Fall 2017
Physics 441
Exam 3
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40111868

No time limit. Student calculators are allowed. One page of handwritten notes allowed (front & back). Books not allowed.

Name Solutions

Instructions: Please label & circle/box your answers. **Show your work**, where appropriate! And remember: **in any problems involving Gauss's Law/Ampere's Law, you should explicitly show your Gaussian surface/Amperian loop**. For all problems, unless otherwise specified you may assume that you are dealing with **electrostatics**, i.e. the charges are not moving and the fields have come to equilibrium, and that all **dielectrics are linear and isotropic**.

Griffiths front and back covers

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Curl: $\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

Special case derivatives:
(similar things true for $\mathscr{r}$)

$\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta(\mathbf{r})$ $\nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r})$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^8\ \mathrm{m/s}$	(speed of light)
$e = 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Some miscellaneous mathematical stuff:

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{a}{2}, & \text{if } n = m \end{cases}$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$
$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\int_{-1}^{1} P_\ell(x)\,P_m(x)dx = \begin{cases} 0, & \text{if } \ell \neq m \\ \frac{2}{2\ell+1}, & \text{if } \ell = m \end{cases}$$

$$\int_0^\pi P_\ell(\cos\theta)\,P_m(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0, & \text{if } \ell \neq m \\ \frac{2}{2\ell+1}, & \text{if } \ell = m \end{cases}$$

$P_\ell(x)$ are the Legendre polynomials; the first few are these:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

Some definite integrals:

$\int_0^\pi \sin^2 x\,dx = \frac{\pi}{2}$ $\quad\int_0^\pi \cos^2 x\,dx = \frac{\pi}{2}$ $\quad\int_0^{2\pi} \sin^2 x\,dx = \pi$ $\quad\int_0^{2\pi} \cos^2 x\,dx = \pi$

$\int_0^\pi \sin^3 x\,dx = \frac{4}{3}$ $\quad\int_0^\pi \cos^3 x\,dx = 0$ $\quad\int_0^{2\pi} \sin^3 x\,dx = 0$ $\quad\int_0^{2\pi} \cos^3 x\,dx = 0$

$\int_0^\pi \sin^4 x\,dx = \frac{3\pi}{8}$ $\quad\int_0^\pi \cos^4 x\,dx = \frac{3\pi}{8}$ $\quad\int_0^{2\pi} \sin^4 x\,dx = \frac{3\pi}{4}$ $\quad\int_0^{2\pi} \cos^4 x\,dx = \frac{3\pi}{4}$

(20 pts) **Problem 1:** Multiple choice, 2 pts each. Circle the correct answer.

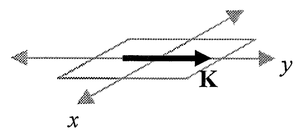
1.1. A surface current $\mathbf{K} = K\,\hat{\mathbf{y}}$ flows within a finite rectangle in the x-y plane as shown. The rectangle goes from $-a$ to $+a$ in the x-direction and $-b$ to $+b$ in the y-direction. How should you set up the integral to calculate the amount of current flowing?

(a) $I = \int_{-a}^{a} K dx$

(b) $I = \int_{-b}^{b} K dy$

(c) $I = \int_{-a}^{a} \int_{-b}^{b} K dy dx$

$I = \int K \, d\ell_{\perp}$



1.2. True/False: $\nabla \cdot \mathbf{J}$ is nonzero and positive for a region of space, the charge density in that region must necessarily be decreasing.

(a) True

(b) False

$\nabla \cdot \vec{J} = -\frac{d\rho}{dt}$ (Eqn of continuity)

1.3. A trajectory of a positive particle is shown. Please forgive the artwork. To help you see the 3D nature of the path the way I intend, I have indicated some places where the particle's path is behind the y-axis and some places where the path is in front. Alternately, viewed from the right, the path is counter-clockwise. There is no electric field present (aside from that created by the particle itself). In which direction is the magnetic field?

(a) $+\hat{\mathbf{x}}$

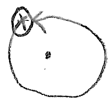
(b) $-\hat{\mathbf{x}}$

(c) $+\hat{\mathbf{y}}$

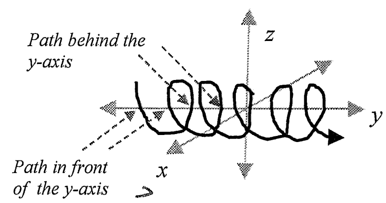
(d) $-\hat{\mathbf{y}}$

(e) $+\hat{\mathbf{z}}$

(f) $-\hat{\mathbf{z}}$



positive y-axis coming out of page



$\vec{F} = q\vec{v} \times \vec{B}$ → you can verify that if B is into the page, the force is correct to give you this circle

1.4. Same coordinate axes as the previous problem. According to the Lorentz force, what general direction is the force of wire 1 on wire 2? The wires are finite; the arrows indicate both the length of the segment of wire as well as the direction of positive current flow.

(a) $+\hat{\mathbf{x}}$

(b) $-\hat{\mathbf{x}}$

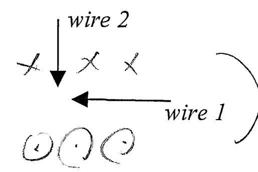
(c) $+\hat{\mathbf{y}}$

(d) $-\hat{\mathbf{y}}$

(e) $+\hat{\mathbf{z}}$

(f) $-\hat{\mathbf{z}}$

$\vec{F}_{on\,2} = \underbrace{I\vec{\ell} \times \vec{B}}_{\text{to the right via right hand rule}}$



field from wire 1 as indicated via right hand rule

1.5. Same situation, what direction is the force of wire 2 on wire 1? Hint: as we will learn next semester Newton's Third Law doesn't apply in electricity and magnetism quite the way you learned it in Physics 121.

(a) $+\hat{\mathbf{x}}$

(b) $-\hat{\mathbf{x}}$

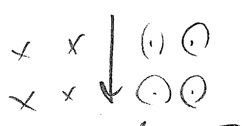
(c) $+\hat{\mathbf{y}}$

(d) $-\hat{\mathbf{y}}$

(e) $+\hat{\mathbf{z}}$

(f) $-\hat{\mathbf{z}}$

$\vec{F}_{on\,1} = \underbrace{I\vec{\ell} \times \vec{B}}_{\text{upwards via right hand rule}}$



field from wire 2 as indicated

1.6. A current runs in the $+\hat{\mathbf{z}}$ direction. In which direction is the vector potential **A**?

(a) $+\hat{\mathbf{s}}$
(b) $-\hat{\mathbf{s}}$
(c) $+\hat{\boldsymbol{\phi}}$
(d) $-\hat{\boldsymbol{\phi}}$
(e) $+\hat{\mathbf{z}}$
(f) $-\hat{\mathbf{z}}$

$\vec{A}$ generally is same direction as $\vec{J}$

(example $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{r}\, d\tau'$)

1.7. What are the units of the magnetization **M**? (T = tesla, A = ampere)

(a) T
(b) T/m
(c) T/m^2
(d) T·m
(e) $T\cdot m^2$
(f) A
(g) A/m
(h) A/m^2
(i) A·m
(j) $A\cdot m^2$
(k) None of the above
(l) More than one of the above

$\vec{M} = \frac{m \text{ (dipole moment)}}{\text{volume}} = \frac{A\cdot m^2}{m^3}$

1.8. True/False: Ampere's Law for **H** means that whenever you have $I_{free} = 0$, **H** will also be 0.

(a) True
(b) False

$\oint \vec{H}\cdot d\vec{l} = I_{free}$

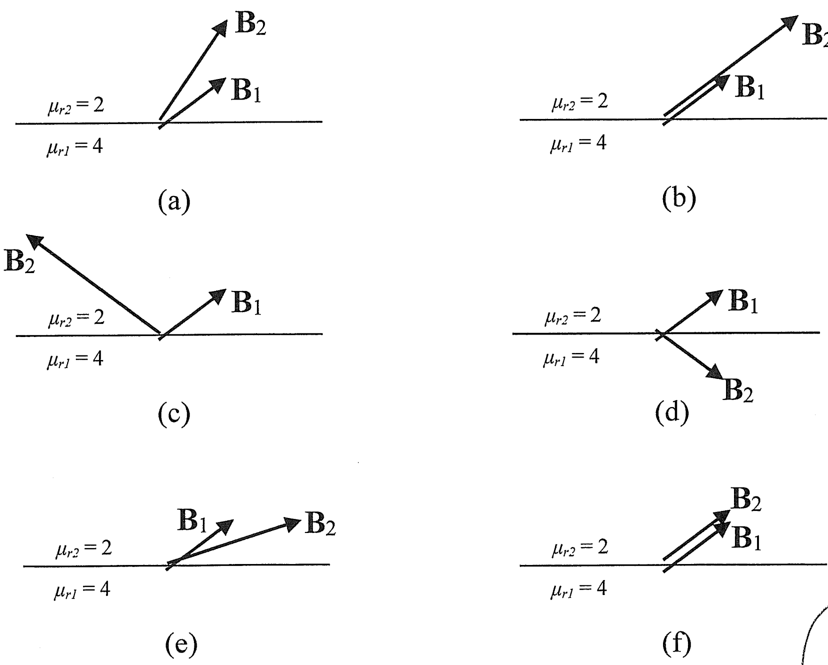
$\rightarrow \oint \vec{H}\cdot d\vec{l} = 0$

→ this doesn't mean H will be zero. Only in high symmetry situations where you can pull H out of the integral will it force H to 0.

1.9. Assuming that there is no free surface current on the boundary between the two magnetic media shown, which of the figures represents possible magnetic field intensity vectors on the two sides of the boundary? The B_1 and B_2 arrows represent magnetic fields just below and just above the boundary, respectively (where the arrows begin).

Sorry, No correct answer :(

Free +2 pts for everyone! :)



correct

$H_{1\parallel} - H_{2\parallel} = K_{free} = 0$ for this case

$H_{1\parallel} = H_{2\parallel}$

$\frac{B_{1\parallel}}{\mu_{r1}} = \frac{B_{2\parallel}}{\mu_{r2}}$

$B_{2\parallel} = \frac{\mu_{r2}}{\mu_{r1}} B_{1\parallel} = \frac{1}{2} B_{1\parallel}$

Also $B_{1\perp} = B_{2\perp}$

Should be $\vec{B}_1$ $\vec{B}_2$

1.10. Ampere's Law, written as $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, is __________ with the rule that the divergence of the curl of a vector field must necessarily be zero.

(a) always consistent
(b) never consistent
(c) consistent if ρ = constant in time
(d) not consistent if ρ = constant in time

$$\nabla \cdot (\nabla \times \vec{B}) \overset{\text{must}}{=} 0$$

$$\nabla \cdot (\mu_0 \vec{J}) \overset{\text{must}}{=} 0$$

$$\cancel{\mu_0}\ \underbrace{\nabla \cdot \vec{J}}_{-\frac{\partial \rho}{\partial t}} \overset{\text{must}}{=} 0$$

$$\boxed{\frac{\partial \rho}{\partial t} \overset{\text{must}}{=} 0}$$

unless you modify Ampere's law to be $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$,

then it is always consistent.

(9 pts) **Problem 2**: Short answers.

(a) Write out explicitly in Cartesian coordinates what is mean by $\vec{\nabla}(\mathbf{m}\cdot\mathbf{B})$. Simplify as much as possible.

$$= \vec{\nabla}(m_x B_x + m_y B_y + m_z B_z)$$

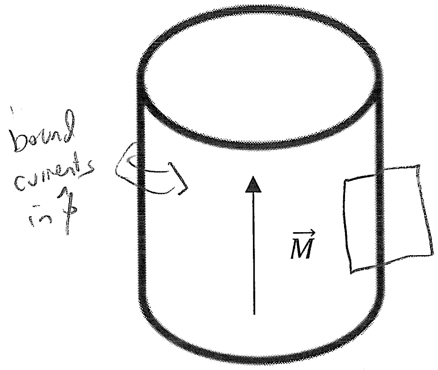
$$= m_x \vec{\nabla} B_x + m_y \vec{\nabla} B_y + m_z \vec{\nabla} B_z$$

$$= m_x\left(\frac{\partial B_x}{\partial x}\hat{x} + \frac{\partial B_x}{\partial y}\hat{y} + \frac{\partial B_x}{\partial z}\hat{z}\right) + m_y\left(\frac{\partial B_y}{\partial x}\hat{x} + \frac{\partial B_y}{\partial y}\hat{y} + \frac{\partial B_y}{\partial z}\hat{z}\right) + m_z(\ldots)$$

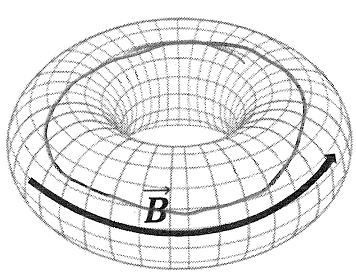
$$= \boxed{\hat{x}\left(m_x\frac{\partial B_x}{\partial x} + m_y\frac{\partial B_y}{\partial x} + m_z\frac{\partial B_z}{\partial x}\right) + \hat{y}\left(m_x\frac{\partial B_x}{\partial y} + m_y\frac{\partial B_y}{\partial y} + m_z\frac{\partial B_z}{\partial y}\right) + \hat{z}\left(m_x\frac{\partial B_x}{\partial z} + m_y\frac{\partial B_y}{\partial z} + m_z\frac{\partial B_z}{\partial z}\right)}$$

(b) Draw an appropriate Amperian loop on each figure for the indicated magnetization **M**, magnetic field **B**, and current *I*.

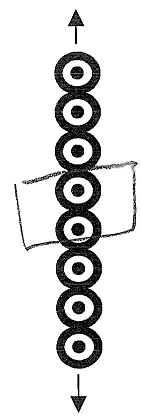
must "catch" the current and be parallel (and possibly in some sections ⊥) to $\vec{B}$ and/or $\vec{H}$



i. cylinder with magnetization $\vec{M}$ in the $\hat{\mathbf{z}}$ direction.



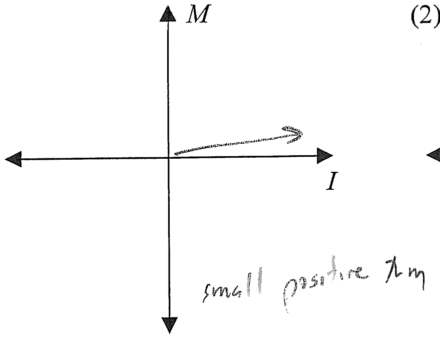
ii. torus with magnetic field $\vec{B}$ in the $\hat{\boldsymbol{\phi}}$ direction.



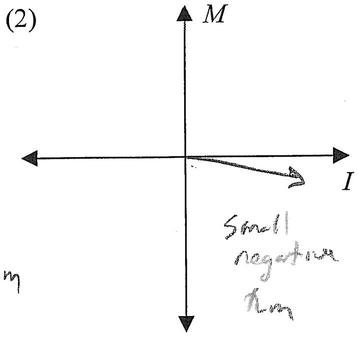
iii. infinite plane of wires with current *I* coming out of the page.

(c) A piece of initially unmagnetized material is placed inside a solenoid. Sketch the general shape of the magnetization as a function of solenoid current as the current is increased from zero, for (1) paramagnetic, (2) diamagnetic, and (3) ferromagnetic materials. → this will be proportional to H field

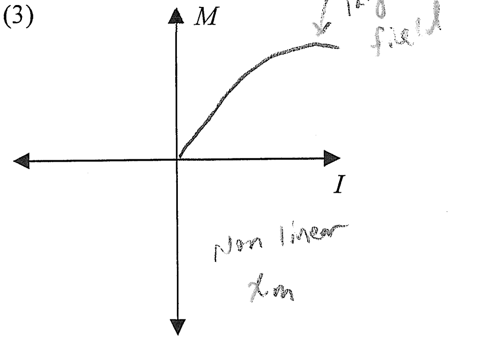
(1)



(2)

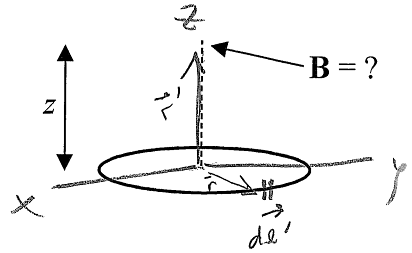


(3)



$\vec{M} = \chi_m \vec{H}$

(14 pts) **Problem 3:** Use the Biot-Savart law to determine the magnetic field of a current loop (radius *R*, current *I*) a distance *z* above its center along the axis of symmetry. The current is counter-clockwise as viewed from above.



$$\vec{r} = z\hat{z}$$
$$\vec{r}' = R\hat{s}$$
$$\vec{\mathscr{r}} = z\hat{z} - R\hat{s}$$
$$\mathscr{r} = \sqrt{z^2 + R^2}$$
$$d\vec{l}' = R\, d\phi'\, \hat{\phi}$$

By symmetry answer will be in z-direction

Biot Savart:

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$

can cross this out because $\hat{\phi} \times \hat{z}$ is not in z direction

$$= \frac{\mu_0}{4\pi} I \int \frac{(R\, d\phi'\, \hat{\phi}) \times (z\hat{z} - R\hat{s})}{(z^2 + R^2)^{3/2}}$$

triplet $s\ \phi\ z$ $\rightarrow \hat{\phi} \times \hat{s} = -\hat{z}$

$$= \frac{\mu_0}{4\pi} I\, R(-R) \int \frac{(\hat{\phi} \times \hat{s})\, d\phi'}{(z^2 + R^2)^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi} \frac{R^2\, \hat{z}}{(z^2 + R^2)^{3/2}} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi}$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2} \frac{R^2\, \hat{z}}{(z^2 + R^2)^{3/2}}}$$

(14 pts) **Problem 4**: Later in life you make the (questionable?) career choice of becoming a university professor and you are assigned to teach Physics 441. A student comes to you after you teach the unit on the vector potential **A**, and says, "I think I have figured out the vector potential field for an infinite solenoid!" The solenoid has radius R and the magnetic field inside the solenoid is a constant $B_0\hat{\mathbf{z}}$ (field outside the solenoid is zero). The student shows you this:

$$\mathbf{A} = \begin{cases} \dfrac{B_0 s}{2}\hat{\boldsymbol{\phi}}, & s < R \\ \dfrac{B_0 R^2}{2s}\hat{\boldsymbol{\phi}}, & s > R \end{cases}$$

$A = A_\phi(s)$

(a) Prove whether or not this is a correct vector potential for the solenoid field.

Does $\vec{B} = \nabla \times \vec{A}$?

inside

$$\nabla \times \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(s A_\phi)\hat{z} + \dots$$

unimportant terms because

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(\frac{B_0}{2}s^2\right)\hat{z}$$

$$= \frac{1}{\cancel{s}}\frac{B_0}{\cancel{2}}\cancel{2}\cancel{s}\,\hat{z}$$

$$= B_0\hat{z} \quad \checkmark \quad \text{yes!}$$

yes, it's correct!

outside

$$\nabla \times \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}\left(\cancel{s}\frac{B_0 R^2}{2\cancel{s}}\right)$$

$$= 0 \quad \checkmark \quad \text{yes!}$$

Boundary Also should check that $\vec{A}$ is continuous: $\left.\frac{B_0 s}{2}\right|_{s=R} \stackrel{?}{=} \left.\frac{B_0 R^2}{2s}\right|_{s=R}$

$B_0 R/2 = \frac{B_0 R}{2}$ ✓ yes

(b) Regardless of whether this is the correct **A** for the situation, prove whether or not this vector potential is in the Coulomb gauge.

Does $\nabla \cdot \vec{A} = 0$?

$$\nabla \cdot \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(s A_s) + \frac{\partial}{\partial \phi}A_\phi + \frac{\partial}{\partial z}A_z$$

$$= 0 \quad \checkmark$$

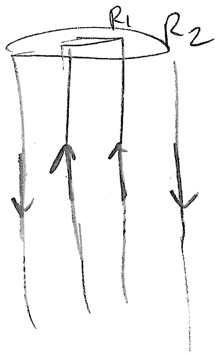
$\boxed{\text{Yes, it's Coulomb gauge}}$

$\vec{K}_b = \vec{M} \times \hat{n}$ triplet s ϕ z

$\vec{J}_b = \nabla \times \vec{M}$

(14 pts) **Problem 5**: A thick cylindrical shell with inner radius R_1 and outer radius R_2 and extending infinitely in the z-direction has a built-in magnetization between R_1 and R_2 given by: $\mathbf{M} = M_0 \frac{R_1}{s} \hat{\boldsymbol{\phi}}$.

(a) Calculate the bound currents on the two surfaces and inside the volume of the shell.



inner surface $\vec{K}_b = \left(M_0 \frac{R_1}{R_1} \hat{\phi}\right) \times (-\hat{s})$

$$\boxed{\vec{K}_b = +M_0 \hat{z}}$$

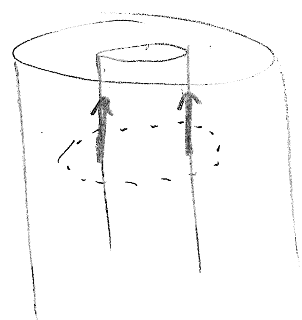
outer surface $\vec{K}_b = \left(M_0 \frac{R_1}{R_2} \hat{\phi}\right) \times (+\hat{s})$

$$\boxed{\vec{K}_b = -M_0 \frac{R_1}{R_2} \hat{z}}$$

volume $\vec{J}_b = \nabla \times \vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(s M_\phi)\hat{z} + \ldots$ unimportant terms

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(\cancel{s}\, M_0 \frac{R_1}{\cancel{s}}\right)\hat{z} = \boxed{0}$$

(b) Using either Ampere's law for **B** or for **H** (or both, if you want to check yourself), calculate the magnetic field as a function of s inside the shell, i.e. for $R_1 < s < R_2$.



Ampere's Law for $\vec{B}$

$\underbrace{\oint \vec{B} \cdot d\vec{l}}_{= B \cdot 2\pi s} = \mu_0 I_{enc}$

$\downarrow$

$= \int K \, dl$

$= (M_0)(2\pi R_1)$

$B \cdot 2\pi s = \mu_0 M_0 2\pi R_1$

$$\boxed{\vec{B} = \mu_0 \frac{M_0 R_1}{s} \hat{\phi}}$$

for $\vec{H}$

$\oint \vec{H} \cdot d\vec{l} = I_{free\ enc.}$

$H \cdot 2\pi s = 0$

$\underline{H = 0}$

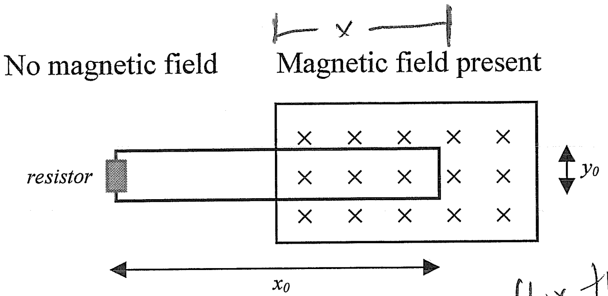
then $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

So $\vec{B} = \mu_0 \vec{M}$

$$\boxed{\vec{B} = \mu_0 M_0 \frac{R_1}{s} \hat{\phi}}$$

Yay, they agree! :)

(15 pts) **Problem 6**. A rectangular loop of wire spans a region of space where the magnetic field abruptly goes from being zero to being constant in space, as shown. The dimensions of the loop are shown and there is a resistor, resistance R, as part of the loop.



increases flux through loop

The rectangular loop is pushed to the right at speed v without deforming it. At the same time the B-field increases in magnitude with time according to $\mathbf{B} = B_0 \left(\frac{t}{\tau}\right)^2 \hat{\mathbf{z}}$ where $\hat{\mathbf{z}}$ is into the page and τ is a positive constant.

also increases flux through loop.

(a) What direction is the current through the resistor? (up or down) Show your work and/or explain your logic; no credit for answers with no work/explanation.

By Lenz's law, an opposing current will set up to counter the change in flux.

By right hand rule, this will be CCW

So current through resistor is $\boxed{\text{down}}$

(b) What is the current through the resistor as a function of time, assuming that at time $t = 0$ exactly half of the rectangle is inside the field (as shown)?

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

$\Phi = \int \vec{B}\cdot d\vec{a} = BA$ since the field is constant in space (within the field region)

$$= -\frac{d}{dt}(BA)$$

$$= -\frac{d}{dt}\left(B_0 \frac{t^2}{\tau^2} \cdot x y_0\right)$$

$$= -\frac{B_0 y_0}{\tau^2} \underbrace{\frac{d}{dt}(t^2 x)}_{t^2 \frac{dx}{dt} + 2tx}$$

$\frac{dx}{dt} = v$

$x = \frac{x_0}{2} + vt$ from this condition

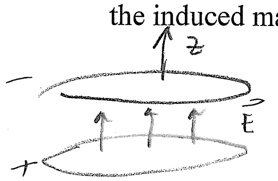
$$= -\frac{B_0}{\tau^2} y_0 \left(t^2 v + 2t\left(\frac{x_0}{2} + vt\right)\right)$$

$$= -\frac{B_0}{\tau^2} y_0 \left(3t^2 v + x_0 t\right)$$

the current is $\frac{\mathcal{E}}{R}$ and the negative sign just indicates direction, so

$$\boxed{I = \frac{B_0 y_0}{\tau^2 R}\left(3t^2 v + x_0 t\right)}$$

(14 pts) **Problem 7**. The charge of a capacitor (parallel circular plates with area A, electric field in the $\hat{\mathbf{z}}$ direction) is decreasing according to $Q = Q_0 e^{-t/\tau}$, where Q_0 and τ are positive constants). Determine the induced magnetic field, both magnitude and direction.



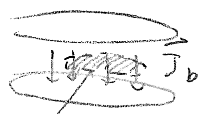
$\vec{E}$ in a cap $= \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0}$ (could prove via Gauss's Law)

$$\vec{E}(t) = \frac{Q_0 e^{-t/\tau}}{\epsilon_0 A} \hat{z}$$

Full Ampere's Law: $\nabla \times \vec{B} = \mu_0 \vec{J} + \underbrace{\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$

"J_D", creates a magnetic field as if it were a $\vec{J}$

$$\vec{J}_D = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \cancel{\epsilon_0} \frac{Q_0}{\cancel{\epsilon_0} A} \left(-\frac{1}{\tau}\right) e^{-t/\tau} \hat{z}$$



Amperian loop

$\vec{B}$ from that "current" will be in $-\hat{\phi}$ direction (right hand rule)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$= B \cdot 2\pi s$

$\hookrightarrow$ the "I" from J_D would be $\int J\, da = J_D \pi s^2$ since $\vec{J}_D$ is constant in space

$$= \frac{Q_0}{A}\left(\frac{1}{\tau} e^{-t/\tau}\right)(\pi s^2)$$

$$B \cdot 2\pi s = \mu_0 \frac{Q_0}{A\tau} e^{-t/\tau} \pi s^2$$

$$\boxed{\vec{B} = \frac{\mu_0 s Q_0}{2A\tau} e^{-t/\tau} (-\hat{\phi})}$$

(inside the capacitor)