposite direction
(e) $E_{tot} = E_0$, and they are in the opposite direction
(f) $E_{tot} > E_0$, and they are in the opposite direction

$\vec{E}_{\text{total inside}} = \frac{\vec{E}_0}{\epsilon_r}$

as discussed in class

(the field from polarized charges partially cancels out E_0)

1.10. Same situation. D_0 is the magnitude of the **D** field before the dielectric was put in the region. What is true about the total **D** field inside the dielectric, $\mathbf{D_{tot}}$? (Assume no edge effects, if it matters.)

(a) $D_{tot} < D_0$, and they are in the same direction
(b) $D_{tot} = D_0$, and they are in the same direction
(c) $D_{tot} > D_0$, and they are in the same direction
(d) $D_{tot} < D_0$, and they are in the opposite direction
(e) $D_{tot} = D_0$, and they are in the opposite direction
(f) $D_{tot} > D_0$, and they are in the opposite direction

$\vec{D} = \epsilon_0\epsilon_r\vec{E}$ in general

Before: $\vec{D}_0 = \epsilon_0\vec{E}_0$

After: $\vec{D}_{tot} = \epsilon_0\epsilon_r\vec{E}_{tot}$

$= \epsilon_0\epsilon_r\left(\vec{E}_0/\epsilon_r\right)$

$= \epsilon_0\vec{E}_0$ same!

1.11. Same situation. What is true about how the **D** field compares to the **E** field inside the dielectric?

(a) $D_{tot} < \epsilon_0 E_{tot}$, and they are in the same direction
(b) $D_{tot} = \epsilon_0 E_{tot}$, and they are in the same direction
(c) $D_{tot} > \epsilon_0 E_{tot}$, and they are in the same direction
(d) $D_{tot} < \epsilon_0 E_{tot}$, and they are in the opposite direction
(e) $D_{tot} = \epsilon_0 E_{tot}$, and they are in the opposite direction
(f) $D_{tot} > \epsilon_0 E_{tot}$, and they are in the opposite direction

$\vec{D}_{tot} = \epsilon_0\epsilon_r\vec{E}_{tot}$ which is $> \epsilon_0\vec{E}_{tot}$

$\nabla \times \vec{E} = 0 \rightarrow E_{\parallel 1} = E_{\parallel 2}$ $\nabla \cdot \vec{D} = \rho_{free} \rightarrow D_{\perp 1} - D_{\perp 2} = \sigma_{free}$

$= 0$ in this case

1.12. A boundary exists between two linear dielectric materials, i.e. ϵ_r has two different values above and below the boundary. There are no free charges in the region considered. What must be continuous across the boundary?

i) $E_{//}$ ii) $E_\perp$ iii) $D_{//}$ iv) $D_\perp$

(a) (i) only
(b) (ii) only
(c) (iii) only
(d) (iv) only
(e) (i) and (ii)
(f) (i) and (iii)
(g) (i) and (iv)
(h) (ii) and (iii)
(i) (ii) and (iv)
(j) (iii) and (iv)

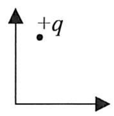
1.13. Which of the following is true of bound charges?

~~(a)~~ Bound charges mark the start and end of the displacement field.
~~(b)~~ Bound charges are free to move within an object but cannot be transferred from one object to another object.
~~(c)~~ Bound charges are normally found in conductors.
(d) Bound charges are the result of dipole moments related to a material.
~~(e)~~ Bound charges can exist in the absence of a polarization field.
(f) More than one of the above.

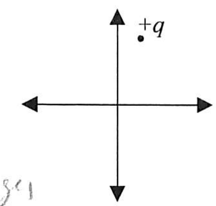
$\left.\begin{array}{l}\rho_B = -\nabla \cdot \vec{P} \\ \sigma_B = \vec{P} \cdot \hat{n}\end{array}\right\}$ if $\vec{P} = 0$ then no bound charges

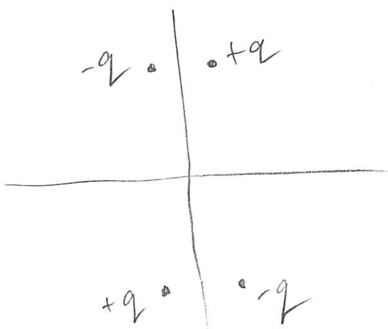
(11 pts) **Problem 2**. Short answers.

(a) A positive charge $+q$ is close to two grounded conducting semi-infinite planes, as shown (the arrows are conductors which extend infinitely into and out of the page, as well as infinitely in the direction of the arrows).

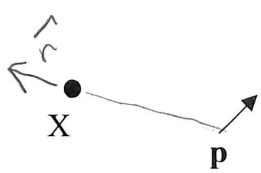


Draw the appropriate image charge configuration that could be used to determine the electric potential near the charge q. How do you know your answer is the appropriate configuration? Be specific. *Hint*: think about the drawing to the right.

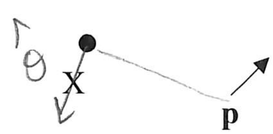




- By symmetry, these 4 charges will cause $V=0$ on both sets of lines
- It matches given situation in Region I
- By uniqueness, it will give solution to given situation

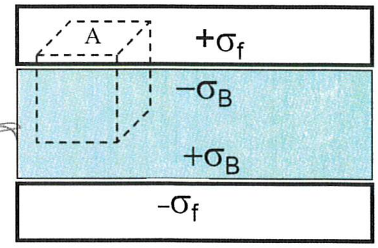


(b) Given a dipole moment **p** and point X as shown in the figure, what is the direction of $\hat{\mathbf{r}}$ that must be used in the dipole electric field equation, namely $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\ \hat{\mathbf{r}} + \sin\theta\ \hat{\boldsymbol{\theta}}\right)$? Draw the $\hat{\mathbf{r}}$ direction on point X in the figure just above.



(c) Same situation… what is the direction of $\hat{\boldsymbol{\theta}}$? Draw the $\hat{\boldsymbol{\theta}}$ direction on the figure just above.

(8 pts) **Problem 3**. A parallel plate capacitor (separation d) is charged up (total charge Q, charge per area σ_f) with a dielectric material (dielectric constant ε_r) present between the plates. The dielectric polarizes, and bound charge densities $+\sigma_B$ and $-\sigma_B$ are produced as shown. What is the magnitude of the **D** field inside the dielectric? (Ignore fringing fields as usual.) *Hint*: use the Gaussian surface shown to analyze the situation.



$\oint \vec{D} \cdot d\vec{a} = q_{free\ enc}$, using this Gaussian surface

$\cancel{\int_{top}} + \int_{bottom} + \cancel{\int_{sides}} = \sigma_f A$

$=0$ because $D=0$ in the conductor

$=0$ because no flux going through sides

$D \cdot A = \sigma_f A$

$$\boxed{\vec{D} = \sigma_f (-\hat{z})}$$

only magnitude is asked for, but direction in $-\hat{z}$ because same direction as $\vec{E}$ field.

Method 2 Could have also used Gauss's Law for $\vec{E}$:

$\oint \vec{E} \cdot d\vec{a} = q_{enc}/\epsilon_0$

$E \cdot A = (\sigma_f A - \sigma_b A)/\epsilon_0$

$E = \frac{1}{\epsilon_0}(\sigma_f - \sigma_b)$

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ $\quad \sigma_b = \vec{P} \cdot \hat{n} \rightarrow \sigma_b(\text{top surface}) = P$ (magnitude)

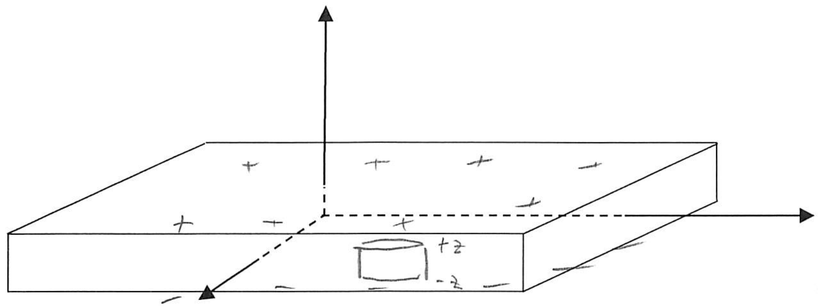
direction of $\vec{P}$ is $+\hat{z}$

So $\vec{P} = +\sigma_b \hat{z}$

$= \epsilon_0 \frac{1}{\epsilon_0}(\sigma_f - \sigma_b) + \sigma_b$

$\vec{D} = \sigma_f (-\hat{z})$

(12 pts) **Problem 4**. A ferroelectric infinite slab with thickness $2d$ is centered on the *x-y* plane. A section is shown. It has a polarization $\mathbf{P} = P_0 \sin\left(\frac{\pi z}{2d}\right)\hat{\mathbf{z}}$. Determine the electric field *inside* the slab, both magnitude and direction, as a function of *z*, for $0 < z < d$.



Hard way

$\sigma_B = \vec{P}\cdot\hat{n}$

top $\hat{n}$ is $+\hat{z} \rightarrow \sigma_{B\,top} = +P_0 \sin\frac{\pi z}{2d}$

bottom $\hat{n}$ is $-\hat{z} \rightarrow \sigma_{B\,bottom} = -P_0 \sin\frac{\pi z}{2d}$

$\rho_B = -\nabla\cdot\vec{P} = -\frac{\partial}{\partial z}\left(P_0 \sin\frac{\pi z}{2d}\right)$

$= -P_0 \frac{\pi}{2d}\cos\frac{\pi z}{2d}$ = symmetric about xy plane

Gaussian surface as shown, by symm E at top surface = E at bottom surface

$\oint \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$ — q_{enc} only from ρ_B, not σ_B's

$q_{enc} = \int \rho\, d\tau = A\int_{-z}^{z}\left(-P_0 \frac{\pi}{2d}\cos\frac{\pi z'}{2d}\right)dz'$

$\int_{top} + \int_{bottom} + \cancel{\int_{wrapper}} = \frac{q_{enc}}{\epsilon_0}$

no flux!

$= -A \frac{\cancel{\pi}}{\cancel{2d}} P_0 \left.\frac{\sin\frac{\pi z'}{2d}}{\cancel{\pi}/\cancel{2d}}\right|_{-z}^{z}$

$= -2AP_0 \sin\frac{\pi z}{2d}$

$EA + EA = \frac{-2AP_0}{\epsilon_0}\sin\frac{\pi z}{2d}$

$\cancel{2}E\cancel{A} = -\frac{\cancel{2A}P_0}{\epsilon_0}\sin\frac{\pi z}{2d}$

$$\boxed{\vec{E} = -\frac{P_0}{\epsilon_0}\sin\frac{\pi z}{2d}\,\hat{z}}$$ (direction from symmetry)

Easy Way

$\oint \vec{D}\cdot d\vec{a} = \underbrace{q_{free\ enclosed}}_{=0}$, same Gaussian surface and same symmetry

$\int_{top} + \int_{bottom} + \int_{wrapper} = 0$

$DA + DA + 0 = 0$

$D = 0$

$\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$0 = \epsilon_0\vec{E} + \vec{P}$

$\vec{E} = -\frac{\vec{P}}{\epsilon_0}$

$$\boxed{\vec{E} = -\frac{P_0}{\epsilon_0}\sin\frac{\pi z}{2d}\,\hat{z}}$$

same! :)

$V(r=R) = V_0(5\cos^3\theta + 6\cos^2\theta - 2\cos\theta - 2)$

R

We'll use Sep. of Var. in spherical coords (ϕ no dependence)

(19 pts) **Problem 5**. A sphere has the following potential on its surface at $r = R$: $V = V_0(5\cos^3\theta + 6\cos^2\theta - 2\cos\theta - 2)$. Find the potential outside the sphere, as a function of R and θ.

Step 1: express bndy cond V in terms of Legendre polynomials.

$$5x^3 + 6x^2 - 2x - 2 = ?\,P_3 + ?\,P_2 + ?\,P_1 + ?\,P_0$$

Start with P_3... we need $\underline{2\,P_3\text{'s}}$ to give $5x^3$

$$\cancel{5x^3} + 6x^2 - 2x - 2 = 2\left(\tfrac{5}{2}x^3 - \tfrac{3}{2}x\right) + ?\,P_2 + ?\,P_1 + ?\,P_0$$

$\cancel{5x^3} - 3x$ ↳ bring to LHS

$$6x^2 + x - 2 = ?\,P_2 + ?\,P_1 + ?\,P_0$$

Now, we need $\underline{4\,P_2\text{'s}}$ to give $6x^2$

$$\cancel{6x^2} + x - 2 = 4\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right) + ?\,P_1 + ?\,P_0$$

$\cancel{6x^2} - 2$ ↳ bring to LHS

$$x = ?\,P_1 + ?\,P_0$$

obviously we need $\underline{1\,P_1}$ and $\underline{0\,P_0}$

So $\underline{5x^3 + 6x^2 - 2x - 2 = 2P_3 + 4P_2 + 1P_1}$

there are other ways to get this result, but this method seemed most straightforward to me

Now, S. of Var of Laplace's Eqn $\nabla^2 V = 0$ gives us $V = R\,\Theta$

$$R = \cancel{A r^\ell} + \frac{B}{r^{\ell+1}}$$

we want outside sphere, so $A = 0$ (must be finite at $r = $ infinity)

$$\Theta = P_\ell(\cos\theta)$$

(throw out Q solutions as always)

General soln: $\underline{V = \sum_\ell B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)}$

Apply $r = R$ bndy cond on next page

(extra page for work in case you need it)

$$V(r=R) = V_0\left(2P_3(\cos\theta) + 4P_2(\cos\theta) + 1\,P_1(\cos\theta)\right)$$

$$\sum_l B_l \frac{1}{R^{l+1}} P_l(\cos\theta) = \nearrow$$

only $l = 1, 2,$ and 3 terms survive ins-mmation. Equate coefficients

$\underline{l=1}$ $\quad \frac{B_1}{R^2}\cancel{P_1} = V_0 \cancel{P_1} \rightarrow B_1 = V_0 R^2$

$\underline{l=2}$ $\quad \frac{B_2}{R^3}\cancel{P_2} = 4V_0 \cancel{P_2} \rightarrow B_2 = 4V_0 R^3$

$\underline{l=3}$ $\quad \frac{B_3}{R^4}\cancel{P_3} = 2V_0 \cancel{P_3} \rightarrow B_3 = 2V_0 R^4$

answer is

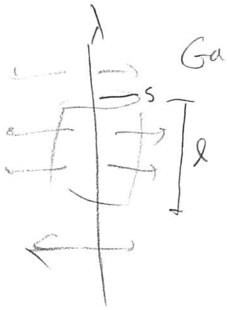
$$V = (V_0R^2)\frac{1}{r^2}P_1(\cos\theta) + (4V_0R^3)\frac{1}{r^3}P_2(\cos\theta) + (2V_0R^4)\frac{1}{r^4}P_3(\cos\theta)$$

or

$$\boxed{V = \frac{V_0R^2}{r^2}\cos\theta + \frac{4V_0R^3}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \frac{2V_0R^4}{r^4}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)}$$

for potential outside the sphere, $r > R$

(18 pts) **Problem 6**. An infinite line of charge (charge density λ) goes along the z-axis.
(a) What is the electric field from the line of charge?



Gauss's Law $\oint \vec{E} \cdot d\vec{a} = q_{enc}/\epsilon_0$

$$\cancel{\int_{top}} + \cancel{\int_{bottom}} + \int_{wrapper} = q_{enc}/\epsilon_0$$

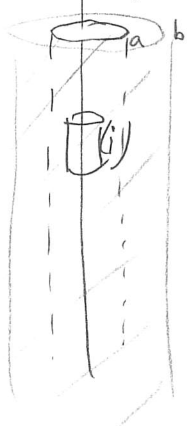
(no flux)

$$E \cdot 2\pi s \cancel{l} = \lambda \cancel{l}/\epsilon_0$$

$$\boxed{\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{s}}$$

(b) An infinitely long thick cylindrical shell (inner radius a, outer radius b, dielectric constant ϵ_r) is centered on the line charge. What is the electric field in the three regions, (i) $s < a$, (ii) $a < s < b$, and (iii) $s > b$?

thick dielectric shell, ϵ_r



Use Gauss's Law for $\vec{D}$

$$\left.\begin{aligned} \oint \vec{D} \cdot d\vec{a} &= q_{free\ enc} \\ D \cdot 2\pi s l &= \lambda l \\ \vec{D} &= \frac{1}{2\pi}\frac{\lambda}{s}\hat{s} \end{aligned}\right\} \rightarrow \text{same for all three regions}$$

Then use $\vec{D} = \epsilon_0 \epsilon_r \vec{E}$ for each region

(i) $\vec{E} = \dfrac{\vec{D}}{\epsilon_0}$ (because $\epsilon_r = 1$ here)

$$\boxed{\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{s}}$$

(ii) $\vec{E} = \dfrac{\vec{D}}{\epsilon_0 \epsilon_r}$

$$\boxed{\vec{E} = \frac{1}{2\pi\epsilon_0\epsilon_r}\frac{\lambda}{s}\hat{s}}$$

(iii) $\vec{E} = \dfrac{\vec{D}}{\epsilon_0}$ (because $\epsilon_r = 1$ here)

$$\boxed{\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{s}}$$

Method 2
alternate quicker method - just recognize that $\vec{E}$ field will be reduced by ϵ_r (in each region) compared to answer for (a)

$\rho_b = -\nabla \cdot \vec{P}$ $\qquad\qquad$ $\sigma_b = \vec{P} \cdot \hat{n}$

(c) Find the bound volume charge density within the dielectric and also the bound surface charge densities at the inner and outer surfaces of the dielectric shell.

first need to find $\vec{P}$ inside the dielectric

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \rightarrow \vec{P} = \vec{D} - \epsilon_0 \vec{E}$$

$$= \left(\frac{1}{2\pi} \frac{\lambda}{s} - \cancel{\epsilon_0} \frac{1}{2\pi \cancel{\epsilon_0} \epsilon_r} \frac{\lambda}{s} \right) \hat{s}$$

$$\vec{P} = \frac{1}{2\pi} \frac{\lambda}{s} \hat{s} \underbrace{\left(1 - \frac{1}{\epsilon_r} \right)}_{\text{or } \frac{\epsilon_r - 1}{\epsilon_r}} \hat{s}$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\left(\frac{1}{s} \frac{\partial}{\partial s} (s P_s) + 0 + 0 \right)$$

$$= -\frac{1}{s} \frac{\partial}{\partial s} \left(\frac{1}{2\pi} \lambda \frac{\epsilon_r - 1}{\epsilon_r} \right)$$

$$\boxed{\rho_b = 0} \quad \text{for } a < s < b$$

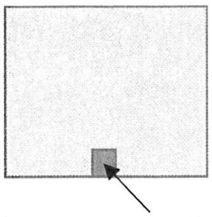
$s = a$ surface : $\hat{n} = -\hat{s}$

$$\sigma_b = \vec{P} \cdot \hat{n} \Big|_{s=a} = \boxed{\frac{-\lambda}{2\pi a} \frac{\epsilon_r - 1}{\epsilon_r}} \quad \text{at } s = a$$

$s = b$ surface $\hat{n} = +\hat{s}$

$$\sigma_b = \vec{P} \cdot \hat{n} \Big|_{s=b} = \boxed{\frac{+\lambda}{2\pi b} \frac{\epsilon_r - 1}{\epsilon_r}} \quad \text{at } s = a$$

(12 pts) **Problem 7**. Derive the specific condition you could use in a computational relaxation problem with a square dielectric to set the potential of the (i, j) cell which is located at the bottom boundary of the dielectric (dielectric constant ϵ_r) as shown. Write your answer as "$V_{i,j} = \ldots$" where the right hand side is given in terms of the potential of neighboring cells. Yes, this is just like part of the computational homework problem.



This cell is (i, j). Cell just below is $(i+1, j)$.

$$\nabla \cdot \vec{D} = \rho_{free} \rightarrow D_{\perp 1} - D_{\perp 2} = \sigma_{free} = 0$$

$$\epsilon_{r1} E_{\perp 1} - \epsilon_{r2} E_{\perp 2} = 0$$

This case $\epsilon_{r1} = \epsilon_r$
$\epsilon_{r2} = 1$

$$\epsilon_r E_{\perp 1} = E_{\perp 2}$$

$$E_\perp = -\frac{\partial V}{\partial z} \simeq -\frac{\Delta V}{\Delta z}$$

saying z coordinate is represented by i, i+1, etc.
(in some unit system)

$$-\epsilon_r \frac{V_{i-1} - V_i}{(i-1)-(i)} = -\frac{V_{i+1} - V_{i+2}}{(i+1) - (i+2)}$$

$$\cancel{\epsilon_r}(V_{i-1} - V_i) = \frac{V_{i+1} - V_{i+2}}{\epsilon_r}$$

adding in j's because this is true for each column

$$\boxed{V_{i,j} = V_{i-1,j} - \frac{1}{\epsilon_r}\left(V_{i+1,j} - V_{i+2,j}\right)}$$

or $V_{i-1,j} + \frac{1}{\epsilon_r}\left(V_{i+2,j} - V_{i+1,j}\right)$