Physics 441 Final Exam - due Thurs 12/15/16, 5 pm

Rules/Guidance:

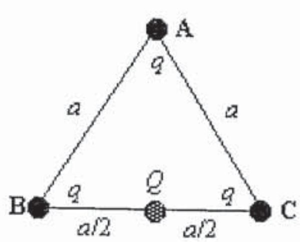
- The exam is completely open notes/books. You may use the textbook, other textbooks, your own class notes, websites, etc.
- You may *not* communicate with other people about the exam (classmates, classmates' notes, other current or past Physics Department students, relatives, internet forums or chat rooms, Facebook, etc.).
- If the wording of any of the exam problems seems unclear, please talk to me and I will clarify what is meant.
- Feel free to ask me or Spencer any questions about homework, exam, or in-class worked problems. But limit it to actual problems we've already done, rather than hypothetical problems that might be similar to the exam problems.
- Please work neatly and start each problem on a new page.
- The exam is out of **140 total points.**
- Please turn in this printed out exam along with your work.

Name Solutions

Additional Instructions: Please label & circle/box your answers. **Show your work,** where appropriate! And remember: **in any problems involving Gauss's or Ampere's Law, you should explicitly show your Gaussian surface/Amperian loop.**

(18 pts) **Problem 1:** Multiple choice, 2 pts each. Circle the correct answer.

1.1. The figure shows an equilateral triangle ABC. A positive point charge $+q$ is located at each of the three vertices A, B, and C. Each side of the triangle is of length a. A positive point charge Q is placed at the mid-point between B and C.



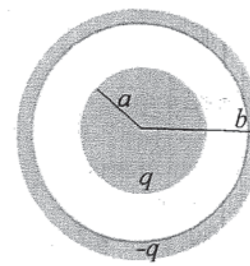
What is the initial direction the point charge Q will move once initially placed?

A) ↑ B) ↗ C) ↓

D) → E) ↖ F) ↙

Force from A is down
Force from B is right } these cancel out
Force from C is left

1.2. A conducting sphere of radius a with total charge q is surrounded by a spherical shell of inner radius b and total charge $-q$. What is the electrostatic potential energy of the system?



(a) $\frac{1}{4\pi\epsilon_0}\int_0^\infty \frac{q}{r^2}\,dr$
(b) $\frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}\,dr$
(c) $\frac{1}{8\pi\epsilon_0}\int_a^b \frac{q^2}{r^2}\,dr$
(d) $\frac{1}{32\pi^2\epsilon_0}\int_0^\infty \frac{q^2}{r^4}\,dr$
(e) $\frac{1}{32\pi^2\epsilon_0}\int_a^b \frac{q^2}{r^4}\,dr$
(f) 0

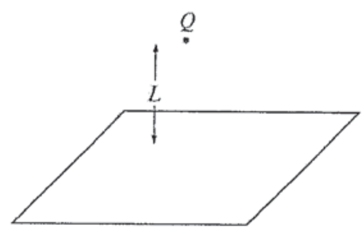
$$U = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2\, d\tau$$

$$= \frac{\epsilon_0}{2}\int_a^b \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2 \cdot 4\pi r^2\, dr$$

$$= \frac{\cancel{\epsilon_0}}{2}\cdot\frac{1}{(4\pi)^{\cancel{2}}\epsilon_0^{\cancel{2}}}\,\cancel{4\pi}\int_a^b \frac{q^2}{r^2}\,dr$$

1.3. Which of the following is true of Laplace's equation?
(a) It can have more than one solution for a given set of boundary conditions. Not true (uniqueness)
(b) The solutions in one dimension must be sines and cosines (or a linear combination). Not true (in 1D you get $f = Ax + B$)
(c) It is only valid in regions of space that contain no charges. ✓ $\nabla^2 V = -\rho/\epsilon_0$ otherwise
(d) It requires that nowhere within the region of interest can the potential be zero Not true; if you have $+V_0$ on one boundary and $-V_0$ opposite, it'll go to zero somewhere in between
(e) More than one of the above. Nope!

1.4. A positive charge Q is located at a distance L above an infinite grounded conducting plane:



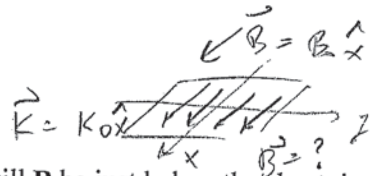
What is the total charge that will be induced on the plane?

(a) $2Q$
(b) Q
(c) 0
(d) $-Q$ We worked this out in class
(e) $-2Q$

1.5. A localized charge distribution has no net charge, zero dipole moment, but a nonzero quadrupole moment. At a large distance r from the distribution, the electric potential will fall off like:
(a) $1/r$
(b) $1/r^2$
(c) $1/r^3$
(d) $1/r^4$
(e) $1/r^8$

monopole $V \sim \frac{1}{r}$
dipole $V \sim \frac{1}{r^2}$
quadrupole $V \sim \frac{1}{r^3}$

1.6. A sheet of current with surface current density $\mathbf{K} = \boldsymbol{K_0}\,\hat{\mathbf{x}}$ is in the x-y plane. Just above the sheet, i.e., when z = a very small positive number, the magnetic field $\mathbf{B} = \boldsymbol{B_0}\hat{\mathbf{x}}$. K_0 and B_0 are both positive



constants. In what direction will **B** be just below the sheet, i..e., when z = a very small negative number?

(a) +x
(b) –x
(c) +z
(d) -z
(e) +y
(f) –y
(g) Some other direction
(h) It cannot be determined from the information given

BC 1) $B_{\perp 1} = B_{\perp 2}$ $\longrightarrow B_{2z} = 0$

BC 2) $B_{\parallel 1} - B_{\parallel 2} = \mu_0 K_{\parallel}$ where "$\parallel$" means $\parallel$ to surface but $\perp$ to $\vec{K}$. That's y-direction $\Big\}$ $B_{2y} = 0 - \mu_0 K_0$

I think if you're careful this turns out to be a + sign

BC 3) (From book, not discussed much in class) $B_{\parallel 1} = B_{\parallel 2}$ where $\parallel$ = $\parallel$ to surface and $\parallel$ to K $\Big\}$ $B_{2x} = B_{1x} = B_0$

$\therefore \vec{B}_2$ has components in both x and y!

1.7. An infinite cylinder has a permanent magnetization M_0 in the z-direction (along the axis of the cylinder). Which of the following are true about the fields inside the cylinder?

no I_{free} enclosed
(a) The **B** field is zero; the **H** field is non-zero
Amp Law (b) The **B** field is non-zero; the **H** field is zero
$\rightarrow H = 0$ (c) Both **B** and **H** fields are zero
(d) Both **B** and **H** fields are non-zero, and pointing in the same direction
(e) Both **B** and **H** fields are non-zero, but pointing in opposite directions

And since $B = \mu_0(H + M)$, $B \neq 0$

1.8. A table of resistivities is given. What would be the approximate resistance of a 10 m long section of 32 gauge (0.2 mm diameter) copper wire?

(a) 0.01 Ω
(b) 0.05 Ω
(c) 0.2 Ω
(d) 1 Ω
(e) 5 Ω
(f) 20 Ω

$R = \rho \frac{l}{A}$

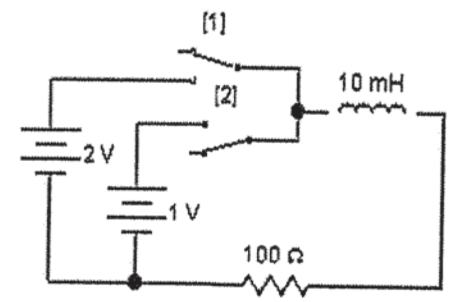
$= \frac{(1.68 \cdot 10^{-8})(10)}{\pi (.1 \cdot 10^{-3})^2}$

$= 5.3 \Omega$

Material	Resistivity	Material	Resistivity
Conductors:		*Semiconductors:*	
Silver	1.59×10^{-8}	Sea water	0.2
Copper	1.68×10^{-8}	Germanium	0.46
Gold	2.21×10^{-8}	Diamond	2.7
Aluminum	2.65×10^{-8}	Silicon	2500
Iron	9.61×10^{-8}	*Insulators:*	
Mercury	9.61×10^{-7}	Water (pure)	8.3×10^{3}
Nichrome	1.08×10^{-6}	Glass	$10^{9} - 10^{14}$
Manganese	1.44×10^{-6}	Rubber	$10^{13} - 10^{15}$
Graphite	1.6×10^{-5}	Teflon	$10^{22} - 10^{24}$

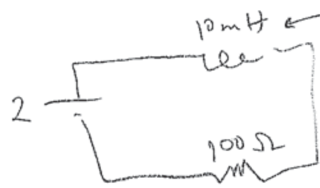
TABLE 7.1 Resistivities, in ohm-meters (all values are for 1 atm, 20° C). *Data from Handbook of Chemistry and Physics*, 91st ed. (Boca Raton, Fla.: CRC Press, 2010) and other references.

1.9. Switch 2 is closed and the system comes to equilibrium. Then, switch 2 is opened while switch 1 is closed simultaneously, and the system comes to another equilibrium. What is the final current through the inductor?



(a) 0.01 A
(b) 0.02 A
(c) 0.03 A
(d) 0.04 A
(e) 0.05 A
(f) 0.06 A

First equilibrium doesn't really matter. Second circuit is

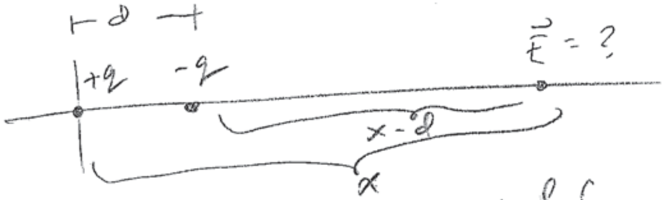


Long time, acts like wire

current is $\frac{V}{R} = \frac{2}{100} = .02$ A

Worked problems – please write on your own paper, no more than one problem per page.

(10 pts) **Problem 2.** Two point charges of equal but opposite charge (i.e. q and $-q$) are separated by a distance d, the positive charge being on the left and the negative one on the right. Find the electric potential along the line connecting the two charges as a function of distance from the positive charge, x.



it's unclear if x can be negative. I'll assume neg x = left of q

V = potential from q + potential from $-q$

$$\boxed{V(x) = \frac{1}{4\pi\epsilon_0}\frac{q}{x} - \frac{1}{4\pi\epsilon_0}\frac{q}{|x-d|}}$$

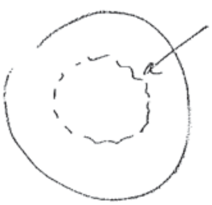
I used absolute value so that the formula works when x = negative

(13 pts) **Problem 3.** A sphere (radius R) has a volume charge density which increases with the distance from the center of the sphere: $\rho = k\,r$. (a) What are the units of k? (b) Determine the electric field in terms of k and r for points inside and outside the sphere. (c) Determine the electric potential for points inside the sphere, using the normal convention that $V(r=\infty)=0$.

(a) charge density is units of C/m^3, so we have $C/m^3 = [k]\cdot m$

$$\boxed{[k] = C/m^4}$$

(b)



Gaussian surface for $r<R$

$$\oint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

$q_{enc} = \int \rho \, d\tau$

$= \int_0^r kr \cdot 4\pi r^2 dr$

$= 4\pi k \cdot \frac{1}{4} r^4$

$= \pi k r^4$

E = constant, $\vec{E}$ parallel to $d\vec{a}$

$$\rightarrow E\cdot 4\cancel{\pi} r^2 = \frac{\cancel{\pi} k r^4}{\epsilon_0}$$

$$\boxed{\vec{E} = \frac{k r^2}{4\epsilon_0}\hat{r}}$$

Gaussian surface for $r>R$

only difference is q_{enc} integral is 0 → R

$\rightarrow q_{enc} = \pi k R^4$

$$\oint \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$$

$$E\, 4\cancel{\pi} r^2 = \frac{\cancel{\pi} k R^4}{\epsilon_0}$$

$$\boxed{\vec{E} = \frac{kR^4}{4\epsilon_0}\frac{1}{r^2}\hat{r}}$$

(c) $V = -\int \vec{E}\cdot d\vec{l}$

for pts inside the sphere

$$V = -\int_\infty^R \vec{E}_{outside}\cdot d\vec{r} + -\int_R^r \vec{E}_{inside}\cdot d\vec{r}$$

$$= -\int_\infty^R \frac{kR^4}{4\epsilon_0}\frac{1}{r^2}dr \quad - \int_R^r \frac{kr^2}{4\epsilon_0}dr$$

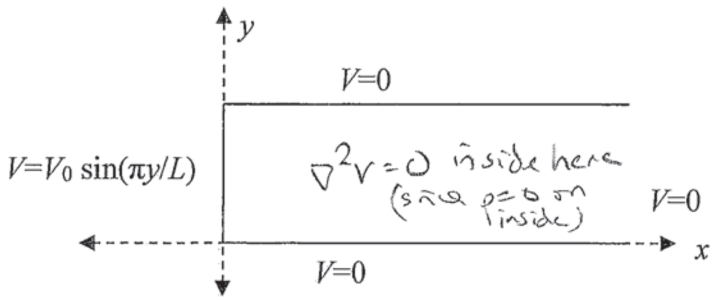
$$= \frac{kR^{\cancel{4}\,3}}{4\epsilon_0}\frac{1}{\cancel{R}} \quad - \frac{k}{4\epsilon_0}\frac{1}{3}(r^3 - R^3)$$

$$\boxed{V(r) = \frac{kR^3}{4\epsilon_0} + \frac{k}{12\epsilon_0}(R^3 - r^3)}$$

or since $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$

could say $\boxed{= \frac{kR^3}{3\epsilon_0} - \frac{k}{12\epsilon_0}r^3}$

(13 pts) **Problem 4.** The figure below extends infinitely in the + and – z-directions (not shown), and in the + x-direction. The potential is held fixed along the sides as indicated: the upper and lower sides are held at 0, whereas the left-hand side is help at a potential of $V_0\sin(\frac{\pi y}{L})$, where V_0 is a constant and L is the length of the object in the y-direction. Find the potential $V(x,y)$ everywhere inside the boundary. Then use Mathematica or similar program to make a 3D plot of the potential (set V_0 and L equal to 1).



Boundary conditions
1. V = 0 at $y = 0$
2. V = 0 at $y = L$
3. V = 0 at x = ∞
4. V = $V_0\sin(\frac{\pi y}{L})$ at $x = 0$

Laplace eqn $\nabla^2 V = 0$

Sep. of vars. Guess $V = X(x)Y(y)$

plug in $X''Y + XY'' = 0$

divide by XY $\frac{X''}{X} + \frac{Y''}{Y} = 0$

$\frac{X''}{X} = -\frac{Y''}{Y}$

Each side must equal a constant, call it k^2.

Due to symmetry of problem I know X must be exponential, Y must be sines/cosines

$X'' = k^2 X \rightarrow X = e^{-kx}$ or e^{kx}

$Y'' = -k^2 Y \rightarrow Y = \sin ky$ or $\cos ky$

BC 3: rule out e^{kx} terms

BC 1: rule out cosine terms

BC 2: k must be $\frac{n\pi}{L}$

piece together, use linear combinations

$V = \sum_n C_n e^{-\frac{n\pi x}{L}} \sin\frac{n\pi y}{L}$

BC 4: $V_0 \sin\frac{\pi y}{L} = \sum_n C_n e^0 \sin\frac{n\pi y}{L}$

Equate like terms $\rightarrow$ only $n=1$ survives! and $C_1 = V_0$

Final answer $\boxed{V(x,y) = V_0 e^{-\frac{n\pi x}{L}} \sin\frac{n\pi y}{L}}$

plotted on next page

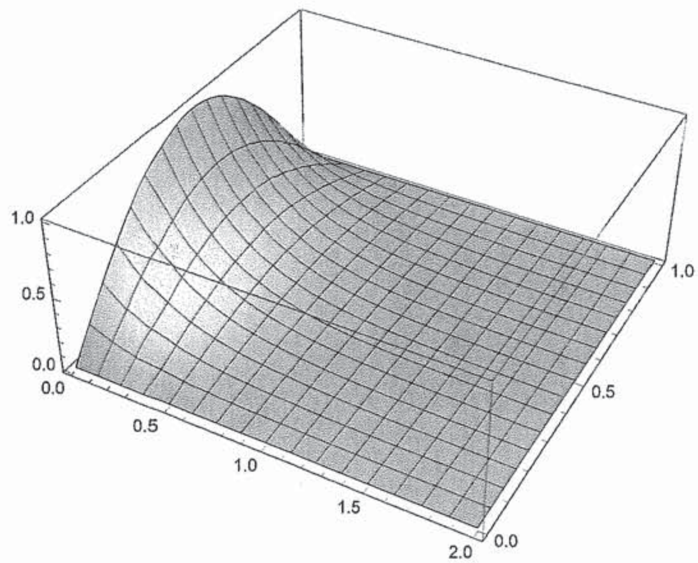
Problem 4 Plot

In[1]:= (* for V0 = 1, L = 1 *)

```
V[x_, y_] = Exp[-Pi x] Sin[Pi y]
Plot3D[V[x, y], {x, 0, 2}, {y, 0, 1}, PlotRange → All]
```

Out[1]= $e^{-\pi x}$ Sin[π y]

Out[2]=



(15 pts) **Problem 5**. (a) A molecule with a polarizability α is placed in an electric field and polarizes. If the field has the functional form, $\mathbf{E} = 5x\hat{\mathbf{x}} + 2z^3\hat{\mathbf{z}}$ and the molecule is located at the point (1, 2, 3) (ignore units for this problem) what are the force and torque on it? (b) What charge distribution $\rho(x, y, z)$ would give rise to such a field?

(a) $\vec{p} = \alpha \vec{E}$

$\vec{p} = \alpha\left(5 \cdot 1\hat{x} + 2\,3^3\hat{z}\right)$

$\vec{p} = \alpha\left(5\hat{x} + 54\hat{z}\right)$

Force : $\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$

$= p_x \frac{\partial \vec{E}}{\partial x} + p_y \frac{\partial \vec{E}}{\partial y} + p_z \frac{\partial \vec{E}}{\partial z}$

$= (5\alpha)(5\hat{x}) + 0 + (54\alpha)\left(6z^2\Big|_{\text{the point } (1,2,3)} \hat{z}\right)$

$\uparrow$ $6 \cdot 3^2$

$$\boxed{\vec{F} = 25\alpha\hat{x} + 2916\alpha\hat{z}}$$

Torque : $\vec{\tau} = \vec{p} \times \vec{E}$

$\uparrow$

$\vec{p}$ is in the $\vec{E}$ direction for this situation

so cross product is zero

$$\boxed{\vec{\tau} = 0}$$

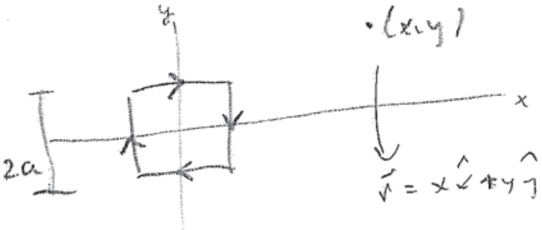
(b) From Gauss's Law

$\nabla \cdot \vec{E} = \rho/\epsilon_0 \longrightarrow \rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}$

$\rho = \epsilon_0\left(\frac{\partial E_x}{\partial x} + \cancel{\frac{\partial E_y}{\partial y}} + \frac{\partial E_z}{\partial z}\right)$

$$\boxed{\rho = \epsilon_0\left(5 + 6z^2\right)}$$

(15 pts) **Problem 6.** A current flows clockwise in a square loop of side length $2a$ that lies flat in the plane of this page. Write down several integral expressions that you could use to find the magnetic field at a point (x, y) that lies in the plane of the loop, where the distances x and y are measured from the center of the square. (You don't have to actually do the integrals.)



Biot-Savart Law: $\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l} \times \vec{\mathscr{r}}}{\mathscr{r}^3}$

For top section

$\vec{r}\,' = x'\hat{x} + a\hat{y}$

$\vec{\mathscr{r}} = (x-x')\hat{x} + (y-a)\hat{y}$

$\mathscr{r} = \left((x-x')^2 + (y-a)^2\right)^{1/2}$

$d\vec{l} = dx'\hat{x}$

$$\vec{B}_{top} = \frac{\mu_0 I}{4\pi}\int_{-a}^{a} \frac{dx'\hat{x} \times \left(\cancel{(x-x')\hat{x}} + (y-a)\hat{y}\right)}{\left((x-x')^2 + (y-a)^2\right)^{3/2}}$$

0 from cross product

$$\boxed{\vec{B}_{top} = \frac{\mu_0 I}{4\pi}(y-a)\hat{z}\int_{-a}^{a} \frac{dx'}{\left((x-x')^2 + (y-a)^2\right)^{3/2}}}$$

For bottom section

$\vec{r}\,' = x'\hat{x} - a\hat{y}$

$\vec{\mathscr{r}} = (x-x')\hat{x} + (y+a)\hat{y}$

$\mathscr{r} = \left((x-x')^2 + (y+a)^2\right)^{1/2}$

$d\vec{l} = dx'\hat{x}$

I'll take care of negative sign via limits of integration

$$\vec{B}_{bott} = \frac{\mu_0 I}{4\pi}\int_{a}^{-a} \frac{dx'\hat{x} \times \left(\cancel{(x-x')\hat{x}} + (y+a)\hat{y}\right)}{\left((x-x')^2 + (y+a)^2\right)^{3/2}}$$

0

$$\boxed{\vec{B}_{bottom} = \frac{\mu_0 I}{4\pi}(y+a)(-\hat{z})\int_{-a}^{a} \frac{dx'}{\left((x-x')^2 + (y+a)^2\right)^{3/2}}}$$

For left section

$\vec{r}\,' = -a\hat{x} + y'\hat{y}$

$\vec{\mathscr{r}} = (x+a)\hat{x} + (y-y')\hat{y}$

$\mathscr{r} = \left((x+a)^2 + (y-y')^2\right)^{1/2}$

$d\vec{l} = dy'\hat{y}$

$$\vec{B}_{left} = \frac{\mu_0 I}{4\pi}\int_{-a}^{a} \frac{dy'\hat{y} \times \left((x+a)\hat{x} + \cancel{(y-y')\hat{y}}\right)}{(\text{stuff})^{3/2}}$$

0

$$\boxed{\vec{B}_{left} = \frac{\mu_0 I}{4\pi}(x+a)(-\hat{z})\int_{-a}^{a} \frac{dy'}{\left((x+a)^2 + (y-y')^2\right)^{3/2}}}$$

For right section

$\vec{r}\,' = a\hat{x} + y'\hat{y}$

$\vec{\mathscr{r}} = (x-a)\hat{x} + (y-y')\hat{y}$

$\mathscr{r} = \left((x-a)^2 + (y-y')^2\right)^{1/2}$

$d\vec{l} = dy'\hat{y}$

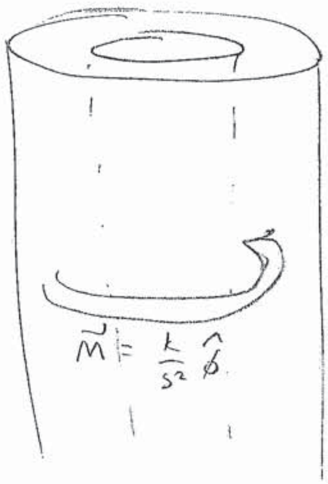
$$\vec{B}_{right} = \frac{\mu_0 I}{4\pi}\int_{a}^{-a} \frac{dy'\hat{y} \times \left((x-a)\hat{x} + \cancel{(y-y')\hat{y}}\right)}{(\text{stuff})^{3/2}}$$

0

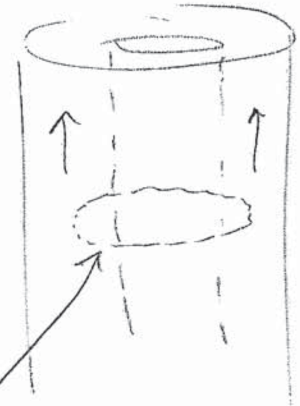
$$\boxed{\vec{B}_{right} = \frac{\mu_0 I}{4\pi}(y-a)\hat{z}\int_{-a}^{a} \frac{dy'}{\left((x-a)^2 + (y-y')^2\right)^{3/2}}}$$

then $\boxed{\vec{B}_{tot} = \vec{B}_{top} + \vec{B}_{bottom} + \vec{B}_{left} + \vec{B}_{right}}$

(10 pts) **Problem 7**. An infinitely long, thick cylindrical shell of inner radius a and outer radius b has a magnetization of $\mathbf{M} = \frac{k}{s^2}\,\hat{\boldsymbol{\phi}}$. Calculate the magnetic field in the region $a < s < b$ as a function of distance from the center of the cylinder.



this will have bound surface and/or volume currents in z-direction, like this (or maybe in $-\hat{z}$, doesn't matter)



The easy way

Use Amperian loop

Amp. Law for $\vec{H}$: $\oint \vec{H} \cdot d\vec{l} = I_{free\ enc} \rightarrow = 0$

proper symmetry, H is constant on the loop and $\vec{H} \parallel d\vec{l}$

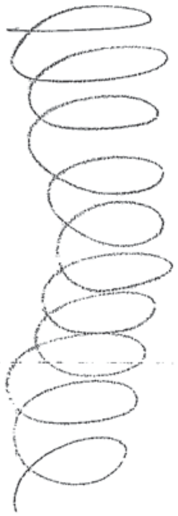
$$H \cdot 2\pi s = 0$$

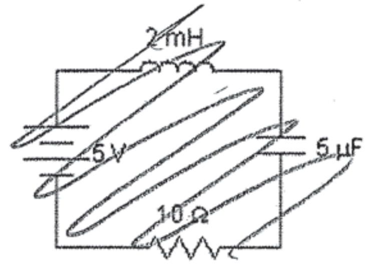
$$H = 0$$

Then $\vec{B} = \mu_0(\vec{H} + \vec{M}) \longrightarrow \vec{B} = \mu_0 \vec{M}$

$$\boxed{\vec{B} = \mu_0 \frac{k}{s^2}\hat{\phi}}$$

(12 pts) **Problem 8**. The current of an infinite solenoid (n turns/length) is increasing linearly according to $I = I_0 \frac{t}{\tau}$, where I_0 and τ are positive constants). Determine the induced electric field, both magnitude and direction.





(next problem)

Assuming "slowly varying current";

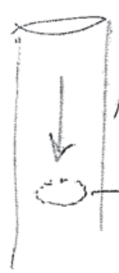
$\vec{B} = \mu_0 n I \hat{z}$

$\vec{B} = \mu_0 n I_0 \frac{t}{\tau} \hat{z}$ for the solenoid interior

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow -\frac{\partial B}{\partial t}$ acts like a "current density" to produce $\vec{E}$ in same way $\mu_0 \vec{J}$ produces $\vec{B}$ when $\nabla \times \vec{B} = \mu_0 \vec{J}$

$-\frac{\partial \vec{B}}{\partial t} = \underbrace{-\mu_0 n \frac{I_0}{\tau} \hat{z}}$ ↳ acts to create $\vec{E}$ in a current sort of way

So it's like a $\vec{B}$ field problem like this



$\mu_0 \vec{J}$ = const, what is $\vec{B}$?

$\longrightarrow$ use Ampere's law analog

$\oint \vec{B} \cdot d\vec{l} = \mu_0 J_{enc}$ becomes $\oint \vec{E} \cdot d\vec{l} = \int \left(\frac{\mu_0 n I_0}{\tau} \right) da$

$= \int \mu_0 J \, da$

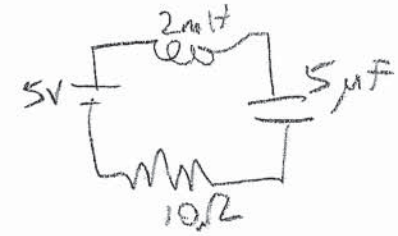
$E \, 2\pi s = \left(\frac{\mu_0 n I_0}{\tau} \right) \pi s^2$

$\boxed{E = \frac{\mu_0 n I_0}{2\tau} s}$ magnitude

direction is $\boxed{-\hat{\phi}}$ from right hand rule and "current" in $-\hat{z}$ direction

(19 pts) **Problem 9**. There is no initial current in the displayed RLC circuit, but the capacitor has an initial voltage of 2 V (the side on the top in the diagram has the positive charge). The battery is connected and switched on at $t = 0$. Find the voltage of the capacitor thereafter as a function of time.

which means relative to the final charge state, $V_{cap}(t=\infty) = +2$



$\alpha = \frac{R}{2L} = \frac{10}{.004} = 2500$

$w_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{.002 \cdot 5 \cdot 10^{-6}}} = 10000$

Since $w_0 > \alpha$, it's underdamped. Use formula from handout

$V_{cap,\ homogeneous} = K_1 e^{-\alpha t} \cos \underbrace{\sqrt{w_0^2 - \alpha^2}}\, t + K_2 e^{-\alpha t} \sin \sqrt{w_0^2 - \alpha^2}\, t$

$\sqrt{w_0^2 - \alpha^2} = 9682$

$V_{homog.} = K_1 e^{-2500t} \cos 9682t + K_2 e^{-2500t} \sin 9682t$

Hold that thought. Now for particular soln

Diff eqn becomes: $\frac{d^2 V_{cap}}{dt^2} + 2\alpha \frac{dV_{cap}}{dt} + w_0^2 V_{cap} = w_0^2 V_{battery}$

Guess $V_{cap} = V_0$ (a constant)

$0 + 0 + \cancel{w_0^2} \times V_0 = \cancel{w_0^2} V_{battery}$

$V_0 = V_{battery}$

$\underline{(= 5V)}$

Combined that gives

$$\boxed{V = 5 + K_1 e^{-2500t} \cos 9682t + K_2 e^{-2500t} \sin 9682t}$$

Now K_1 and K_2 must be chosen to satisfy initial conditions problem 9 cont.

init cond 1 : $V(t=0) = +2$

init cond 2 : $\frac{dv}{dt}(t=0) = 0$ (since no initial current and $I = \frac{d\varphi}{dt} = C\frac{dv}{dt}$)

#1: $+2 = 5 + k_1 \cos 0 + \cancel{k_2 \sin 0}$

$\underline{\underline{k_1 = -3}}$

#2: Derivative done with Mathematica, see attached

Results in

$0 = -2500 k_1 + 9682 k_2$

$k_2 = \frac{2500}{9682} k_1 = \underline{\underline{-.7746}}$

Final answer

$$\boxed{V_{cap} = 5 - 3e^{-2500t}\cos 9682t - 0.7746e^{-2500t}\sin 9682t}$$

I've plotted that on Mathematica as well (not required),

see next page.

problem 9) cont

In[1]:=
```
v[t_] = 5 + k1 Exp[-2500 t] Cos[9682 t] + k2 Exp[-2500 t] Sin[9682 t] ;
v'[t] /. t → 0
```

Derivative at $t = 0$

Out[2]= $-2500\,k1 + 9682\,k2$

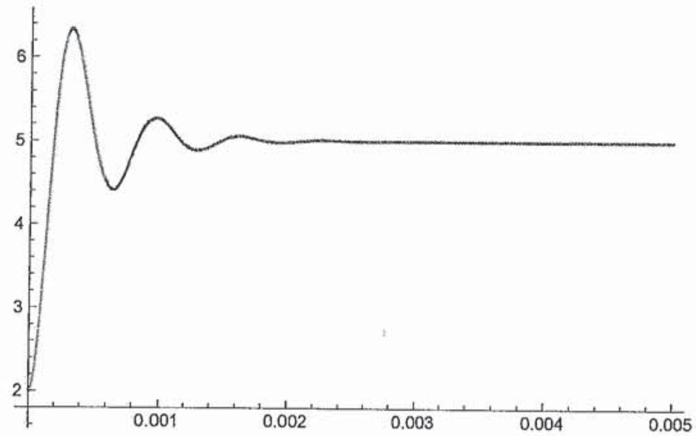
In[3]:=
```
v2[t_] = k1 Exp[-2500 t] Cos[9682 t] + k2 Exp[-2500 t] Sin[9682 t] + 5 /.
  {k1 → -3, k2 → -0.7746}
```

final answer

Out[3]= $5 - 3\,e^{-2500\,t}\,\mathrm{Cos}[9682\,t] - 0.7746\,e^{-2500\,t}\,\mathrm{Sin}[9682\,t]$

In[4]:=
```
Plot[v2[t], {t, 0, .005}, PlotRange → All]
```

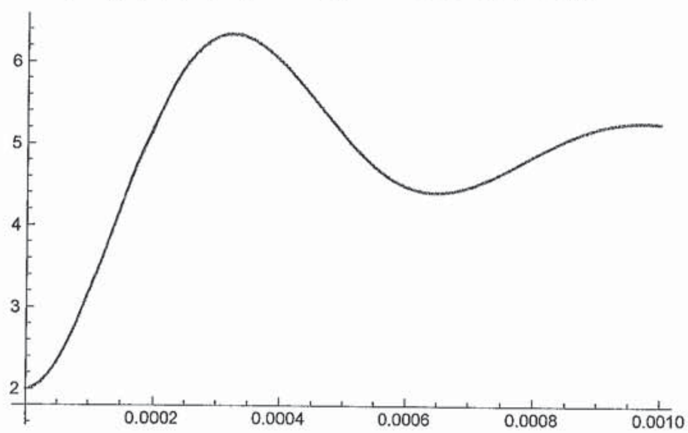
Out[4]=



In[5]:=
```
(* Look how it goes to 5V at t = infinity! *)
```

In[6]:=
```
Plot[v2[t], {t, 0, .001}, PlotRange → All]
```

Out[6]=



In[7]:=
```
(* Look how it satisfies the initial conditions! *)
```

(15 pts) **Problem 10**. As we talked about many times in class, the quantity $1/\sqrt{\epsilon_0\mu_0}$ equals c, the speed of light in a vacuum. We have the tools now to derive that relationship using the Maxwell Equations!

The 1-dimensional "wave equation" is a well known partial differential equation describing waves moving at a velocity v:

$$\frac{\partial^2 f}{dx^2} = \frac{1}{v^2}\frac{\partial^2 f}{dt^2}$$

It's called the wave equation because a basic traveling sine wave $f = A\sin(kx-\omega t)$,* is in fact a solution of the equation as can be seen by taking two spatial derivatives, two time derivatives, and plugging them into the equation:

$$-k^2 A\sin(kx-\omega t) = \frac{1}{v^2}\left(-\omega^2 A\sin(kx-\omega t)\right)$$
$$k^2 = \frac{1}{v^2}(\omega^2)$$

So the wave equation is true for that function, as long as the wave speed $v = \omega/k$.

A very similar equation arises directly from the Maxwell equations, which is the point of this problem.

a. Suppose you have electric and magnetic fields in a vacuum (i.e., no charge/current densities). Write down the 4 Maxwell equations for this case.
b. Show that the **E** field can be decoupled from the **B** field using Vector Identity 11 from the front cover, giving you a single equation for **E**. (The same can be done for **B**.) The equation for **E** that you end up with should be a three dimensional version of the wave equation (a ∇^2 instead of a d^2/dx^2).
c. Show that the wave speed v that you end up is indeed $1/\sqrt{\epsilon_0\mu_0}$. This is still amazing to me, and a central part of the "magic" of Maxwell's equations! Maxwell's equations were derived/discovered by looking at forces between charges and currents—yet this equation describes a traveling electromagnetic wave moving precisely at the measured speed of light!
d. Now suppose you have a dielectric material (linear, isotropic) which has no free charge nor free current, where $\mu_r = 1$, but where ϵ_r is not just equal to 1 anymore. Write out the "in matter" Maxwell's equations for this case. Use Ampere's Law for **H** to determine what the curl of **B** equals, in terms of the **E** field and other given quantities.
e. Repeat steps (b) and (c) to get the wave equation for **E** again. What velocity do you obtain in this case? I taught my Phys 123 students that inside materials the speed of light $v = c/n$, where n is the index of refraction—what does n turn out to be, in terms of the information given in the problem?

(a) Maxwell eqns

$\nabla\cdot\vec{E} = \rho/\epsilon_0$

$\nabla\times\vec{E} = -\partial\vec{B}/\partial t$

$\nabla\cdot\vec{B} = 0$

$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

If ρ and $\vec{J} = 0$

$$\boxed{\begin{aligned}\nabla\cdot\vec{E} &= 0\\ \nabla\times\vec{E} &= -\partial\vec{B}/\partial t\\ \nabla\cdot\vec{B} &= 0\\ \nabla\times\vec{B} &= \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\end{aligned}}$$

* In case it's not obvious, this is a "traveling sine wave" because if you look at it at successive times, the peaks of the sine wave move to the right at a certain speed. Actually, sinusoidal waves are not the only solutions to the wave equation—traveling waves of any shape will solve the equation. But that's beyond what I care about here.

10) cont

(b) Vector identity 11: $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 A$

Applied to $\vec{E}$ field:

$$\nabla \times \underbrace{(\nabla \times \vec{E})}_{= -\frac{\partial \vec{B}}{\partial t}} = \nabla(\underbrace{\nabla \cdot \vec{E}}_{=0}) - \nabla^2 E$$

$$-\nabla \times \left(\frac{\partial \vec{B}}{\partial t}\right) = -\nabla^2 \vec{E}$$

$$\frac{\partial}{\partial t}(\nabla \times \vec{B}) = \nabla^2 E$$

$$\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = \nabla^2 \vec{E}$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$ Wave eqn!

(c) Speed of waves given by: $\frac{1}{v^2} = \mu_0 \epsilon_0$

$$\boxed{v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}} = c!$$

(d) Maxwell Eqns in matter

$$\nabla \cdot \vec{D} = \rho_{free}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$$

$\Rightarrow$

If ρ_{free} and $\vec{J}_{free} = 0$
and if $\vec{D} = \epsilon_0 \epsilon_r \vec{E}$ (linear, isotropic)
and if $\vec{H} = \frac{1}{\mu_0}\vec{B}$ (non magn.)

$\nabla \cdot \vec{D} = 0 \longrightarrow$

$\nabla \times \left(\frac{1}{\mu_0}\vec{B}\right) = \epsilon_0 \epsilon_r \frac{\partial \vec{E}}{\partial t} \longrightarrow$

$$\boxed{\begin{aligned} \nabla \cdot \vec{E} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{B} &= \epsilon_0 \mu_0 \epsilon_r \frac{\partial \vec{E}}{\partial t} \end{aligned}}$$

(e) $$\nabla \times \underbrace{(\nabla \times \vec{E})}_{-\frac{\partial \vec{B}}{\partial t}} = \nabla(\cancel{\nabla \cdot \vec{E}}^{\,0}) - \nabla^2 E$$

$$\frac{\partial}{\partial t}\underbrace{(\nabla \times \vec{B})}_{\epsilon_0 \mu_0 \epsilon_r \frac{\partial \vec{E}}{\partial t}} = \nabla^2 E$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \epsilon_r \frac{\partial^2 \vec{E}}{\partial t^2}}$$

speed $v = \frac{1}{\sqrt{\mu_0 \epsilon_0 \epsilon_r}}$

$$\boxed{v = \frac{c}{\sqrt{\epsilon_r}}} \longrightarrow \boxed{n = \sqrt{\epsilon_r}}$$

index of refraction = sqrt of dielectric constant!