Fall 2018
Physics 441
Exam 3
Dr. Colton, cell: 801-358-1970

40115543

No time limit. Student calculators are allowed. One page of handwritten notes allowed (front & back). Books not allowed.

Name Solutions

Instructions: Please label & circle/box your answers. **Show your work**, where appropriate! Remember: **in any problems involving Gauss's or Ampere's Laws, you should explicitly show your Gaussian surface/Amperian loop**. For all problems, unless otherwise specified you may assume that you are dealing with electrostatics, i.e. the charges are not moving and the fields have come to equilibrium, and that any/all vector potentials are in the Coulomb gauge.

Griffiths front and back covers

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl:
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

Special case derivatives: $\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta(\mathbf{r})$ $\quad\nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r})$
(similar things true for $\mathscr{r}$)

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^{8}\ \mathrm{m/s}$	(speed of light)
$e = 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Some miscellaneous mathematical stuff:

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \qquad \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x \qquad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \qquad \cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

The first few Legendre polynomials:

$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \qquad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x \qquad P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

Some indefinite integrals:

$\int \sin Cx = \frac{-\cos Cx}{C} \qquad \int \sin^2 Cx = \frac{x}{2} - \frac{\sin 2Cx}{4C} \qquad \int \cos Cx = \frac{\sin Cx}{C} \qquad \int \cos^2 Cx = \frac{x}{2} + \frac{\sin 2Cx}{4C}$

Some definite integrals:

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{a}{2}, & \text{if } n = m \end{cases}$$

$\int_0^\pi \sin^2 x\,dx = \frac{\pi}{2} \qquad \int_0^\pi \cos^2 x\,dx = \frac{\pi}{2} \qquad \int_0^{2\pi} \sin^2 x\,dx = \pi \qquad \int_0^{2\pi} \cos^2 x\,dx = \pi$

$\int_0^\pi \sin^3 x\,dx = \frac{4}{3} \qquad \int_0^\pi \cos^3 x\,dx = 0 \qquad \int_0^{2\pi} \sin^3 x\,dx = 0 \qquad \int_0^{2\pi} \cos^3 x\,dx = 0$

$\int_0^\pi \sin^4 x\,dx = \frac{3\pi}{8} \qquad \int_0^\pi \cos^4 x\,dx = \frac{3\pi}{8} \qquad \int_0^{2\pi} \sin^4 x\,dx = \frac{3\pi}{4} \qquad \int_0^{2\pi} \cos^4 x\,dx = \frac{3\pi}{4}$

$$\int_{-1}^1 P_\ell(x)\,P_m(x)dx = \begin{cases} 0, & \text{if } \ell \neq m \\ \frac{2}{2\ell+1}, & \text{if } \ell = m \end{cases} \qquad \int_0^\pi P_\ell(\cos\theta)\,P_m(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0, & if\ \ell \neq m \\ \frac{2}{2\ell+1}, & if\ \ell = m \end{cases}$$

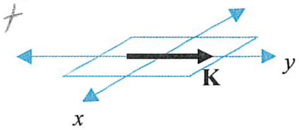
(18 pts) **Problem 1**. Multiple choice, 1.5 pts each. Circle the correct answers for the multiple choice questions.

1.1. A surface current $\mathbf{K} = \boldsymbol{K(x,y)}\,\hat{\mathbf{y}}$ somehow flows within a finite rectangle in the x-y plane as shown. The rectangle goes from $-a$ to $+a$ in the x-direction and $-b$ to $+b$ in the y-direction. How should you set up the integral to calculate the amount of current flowing?

(a) $I = \int_{-a}^{a} K dx$ → this "catches" the current

(b) $I = \int_{-b}^{b} K dy$

(c) $I = \int_{-a}^{a} \int_{-b}^{b} K dy dx$



1.2. To find the magnetic field **B** at the point P due to a current-carrying wire we use the Biot-Savart law. What is the direction of the infinitesimal contribution d**B** created by current in the section of wire, d**l**, that is shown?

Biot-Savart: $\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l} \times \hat{r}}{r^2}$

(a) Upwards

(b) Up and to the left

(c) Up and to the right

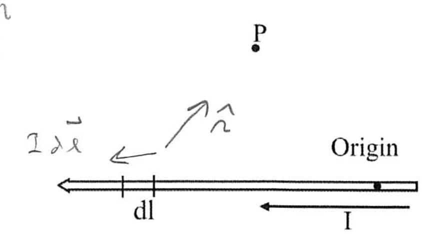
(d) Downwards

(e) Down and to the left

(f) Down and to the right

(g) Into the page

(h) Out of the page



Cross product is into the page.

1.3. A trajectory of a positive particle is shown. Please forgive the artwork. To help you see the 3D nature of the path the way I intend, I have indicated some places where the particle's path is behind the y-axis and some places where the path is in front. Alternately, viewed from the right, the path is counter-clockwise. There is no electric field present (aside from that created by the particle itself). In which direction is the (constant) magnetic field that is influencing the trajectory?

(a) $+\hat{\mathbf{x}}$

(b) $-\hat{\mathbf{x}}$

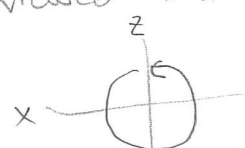
(c) $+\hat{\mathbf{y}}$

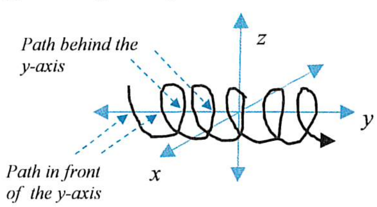
(d) $-\hat{\mathbf{y}}$

(e) $+\hat{\mathbf{z}}$

(f) $-\hat{\mathbf{z}}$

viewed from the right





$F = q\vec{v} \times \vec{B}$ → must be into the page

1.4. A magnetic field is created by running current through a solenoid. A second magnetic field is created by running the same magnitude current through another solenoid which is identical except it has a core made of diamagnetic material. In which case is there more energy stored in the field?

same free current → same H field

(a) First field

(b) Second field

$U = \frac{1}{2\mu_0} \int B^2 \, d\tau = \frac{1}{2}\mu_0 \int H^2 d\tau$ vs. $U = \frac{1}{2} \int \vec{B} \cdot \vec{H} \, d\tau = \frac{1}{2}\mu_0\mu_r \int H^2 d\tau$

diamagnetic: $\mu_r < 1$ so first field is greater

1.5. A current I in a circular loop of radius b produces a magnetic field. At a fixed point far from the loop, the strength of the magnetic field is proportional to which of the following combinations of I and b?

$B \sim$ dipole moment

(a) Ib

(b) Ib^2

(c) $I^2 b$

(d) I/b

(e) I/b^2

(f) I^2/b^2

(g) None of the above

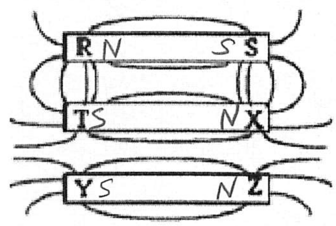
$B \sim \frac{1}{r^2}$ $B \sim \frac{1}{r^3}$

1.6. A localized current distribution has no magnetic monopole moment, no magnetic dipole moment, but a nonzero magnetic quadrupole moment. At a large distance r from the distribution, the magnetic field will fall off like:

$B \sim \frac{1}{r^4}$

(a) No fall off; it's constant
(b) $1/r$
(c) $1/r^2$
(d) $1/r^3$
(e) $1/r^4$
(f) $1/r^5$

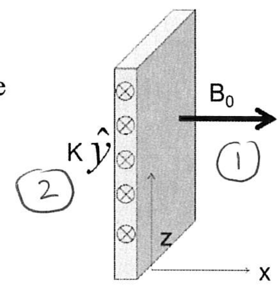
1.7. The diagram depicts iron filings sprinkled around three permanent magnets. Pole R is the same type of pole (i.e. north vs. south) as:



(a) T and Y
(b) T and Z
(c) X and Y
(d) X and Z
(e) T, X, and Y
(f) T, Y, and Z
(g) X, Y, and Z

let's assume R = North, then we can get other poles by inspecting the lines (which go from N to S)

1.8. A sheet of current with surface current density **K** in the y-direction as shown serves as a boundary between two regions of space (the y-axis points into the page). If $\mathbf{B} = B_0\hat{\mathbf{x}}$ just to the RIGHT of the sheet, in what direction will **B** be just to the LEFT of the sheet? (The magnetic fields are not solely due to the sheet itself; they are produced in part by additional currents not shown.)



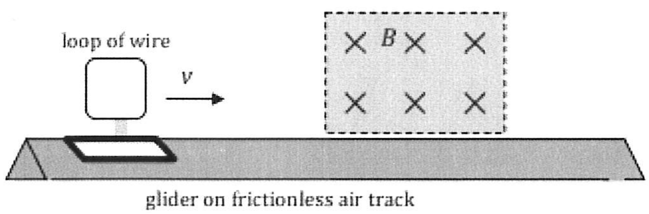
(a) +x
(b) –x
(c) +z
(d) -z
(e) +y
(f) –y
(g) None of the above

$B_{1\perp} = B_{2\perp} \Rightarrow B_2 = B_0\hat{x}$

$\vec{B}_{\parallel 1} - \vec{B}_{\parallel 2} = \vec{K} \times \hat{n}$

$\hat{n}$ points into region 1

$B_{\parallel 2} = -K\hat{y} \times \hat{x} = K\hat{z}$ total field will be $\vec{B} = B_0\hat{x} + K\hat{z}$



1.9. A single, continuous loop of conducting wire is mounted on a glider which travels on a frictionless air track with an initial velocity v. When the front edge of the loop enters the magnetic field region which points into the page as shown...

Faraday / Lenz's law : current in loop will be induced CCW.

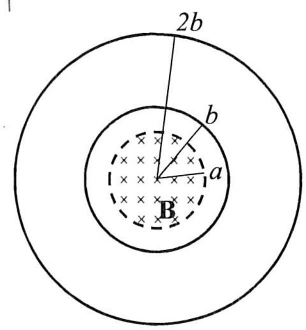
(a) there is a clockwise current in the loop, and the glider slows down.
(b) there is a counterclockwise current in the loop, and the glider slows down.
(c) there is a clockwise current in the loop, and the glider speeds up.
(d) there is a counterclockwise current in the loop, and the glider speeds up.
(e) there is no current in the loop, and the glider travels at constant v.

$\vec{F} = I\vec{l} \times \vec{B}$ means right section of wire (current up) will experience a force to left

1.10. Ampere's Law, written as $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, is ___________ with the rule that the divergence of the curl of a vector field must necessarily be zero.
(a) always consistent
(b) never consistent
(c) consistent if ρ = constant in time
(d) not consistent if ρ = constant in time

$\nabla \cdot (\nabla \times B) = \nabla \cdot (\mu_0 J)$
$= \mu_0 \left(-\frac{\partial \rho}{\partial t}\right)$
will = 0 if ρ = constant in time

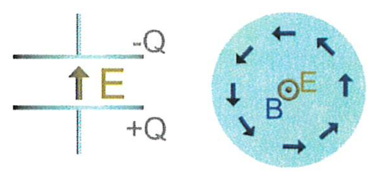
1.11. Two concentric wire loops of radius b and $2b$ lie in the plane of the page as shown. A uniform magnetic field **B** going into the page is confined to a concentric region of radius a, where $a < b$ but is increasing. At a particular moment in time the induced EMF in the wire loop of radius b is ε. What will be the induced EMF in the wire loop of radius $2b$ at that same moment?



(a) $\varepsilon/4$
(b) $\varepsilon/2$
(c) ε
(d) 2ε
(e) 4ε
(f) Zero

$\mathcal{E} = -\frac{d\Phi_B}{dt}$
↳ the changing flux is the same for both loops.

1.12. The figures show a side and top view of a capacitor with charge Q and electric and magnetic fields **E** and **B** at time t. At this time the charge Q must be:



(a) Increasing in time
(b) Decreasing in time.
(c) Constant in time.

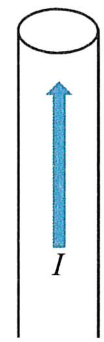
$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$
⇓
induces a B field in the same way as a $\vec{J}$.

By right hand rule, if a $\vec{J}$ were causing the given $\vec{B}$ field, it would be pointing out of the page (right figure). Therefore $\frac{d\vec{E}}{dt}$ is out of the page, and $\vec{E}$ is increasing in time.

χ_m = negative

(15 pts) **Problem 2**. Short answers. No explanations needed. Negative signs required when appropriate. Half credit if wrong sign but otherwise right answer, otherwise questions will be graded pass/fail.

(a) A very long diamagnetic rod carries a uniformly distributed current I along the $+\hat{\mathbf{z}}$ direction. The current sets up a **B** field inside the rod (as well as outside), which then induces a magnetization. Indicate the direction of all of the following (no magnitudes needed). If a given quantity is zero then state that instead of a direction.



1. The (total) **B** field inside the rod $\boxed{\hat{\phi}}$ by right hand rule

2. The **H** field inside the rod $\boxed{\hat{\phi}}$ $\vec{H}$ and $\vec{B}$ are same direction for linear isotropic materials $\left(\vec{H} = \frac{1}{\mu_0\mu_r}\vec{B}\right)$

3. The **M** field inside the rod $\boxed{-\hat{\phi}}$ $\vec{M} = \chi_m \vec{H}$ and χ_m is negative

4. The **A** field inside the rod $\boxed{\hat{z}}$ $\vec{A}$ in same direction as current (in Coulomb gauge, $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{\mathscr{r}}\,d\tau'$)

5. The bound surface current on the "wrapper" $\boxed{\hat{z}}$
$$\vec{K}_b = \vec{M} \times \hat{n} = -\hat{\phi} \times \hat{s} = +\hat{z}$$

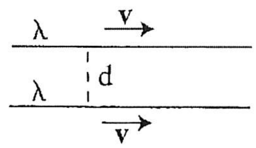
6. The bound volume current $\boxed{-\hat{z}}$
$$\vec{J}_b = \nabla \times \vec{M}\ ; \quad \vec{M} = M(s)(-\hat{\phi})$$
$$= \frac{1}{s}\frac{\partial}{\partial s}(s M_\phi)\,\hat{z}$$
$$= -\hat{z}$$

(b) Units!

1. What are the units of the **B**-field in terms of N = newton, A = ampere, and m = meter?
$\vec{F} = I \vec{\ell} \times \vec{B}$
$N = A \cdot m\,[B]$ $\quad \boxed{[B] = \frac{N}{A \cdot m}}$

2. What are the units of the **H**-field in terms of N = newton, A = ampere, and m = meter?
$H = \frac{B}{\mu_0 \mu_r}$ $\longrightarrow$ $[H] = \frac{\frac{N}{A \cdot m}}{\frac{N}{A^2}} = \boxed{\frac{A}{m}}$
μ_r → dimensionless
μ_0 → units of $\mu_0 = \frac{N}{A^2}$ from list of constants

(13 pts) **Problem 3**. Two parallel infinite line charges (charge density λ) are separated by a distance *d*. Both move in the same direction with velocity *v*, thus creating parallel currents. How great would the speed *v* have to be in order for the magnetic attraction to exactly balance the electrical repulsion? Note: I want a numeric answer in m/s as well as symbolic answer. Hint: the current *I* created by a moving line charge λ is given by $I = \lambda v$.



Magnetic: $\vec{F} = I\vec{l} \times \vec{B}$ where $\vec{B}$ = field of second wire at location of first wire

$B = \mu_0 \frac{I}{2\pi s}$ field from long wire

Note $s = d$, and $I = \lambda v$

$$= \frac{\mu_0 I^2 l}{2\pi d}$$

$$F = \frac{\mu_0 \lambda^2 v^2 l}{2\pi d}$$

Electric: $\vec{F} = q\vec{E}$ where $\vec{E}$ = field of 2^{nd} at location of 1^{st}

$q = \lambda l$

$E = \frac{\lambda}{2\pi\epsilon_0 s}$ from Gauss's law

$$\oint \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$$
$$E\, 2\pi s l = \lambda l/\epsilon_0$$
$$E = \frac{\lambda}{2\pi\epsilon_0 s}$$

$$F = \frac{\lambda^2 l}{2\pi\epsilon_0 d}$$

Equate the two

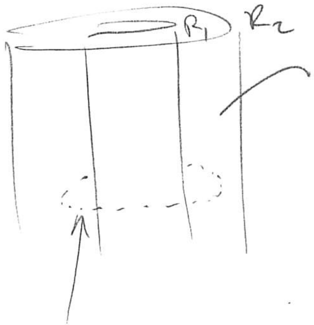
$$\frac{\mu_0 \cancel{\lambda^2} v^2 \cancel{l}}{\cancel{2\pi} \cancel{d}} = \frac{\cancel{\lambda^2} \cancel{l}}{\cancel{2\pi}\epsilon_0 \cancel{d}} \rightarrow \mu_0 v^2 = \frac{1}{\epsilon_0}$$

$$\boxed{v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}} = \boxed{3 \cdot 10^8 \text{ m/s}}$$

this is c!

(15 pts) **Problem 4**: A thick cylindrical shell with inner radius R_1 and outer radius R_2 and extending infinitely in the z-direction has a built-in magnetization between R_1 and R_2 given by: $\mathbf{M} = M_0 \frac{R_1}{s} \hat{\boldsymbol{\phi}}$.

(a) Calculate the bound currents on the two surfaces and inside the volume of the shell.



$\vec{M} = M_0 \frac{R_1}{s} \hat{\phi}$

Amperian loop for part b, to catch current in $\hat{z}$ direction

inner: $\vec{K}_b = \vec{M} \times \hat{n} \big|_{s=R_1}$

$= M_0 \frac{R_1}{R_1} \hat{\phi} \times (-\hat{s})$

$\boxed{\vec{K}_b = M_0 \hat{z}}$ for $s=a$

outer: $\vec{K}_b = \vec{M} \times \hat{n} \big|_{s=R_2}$

$= M_0 \frac{R_1}{R_2} \hat{\phi} \times (+\hat{s})$

$\boxed{\vec{K}_b = M_0 \frac{R_1}{R_2} (-\hat{z})}$

$\vec{J}_b = \nabla \times \vec{M} = \frac{1}{s} \frac{\partial}{\partial s} (s M_\phi) \hat{z} = \frac{1}{s} \frac{\partial}{\partial s} (M_0 R_1 \hat{\phi})$

only non zero term ↑ $\boxed{\vec{J}_b = 0}$

(b) Using either Ampere's law for **B** or for **H** (or both, if you want to check yourself), calculate the magnetic field as a function of s inside the shell, i.e. for $R_1 < s < R_2$.

Method 1 H field

$\oint \vec{H} \cdot d\vec{l} = I_{free\ enc}$

$H \cdot 2\pi s = 0$

$H = 0$

Then $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

so $\vec{B} = \mu_0 \vec{M}$

$\boxed{\vec{B} = \mu_0 M_0 \frac{R_1}{s} \hat{\phi}}$ $R_1 < s < R_2$

Method 2 B field and bound currents

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

in $\hat{\phi}$ direction by symmetry

$I = \int K \, dl$

$K = K$ from inner surface only

$= K_b \, 2\pi R_1$

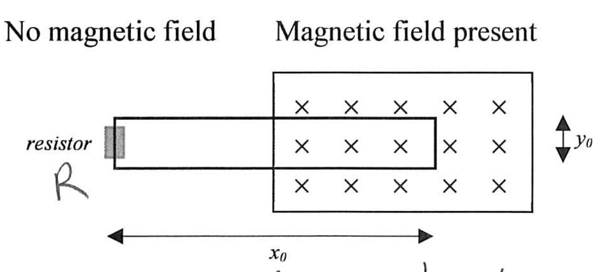
$= M_0 \, 2\pi R_1$

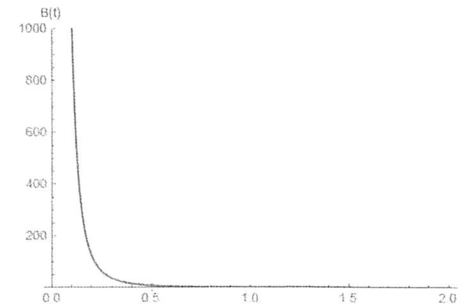
$B \cdot 2\cancel{\pi} s = \mu_0 M_0 \, 2\cancel{\pi} R_1$

$\boxed{\vec{B} = \frac{\mu_0 M_0 R_1}{s} \hat{\phi}}$

$R_1 < s < R_2$

(16 pts) **Problem 5**. A rectangular loop of wire spans a region of space where the magnetic field abruptly goes from being zero to being constant in space, as shown. The dimensions of the loop are shown and there is a resistor, resistance R, as part of the loop.





$x = \frac{x_0}{2} + vt$

The rectangular loop is pushed to the right at speed v without deforming it. At the same time the B-field changes with time according to $\mathbf{B} = B_0\left(\frac{t}{\tau}\right)^{-3}\hat{\mathbf{z}}$ where $\hat{\mathbf{z}}$ is into the page and τ is a positive constant. This is depicted in the right-hand graph for some choice of B_0 and τ. (Ignore the fact that B is infinite at time $t = 0$, which isn't possible.)

(a) What direction is the current in the loop? (CW vs CCW) Show your work and/or explain your logic. If it can't be determined, explain why. No credit for answers with no work/explanation.

B field decreasing → decreases flux → tends to produce CW current

Area increasing → increases flux → " " " CCW current

$\boxed{\text{Can't be determined}}$ (without more details, can't say which effect "wins")

(b) What is the current through the resistor as a function of time, assuming that at time $t = 0$ exactly half of the rectangle is inside the field (as shown)? Hint: that means that if you call x the horizontal distance the loop is immersed in the field, then $x = \frac{x_0}{2} + vt$.

$\mathcal{E} = -\frac{d}{dt}(\Phi_B)$

$\Phi_B = \int \vec{B}\cdot d\vec{a} = BA$ since B is constant in space

$\Phi = B_0\left(\frac{t}{\tau}\right)^{-3}\cdot\left(\frac{x_0}{2} + vt\right) y_0$

$$\frac{d\Phi}{dt} = B_0\tau^3 y_0 \underbrace{\frac{d}{dt}\left(t^{-3}\left(\frac{x_0}{2} + vt\right)\right)}_{t^{-3}(v) + -3t^{-4}\left(\frac{x_0}{2} + vt\right)}$$ using product rule

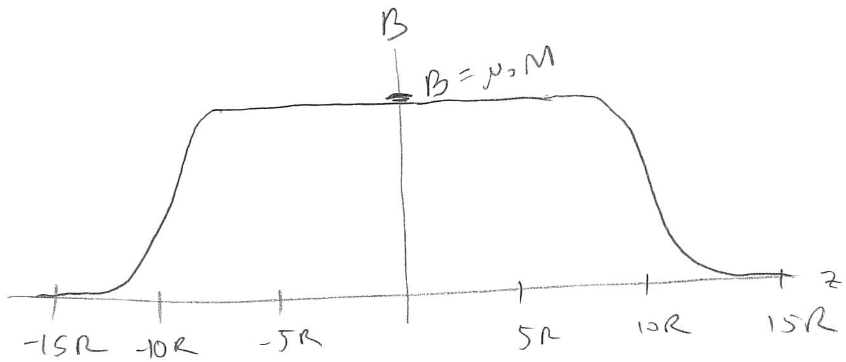
$$\boxed{\mathcal{E} = -B_0\tau^3 y_0\left[v t^{-3} - 3t^{-4}\left(\frac{x_0}{2} + vt\right)\right]}$$

again, since these terms are opposite sign, you can't tell sign of $\mathcal{E}$ / direction of current without more info.

I showed this plot in class and posted it to website. It's a finite solenoid, which is like infinite solenoid but with edge effects

(13 pts) **Problem 6**. A cylindrical bar magnet has a constant magnetization M which points in the z-direction along the cylindrical axis. The magnet is long but not infinite, with its length equal to 20 times its radius R. It is centered at $z = 0$, and therefore extends from $z = -10R$ to $z = +10R$.

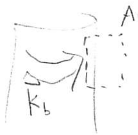
(a) Make a plot of B vs. z for $-15R < z < 15R$. You don't have to work out any equations; a general (but accurate) plot based on your experience/our class discussions is fine. Note that I'm not talking about field *lines*, I'm talking about the actual value of B, with positive and negative B values indicating a field in the $+\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$ directions as appropriate.



(b) The peak value of the B field on your graph should be the same as if it were an infinite cylinder. Figure out that value and label it on your graph above.

$\vec{K}_b = \vec{M} \times \hat{n} = M\hat{\phi}$

Amperian loop



$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

↳ only left hand side is non zero

$B \cdot \cancel{l} = \mu_0 M \cancel{l}$

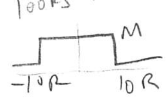
$B = \mu_0 M$

(c) On a new graph, make a plot of H vs. z for the same situation, again with positive and negative H values indicating a field in the $+\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$ directions as appropriate. Hint: $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$. This graph will <u>not</u> be just a flat line at $H = 0$. One of the points of this problem is to demonstrate to you that just because $I_{free} = 0$ for a particular situation does not necessarily mean that $H = 0$ everywhere (although it does for certain symmetric situations where we can use Ampere's law).

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

$\frac{\vec{B}}{\mu_0}$ → plot above, with max of M

$\vec{M}$ → looks like

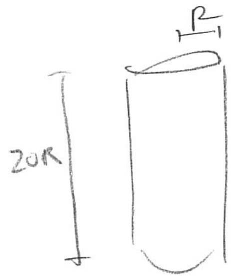


subtract the two graphs and you get

H is non-zero at the edges!

(10 pts) **Problem 7**. Same situation as the last problem. In terms of M, R, and z, what is the **B** field for points along the z-axis where $z \gg R$?

dipole approximation!



$$\vec{m} = \int \vec{M}\, d\tau$$

$$= M\hat{z}\,(\pi R^2)(20R)$$

$$\vec{m} = 20\pi R^3 M \hat{z}$$

Dipole field formula

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

For pts on z-axis,

$$r = z$$

$$\theta = 0$$

$$\hat{r} = \hat{z}$$

$$\hat{\theta} = n/a$$

$$\vec{B} = \frac{\mu_0(20\cancel{\pi}R^3M)}{4\cancel{\pi}z^3}\left(2\hat{z} + 0\right)$$

$$\boxed{\vec{B} = \frac{\mu_0\, 10 R^3 M}{z^3}\hat{z}}$$