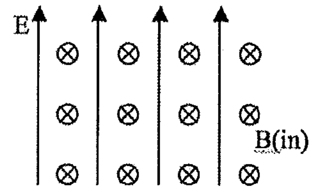
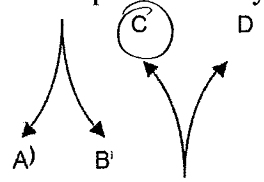
Phys 441 Exam 3 Solutions

(20 pts) **Problem 1:** Multiple choice, 2 pts each. Circle the correct answer.

1.1. A proton (positive charge) is released from rest in uniform **E** and **B** fields. **E** points up, **B** points into the page, as shown. Which of the paths will the proton initially follow?

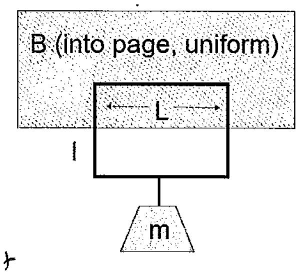




E. It will remain stationary

Electric field accelerates it ↑, then magnetic field deflects it to the left ($\vec{F} = q\vec{v} \times \vec{B}$).

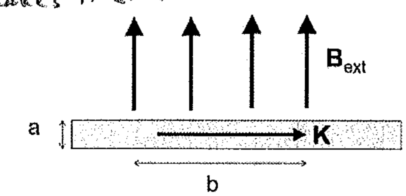
1.2. A wire loop in a **B** field has a current *I*. The magnetic field is localized; it only exists in the hatched region, and is essentially zero everywhere else. Which way must *I* be flowing to hold the mass in place as shown (gravity points down)?



(a) Clockwise
(b) Counter-clockwise
(c) You cannot "levitate" a mass like this

$\vec{F} = I\vec{l} \times \vec{B}$. To give an upward force (to counter downward mg), current must be going to right at top of loop, which makes it CW.

1.3. A "ribbon" (width *a*, length *b*, and infinitely thin in the third dimension) with a uniform surface current density **K** to the right is in a uniform magnetic field $\mathbf{B}_{ext}$, oriented as shown. What is the magnitude of the force on the ribbon?



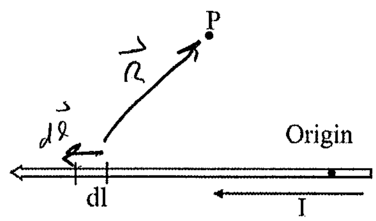
(a) KB_{ext}
(b) aKB_{ext}
(c) $abKB_{ext}$
(d) bKB_{ext}/a
(e) $KB_{ext}/(ab)$

$\vec{F} = I\vec{l} \times \vec{B}$

↳ current from $K = \int K\,dl = K \cdot a$

so $F = (Ka)(b)B_{ext}$

1.4. To find the magnetic field **B** at the point P due to a current-carrying wire we use the Biot-Savart law. What is the direction of the infinitesimal contribution **dB** created by current in the section of wire, **dl**, that is shown?



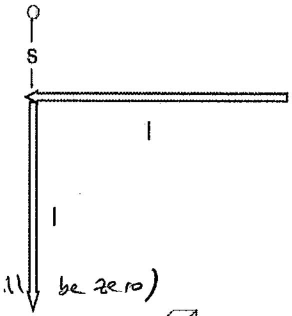
(a) Upwards
(b) Up and to the left
(c) Up and to the right
(d) Downwards
(e) Down and to the left
(f) Down and to the right
(g) Into the page
(h) Out of the page

from Biot Savart,

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}$$

$d\vec{l}$ and $\hat{r}$ in directions shown, cross product via right hand rule gives result into the page.

1.5. A wire makes a 90° bend as shown. The current flows to the left, then downwards. What is the magnitude of the magnetic field created by the wire, at the point shown a distance s above the bend?

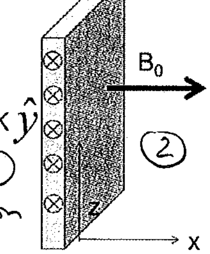


(a) $\mu_0 I/(\pi s)$
(b) $\mu_0 I/(2\pi s)$
(c) $\mu_0 I/(4\pi s)$
(d) $\mu_0 I/(8\pi s)$
(e) None of the above

Field from upper section = $\frac{1}{2}$ field of infinite wire (by symmetry), $= \frac{1}{2} \cdot \frac{\mu_0 I}{2\pi s}$

Field from lower section = 0 since $d\vec{l}$ and $\hat{r}$ are parallel (Biot Savart cross product will be zero)

1.6. A sheet of current with surface current density **K** in the y-direction as shown serves as a boundary between two regions of space. If $\mathbf{B} = \boldsymbol{B_0}\hat{\mathbf{x}}$ just to the RIGHT of the sheet, in what direction will **B** be just to the LEFT of the sheet? (The magnetic fields are not solely due to the sheet itself; they are produced in part by additional currents not shown.)



(a) +x
(b) –x
(c) +z
(d) -z
(e) +y
(f) –y
(g) None of the above

Bndy conds: (1) $B_{1\parallel} - B_{2\parallel} = \mu_0 K$
$\rightarrow B_{1\parallel}$ will be in $\hat{y}$ direction
(2) $B_{1\perp} = B_{2\perp}$
$\rightarrow B_{1\perp}$ will be in $\hat{x}$ direction

Therefore B_1 will have components in both x and y directions

1.7. The vector potential **A** due to a long straight wire with current I along the z-axis, will be in which direction? (Assume Coulomb gauge, and don't worry about the sign.)

(a) $\hat{\mathbf{z}}$
(b) $\hat{\boldsymbol{\phi}}$
(c) $\hat{\mathbf{s}}$

One way of looking at this: $\vec{B} = \nabla \times \vec{A}$, and $\vec{B}$ is in $\hat{\phi}$ direction

ϕ component of curl: $B_\phi = \frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}$

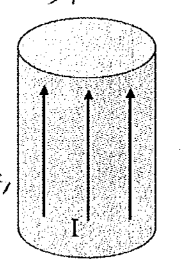
$\rightarrow$ A cannot have z dependence, by symmetry, therefore A_z must be the component giving rise to a B_ϕ.

1.8. With regards to the vector potential **A** (in general), which of the following is continuous as you move past a boundary?

(a) **A**
(b) Not all of **A**, just the perpendicular component
(c) Not all of **A**, just the parallel component
(d) Nothing is guaranteed to be continuous regarding **A**

Since $\vec{B} = \nabla \times \vec{A}$, any discontinuity in A, in any component, would give rise to an infinite magnetic field (in some direction), which is nonphysical

1.9. A very long aluminum (paramagnetic!) rod carries a uniformly distributed current I along the +z direction. What is the direction of the bound <u>surface</u> current? Hint: the current will set up a B field inside the rod (as well as outside); that B field will induce a magnetization.



(a) It points parallel to I
(b) It points antiparallel to I
(c) It points clockwise (viewed from above)
(d) It points counter-clockwise (viewed from above)
(e) It is zero

This I creates a B_ϕ. That will induce an M_ϕ. (Same direction, not opposite, since it's paramagnetic.) Surface current is $\vec{K}_b = \vec{M} \times \hat{n} = M \underbrace{\hat{\phi} \times \hat{s}}_{= -\hat{z}}$ $\rightarrow$ opposite to I

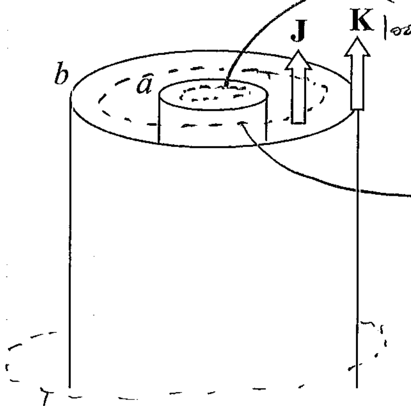
1.10. Same situation. What is the direction of the bound <u>volume</u> current?

(a) It points parallel to I
(b) It points antiparallel to I
(c) It points clockwise (viewed from above)
(d) It points counter-clockwise (viewed from above)
(e) It is zero

Method 1: Total bound current is zero, so $\vec{J}_b$ must be in $+\hat{z}$

Method 2: $\vec{J}_b = \vec{\nabla} \times \vec{M}$
$= -\underbrace{\frac{\partial M_\phi}{\partial z}}_{=0} \hat{s} + \underbrace{\frac{1}{s} \frac{\partial}{\partial s}(s M_\phi)}_{= \frac{1}{s} M} \hat{z}$ (using only ϕ components of M)
= in $+\hat{z}$ direction

(18 pts) **Problem 2.** An infinitely long, thick cylindrical shell of inner radius a and outer radius b carries a current given by the combined effects of a volume current density, $\mathbf{J} = J_0 \frac{s^2}{b^2}\hat{\mathbf{z}}$ that exists between a and b, and a surface current density $\mathbf{K} = K_0\hat{\mathbf{z}}$, that exists at $s = b$. (There is no current inside $s = a$.) Find $\boldsymbol{B}$ everywhere due to these current densities, in terms of the given quantities.



loop 1: $s<a$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$B\cdot 2\pi s = 0$$

$$\boxed{B=0}$$

loop 2: $a<s<b$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\hookrightarrow I_{enc} = \int \vec{J}\cdot d\vec{a}$$

$$= \int_a^s \left(J_0 \frac{s^2}{b^2}\right)\cdot 2\pi s\, ds$$

$$= \frac{J_0}{b^2} 2\pi \cdot \underbrace{\int_a^s s^3\, ds}_{\frac{1}{4}(s^4-a^4)}$$

$$B\cdot 2\pi s = \mu_0 \frac{J_0}{b^2}\frac{2\pi}{4}\frac{1}{4}(s^4-a^4)$$

$$\boxed{\vec{B} = \mu_0 \frac{J_0}{4b^2 s}(s^4-a^4)\ \hat{\phi}}$$

loop 3: $s>B$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\hookrightarrow I_{enc} = \underbrace{\int \vec{J}\cdot d\vec{a}}_{\text{same integral as above, but from } a \text{ to } b} + \underbrace{\int \vec{K}\cdot d\vec{l}}_{= K_0\cdot 2\pi b \text{ (since K is constant)}}$$

$$= \frac{J_0}{b^2} 2\pi \frac{1}{4}(b^4-a^4)$$

$$B\cdot \cancel{2\pi} s = \mu_0 \cancel{2\pi}\left(\frac{J_0}{4b^2}(b^4-a^4) + K_0 b\right)$$

$$\boxed{\vec{B} = \left(\frac{\mu_0 J_0}{4b^2 s}(b^4-a^4) + \frac{K_0 b}{s}\right)\hat{\phi}}$$

(16 pts) **Problem 3**. Suppose you have some vector potential, $\mathbf{A} = k\hat{\boldsymbol{\phi}}$, in a region of space.

(a) Find the magnetic field corresponding to that vector potential.

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

keeping only terms that have A_ϕ in them...

$$= -\underbrace{\frac{\partial A_\phi}{\partial z}}_{=0}\hat{s} + \underbrace{\frac{1}{s}\frac{\partial}{\partial s}(s A_\phi)}_{=\frac{1}{s}k}\hat{z}$$

$$\boxed{\vec{B} = \frac{1}{s}k\,\hat{z}}$$

(b) Find the current density corresponding to that vector potential.

$$\nabla\times\vec{B} = \mu_0\vec{J} \rightarrow \vec{J} = \frac{1}{\mu_0}\nabla\times\vec{B}$$

keeping only terms that have B_z in them...

$$\vec{J} = \frac{1}{\mu_0}\left(\underbrace{\frac{1}{s}\frac{\partial B_z}{\partial \phi}}_{=0}\hat{s} - \underbrace{\frac{\partial B_z}{\partial s}}_{=-\frac{1}{s^2}k}\hat{\phi}\right)$$

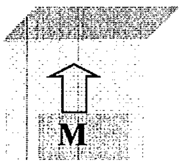
$$\boxed{\vec{J} = +\frac{1}{\mu_0}\frac{k}{s^2}\hat{\phi}}$$

Could also have used $\nabla^2\vec{A} = -\mu_0\vec{J} \rightarrow \vec{J} = -\frac{1}{\mu_0}\nabla^2\vec{A}$

but this seemed easier than dealing with $\nabla^2\vec{A}$

(16 pts) **Problem 4**. A cube of side a is centered on the origin. It is made out of a ferromagnetic material with a permanent magnetization, $\mathbf{M} = M_0\hat{\mathbf{z}}$. (a) What bound currents are implied by this magnetization? Make a sketch and calculate the magnitude(s). (b) Approximately what is the magnetic field produced by this cube at a point, (x, y, z), a large distance away from the cube? You can give your answer in terms of the usual spherical coordinates r, θ, and ϕ.

• (x, y, z)



(a) bound currents:

$\vec{J}_B = \nabla \times \vec{M} = \underline{\underline{0}}$ since $\vec{M}$ is constant

$\vec{K}_B = \vec{M} \times \hat{n}$. This $= 0$ for top and bottom sides.

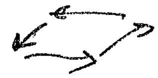
for side facing us, right hand rule gives $\vec{K}_B$ to right

for side on right, RHR gives $\vec{K}_B$ into the page.

for back side, RHR gives $\vec{K}_B$ to left

for left side, RHR gives $\vec{K}_B$ out of page

$\therefore$ K_B wraps around sides like this:



Magnitude is $\boxed{K_B = M_0}$

(b) dipole approximation

$\vec{m} = \int \vec{M}\, d\tau$

$\vec{m} = M_0 a^3 \hat{z}$ since M is constant

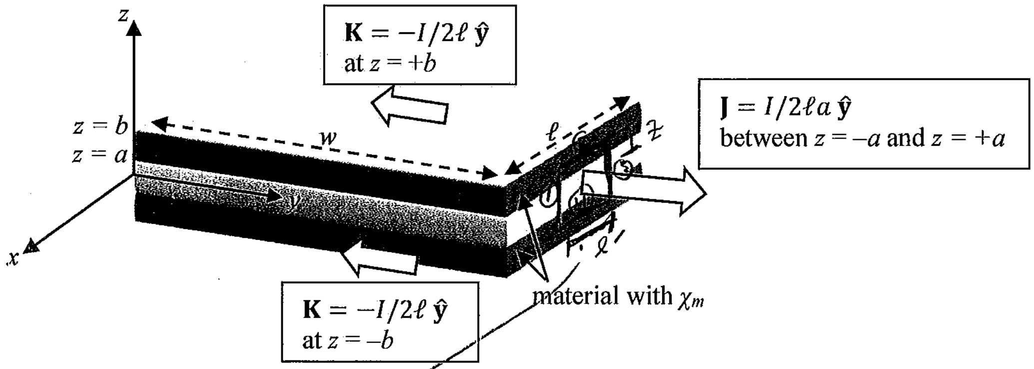
Then $\vec{B} = \dfrac{\mu_0}{4\pi} \dfrac{m}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$ for dipole field (θ & $\hat{\theta}$ measured relative to $\vec{m}$)

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{M_0 a^3}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

where θ and $\hat{\theta}$ are now normal spherical angles, since $\vec{m}$ was in $\hat{z}$ direction

(16 pts) **Problem 5**. Consider a large compound slab (dimensions $w \times \ell \times 2b$, with w and ℓ both much greater than b) consisting of an inner conductor of thickness $2a$ carrying a uniform current (current density $J = I/2\ell a$) in the +y direction, and outer conducting shells located at $z = \pm b$ carrying uniform currents (surface current densities $K = I/2\ell$) in the –y direction. That is, the total current flowing in the +y direction, I, is equal and opposite to the total current flowing in the –y direction. The space between the conductors ($a < z < b$ and $-b < z < -a$). is filled with a linear paramagnetic material with magnetic susceptibility χ_m.

Determine **H**, **B**, and **M** (magnitude and direction) in the region between the inner and upper conductors, i.e. for $a < z < b$. Since the slab is very very thin, you may use the "infinite slab" assumption.



Use Ampere's Law for $\vec{H}$ with this loop

$$\oint \vec{H} \cdot d\vec{\ell} = I_{free\ enc}$$

$$\cancel{\int_1} + \int_2 + \cancel{\int_3} + \int_4$$

By symmetry H will be in x-direction, so integrals 1 and 3 = 0 ($d\vec{\ell}$ is ⊥ to $\hat{x}$)

By symmetry integrals 2 and 4 will have same result, $= H\ell'$

$$I_{free\ enc} = J \cdot area = \frac{I}{2\ell a} 2z\ell'$$

Ampere's Law becomes

$$H \cdot \cancel{\ell'} + H\cancel{\ell'} = \frac{I}{2\ell a} 2z\cancel{\ell'}$$

$$\boxed{\vec{H} = \frac{Iz}{2\ell a}\hat{x}}$$

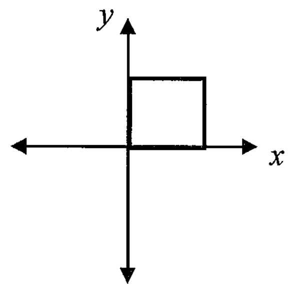
($\hat{x}$ direction for z above x-y plane, would by $-\hat{x}$ for z below plane)

Then $\vec{M} = \chi_m \vec{H}$

$$\boxed{\vec{M} = \frac{Iz\chi_m}{2\ell a}\hat{x}}$$

and $\vec{B} = \mu_0 \mu_r \vec{H} = \mu_0(1+\chi_m)\vec{H} \longrightarrow \boxed{\vec{B} = \mu_0(1+\chi_m)\frac{Iz}{2\ell a}\hat{x}}$

(14 pts) **Problem 6.** A square loop of wire, with sides of length a, lies in the first quadrant of the x-y plane as shown. (The z-direction is out of the page.) In this region there is a non-uniform time-dependent magnetic field $\mathbf{B}(x,t) = kx^2t^3\hat{\mathbf{z}}$ (where k is a positive constant).



(a) Find the magnitude of the EMF induced in the loop.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's Law)}$$

$$\Phi_B = \int \vec{B}\cdot d\vec{a}$$

$$= \int kx^2t^3\,dx\,dy$$

$$= kt^3 \underbrace{\int_0^a dy}_{a} \underbrace{\int_0^a x^2\,dx}_{\frac{1}{3}a^3}$$

$$= \frac{kt^3}{3}a^4$$

$$\mathcal{E} = -\frac{d}{dt}\left(\frac{kt^3}{3}a^4\right)$$

$$\mathcal{E} = -kt^2a^4$$

magnitude $\boxed{\mathcal{E} = kt^2a^4}$

(b) In what direction will the induced current flow (CW vs. CCW)? Justify your answer.

$\boxed{\text{Direction = clockwise}}$

Justify 1: the answer to (a) was negative which via RHR means clockwise.

Justify 2: via Lenz's Law, the current will set up as to oppose the changing flux. In this case the flux is out of the page, and increasing, so to oppose that the current must be clockwise.