Physics 441 Final Exam - due Wed 12/20/17, 5 pm
(extensions to Thursday 12/21/17, 5 pm, are possible if you talk to me in person)

Rules/Guidance:

- The exam is completely open notes/books. You may use the textbook, other textbooks, your own class notes, Wikipedia, the results of Google searches, other websites, etc.
- You may *not* communicate with other people about the exam (classmates, classmates' notes, other current or past Physics Department students, relatives, internet forums or chat rooms, Facebook, etc.).
- If the wording of any of the exam problems seems unclear, please talk to me and I will clarify what is meant.
- Feel free to ask me any questions about homework, exam, or in-class worked problems. But limit it to actual problems we've already done, rather than hypothetical problems that might be similar to the exam problems.
- Please work neatly and start each problem on a new page.
- Please turn in this printed out exam along with your work.
- The exam is out of ~~**150**~~ **total points**. 145 → Due to issue with problem 10, I changed the number of points
- My best guess is that **a well-rested, well-prepared student should take about 4.5 hours** on the exam, not including the extra credit problem. Of course, if you are not yet well-rested and well-prepared, then add in additional time as appropriate—perhaps more-or-less equivalent to what you would have spent studying for a Testing Center exam. Plus sleep time. ☺

Name Solutions

Additional Instructions: Please label & circle/box your answers. **Show your work**, where appropriate! And remember: **in any problems involving Gauss's or Ampere's Law, you should explicitly show your Gaussian surface/Amperian loop.**

(22 pts) **Problem 1:** Multiple choice and short answer, 2 pts each. Circle the correct answers for the multiple choice questions; write the short answers in the space provided.

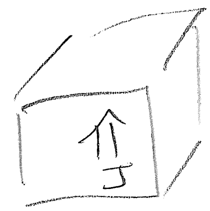
1.1. In the temperature analogy I used in class, if V is like temperature, than **E** is like:
  (a) Constant temperature contours
  (b) Heat capacity
  (c) Heat flow from hot to cold $\vec{E}$ field lines point from high V to low V
  (d) Heat flow from cold to hot
  (e) Specific heat
  (f) Temperature maxima
  (g) Temperature minima

1.2. True/False: Ferroelectric materials that have a polarization field that is constant in space (i.e. same direction, same magnitude everywhere in the material) will only have bound surface charges, not bound volume charges.
  (a) True $\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$ if $\vec{P}$ is constant in space
  (b) False

1.3. A localized charge distribution has no net charge, zero electric dipole moment, but a nonzero electric quadrupole moment. At a large distance r from the distribution, the electric field will fall off like:
(a) No fall off; it's constant
(b) $1/r$
(c) $1/r^2$
(d) $1/r^3$
(e) $1/r^4$
(f) $1/r^5$

monopole $E \sim \frac{1}{r^2}$
dipole $E \sim \frac{1}{r^3}$
quadrupole $E \sim \frac{1}{r^4}$

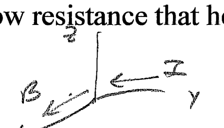
1.4. What would be the best way to calculate the magnetic field produced by a cube of current (side length a, constant current density $\mathbf{J} = J_0\hat{\mathbf{z}}$)?
(a) Coulomb's law
(b) Gauss's law
(c) Gauss's law for **D**
(d) Multipole expansion for V
(e) Images charges
(f) Bound charges
(g) Biot-Savart law
(h) Ampere's law
(i) Ampere's law for **H**
(j) Multipole expansion for **A**
(k) Image currents
(l) Bound currents
(m) Faraday's law
(n) Maxwell's fix to Ampere's law



$\vec{B} = \frac{\mu_0}{4\pi} \int_{cube} \frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3} d\tau$

is only thing that would work

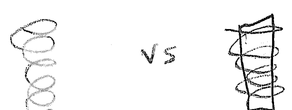
1.5. Gravitational force is in the $-\hat{\mathbf{z}}$ direction. A magnetic field goes in the $\hat{\mathbf{x}}$ direction. A wire lies on a table and a current is run through it in $-\hat{\mathbf{y}}$ direction. True or False: if a sufficiently large current were run through the wire, it could levitate, i.e. overcome the effects of gravity and spring off the table. (Assume that the wire has sufficiently low resistance that heating would not be a limiting factor.)
(a) True
(b) False



$\vec{F} = I \vec{l} \times \vec{B}$ ~ in $\hat{z}$ direction

1.6. A magnetic field is created by running current through a solenoid. A second magnetic field is created by running current through another solenoid, identical (same current, same dimensions) except it has a core made of diamagnetic material. In which case is there more energy stored in the field?
(c) First field
(d) Second field

negative χ_m



$U = \frac{1}{2}\int \vec{B}\cdot\vec{H}\, d\tau$

Same H (set by current in solenoid). Which has bigger $\vec{B}$?

$\vec{B} = \mu_0(1+\chi_m)\vec{H}$

if χ_m is negative, $\vec{B}$ is reduced

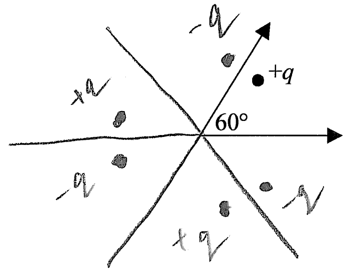
1.7. True/False: In the Coulomb gauge the formula $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\,d\tau'}{\mathscr{r}}$ can always be used to find the vector potential **A** for a given current distribution, **J**.
(a) True
(b) False

Not for infinite distributions

1.8. (Short answer) What is the divergence of $\frac{\hat{\mathbf{r}}}{r^2}$? (in spherical coordinates)

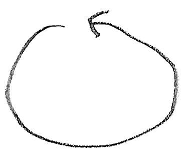
$\boxed{4\pi\delta^3(\vec{r})}$

1.9. (Short answer) Draw the appropriate image(s) to solve this problem (the arrows are semi-infinite grounded conducting planes):



with these charges, all of the planes (including the two original ones) will have $V=0$

1.10.(Short answer) A magnetic field points into the page and is increasing. In what direction is the induced electric field? (Indicate with a picture, no partial credit)

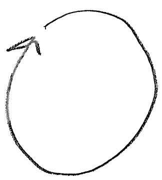


use Lenz's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

the negative sign

1.11.(Short answer) An electric field points into the page and is increasing. In what direction is the induced magnetic field? (Indicate with a picture, no partial credit)



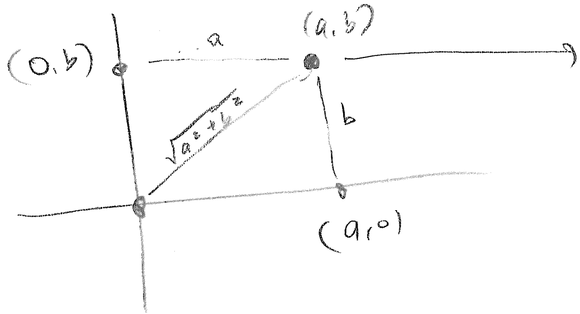
use Maxwell's fix for Ampere's law

$$\nabla \times \vec{B} = \cancel{\mu_0 \vec{J}}^{\,0} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

No negative sign

Worked problems – please work on your own paper, no more than one problem per page.

(8 pts) **Problem 2.** Four charges (all same charge q) are positioned at the corners of a rectangle in the x-y plane: $(0,0)$, $(a,0)$, $(0,b)$, and (a,b). Determine the force on the charge located at (a,b).



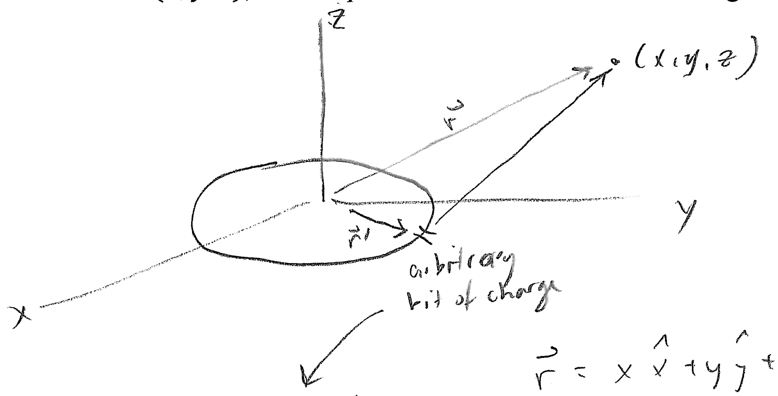
force on this charge = sum of forces from other 3 charges

$$\vec{F} = \frac{q^2}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} \quad \text{for each of the three forces}$$

↳ or $\dfrac{\hat{r}}{r^2}$

$$\boxed{\vec{F} = \frac{q^2}{4\pi\epsilon_0}\left[\frac{\hat{x}}{a^2} + \frac{\hat{y}}{b^2} + \frac{a\hat{x} + b\hat{y}}{(a^2+b^2)^{3/2}}\right]}$$

(10 pts) **Problem 3.** A ring of line charge (radius R, constant linear charge density λ_0) lies in the x-y plane, centered on the z-axis. Set up an integral you could use to calculate the potential at an arbitrary point (x, y, z), with explicit variables and limits of integration.



$d\ell' = R\,d\phi'$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{r}\,' = R\hat{s} = R(\cos\phi'\,\hat{x} + \sin\phi'\,\hat{y})$$

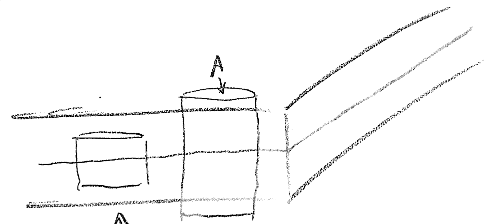
$$\vec{\eta} = \vec{r} - \vec{r}\,' = (x - R\cos\phi')\hat{x} + (y - R\sin\phi')\hat{y} + z\hat{z}$$

$$\eta = \sqrt{(x - R\cos\phi')^2 + (y - R\sin\phi')^2 + z^2}$$

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda\, d\ell'}{\eta}$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\lambda_0 \int_0^{2\pi} \frac{R\,d\phi'}{\left[(x - R\cos\phi')^2 + (y - R\sin\phi')^2 + z^2\right]^{1/2}}}$$

(12 pts) **Problem 4.** An infinite slab has thickness d, going from $-d/2$ to $+d/2$ in the z-direction and being infinite in the x- and y- directions. It has a volume charge density which increases with the distance from the center of the slab according to $= \frac{\rho_0|z|}{d}$. (a) Determine the electric field for points inside and outside the slab. (b) Let the zero of potential be at the surface of the slab. What is the potential at the center, i.e. $z = 0$?



Gaussian surfaces for inside and outside

(a) inside: $\oint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$

$\int_{top} + \int_{bottom} + \cancel{\int_{wrapper}}$ no flux

$E\cancel{A} + E\cancel{A} = \frac{1}{\epsilon_0}\,\rho_0 \frac{\cancel{A} z^2}{d}$

$$\boxed{\vec{E} = \frac{\rho_0 z^2}{2\epsilon_0 d}\begin{cases}\hat{z} \text{ if } z>0\\ -\hat{z} \text{ if } z<0\end{cases}}$$

$q_{enc} = \int \rho\, d\tau$

$= \int_0^z \rho_0 \frac{z'}{d}(A\,dz') + \int_{-z}^0 \frac{\rho_0(-z')}{d}(A\,dz')$

$= \frac{\rho_0 A}{d}\left(\frac{1}{2}z^2 + \frac{1}{2}z^2\right)$

$= \frac{\rho_0 A z^2}{d}$

outside q_{enc} limits go 0 to $\frac{d}{2}$ $\rightarrow$ $q_{enc} = \frac{\rho_0 A \frac{d^2}{4}}{d}$

$= \rho_0 \frac{A d}{4}$

$\oint \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$

$E\cancel{A} + E\cancel{A} = \frac{1}{\epsilon_0}\,\rho_0 \frac{\cancel{A} d}{4}$

$$\boxed{\vec{E} = \frac{\rho_0 d}{8\epsilon_0}\begin{cases}\hat{z} \text{ if } z>0\\ -\hat{z} \text{ if } z<0\end{cases}}$$

(b) $V(\vec{r}) = -\int_{ref}^{\vec{r}} \vec{E}\cdot d\vec{l}$

$V(\text{center}) = -\int_{\text{surface}}^{\text{center}} \vec{E}_{\text{inside}}\cdot d\vec{l}$

$= -\int_{d/2}^{0}\left(\frac{\rho_0 z^2}{2\epsilon_0 d}\right)dz$

$= \frac{-\rho_0}{2\epsilon_0 d}\underbrace{\frac{1}{3}z^3\Big|_{d/2}^{0}}_{-\frac{1}{3}\frac{d^3}{8}}$

$$\boxed{V(\text{center}) = +\frac{\rho_0 d^2}{48\epsilon_0}}$$

(20 pts) **Problem 5**. In class we solved Laplace's equation via separation of variables for rectangular and spherical symmetry, but not for cylindrical symmetry. Those of you with an advanced mathematical intuition may guess that the cylindrical symmetry case will involve Bessel functions, but that surprisingly (to me, anyway) turns out to only be the case for *finite* cylinders. The goal of this problem is to figure out the solution for *infinite* cylinders, i.e. where there is no z-dependence to the potential.

(a) Go through the initial steps of solving Laplace's equation via separation of variables for cylindrical symmetry, assuming V is only a function of s and ϕ. Separate the partial differential equation into two ordinary differential equations for s and ϕ. As usual, you'll need to set two different things equal to the same constant in order to do this.

(b) The ϕ equation: I'll solve this equation for you ☺. Pick the sign of your constant by setting it equal to $+m^2$ or $-m^2$ so that your solutions to the ϕ equation are $\sin m\phi$ and $\cos m\phi$ (and linear combinations, of course). There is an added requirement that the ϕ dependence must be periodic with a period of 2π because e.g. points at $\phi = 34°$ are identical to points at $\phi = 394°$; that means m must be an integer. (You can show that if you want, but it's not required.)

(c) The s equation: Solving the s equation is tricky, and in fact you get a different set of paired solutions when m is zero compared to when it is nonzero. We'll use every physicist's favorite trick of guessing the correct answers. (1) When $m = 0$, show that $\ln s$ and the number 1 are linearly independent solutions that both solve the s equation. (2) When $m \neq 0$, show that s^m and s^{-m} are linearly independent solutions that both solve the s equation.

(d) By taking a summation of linearly independent terms, and considering the $m = 0$ case as a separate term in the summation from the $m \neq 0$ cases (which are infinite), write out the general form of the complete answer to the situation. "General form" means not worrying about any other boundary conditions, which I haven't specified in this problem. Depending on how you write things, you should end up with something like 3-6 arbitrary constants in your answer.

(a) Laplace eqn in cyl. coords:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \cancel{\frac{\partial^2 V}{\partial z^2}} = 0$$

$= 0$ if no z-dependence to V

Assume $V = S(s)\,\Phi(\phi)$

$$\frac{1}{s}\frac{d}{ds}\left(s S' \Phi\right) + \frac{1}{s^2} S \Phi'' = 0$$

$$\frac{\cancel{\Phi}\,\frac{1}{s}\left(s S'' + S'\right)}{S\cancel{\Phi}} + \frac{\frac{1}{s^2}\cancel{S}\Phi''}{\cancel{S}\Phi} = 0 \qquad \text{(also multiply by } s^2\text{)}$$

$$\boxed{\frac{s^2 S'' + s S'}{S} = -\frac{\Phi''}{\Phi} = \text{constant}}$$

5) cont

(b) "constant" must be a positive number, call it m^2

$$\rightarrow \boxed{-\frac{\Phi''}{\Phi} = m^2}$$ "the phi equation"

$$\Phi'' = -m^2 \Phi \longrightarrow \boxed{\Phi = \begin{cases} \sin m\phi \\ \cos m\phi \end{cases} \text{ or linear combs}}$$

$\Phi = A \sin m\phi + B \cos m\phi$ is general soln

Require $\Phi(\phi + 2\pi) = \Phi(\phi)$

$\rightarrow A \sin m(\phi + 2\pi) + B \cos m(\phi + 2\pi) = A \sin m\phi + B \cos m\phi$

$A(\sin m\phi \cos 2\pi m + \cos m\phi \sin 2\pi m) + B(\cos m\phi \cos 2\pi m \ldots$

$- \sin m\phi \sin 2\pi m) = A \sin m\phi + B \cos m\phi$

"you can show that if you want, but it's not required"

$\sin m\phi (A \cos 2\pi m - B \sin 2\pi m) + \cos m\phi (A \sin 2\pi m + B \cos 2\pi m)$

$= A \sin m\phi + B \cos m\phi$

$$\left.\begin{aligned} A \cos 2\pi m - B \sin 2\pi m &= A \\ A \sin 2\pi m + B \cos 2\pi m &= B \end{aligned}\right\}$$ $\cos 2\pi m = 1$ and $\sin 2\pi m = 0$ only possible if m = integer

(c) $\boxed{s^2 S'' + s S' = + m^2 S}$ "the s equation"

(1) When $m = 0$: $s^2 S'' + s S' = 0$

Guess $\left.\begin{aligned} S &= \ln s \\ S' &= \tfrac{1}{s} \\ S'' &= -\tfrac{1}{s^2} \end{aligned}\right\}$ the s equation becomes

$s^2\left(-\frac{1}{s^2}\right) + s\left(\frac{1}{s}\right) \stackrel{?}{=} 0$

$-1 + 1 \stackrel{?}{=} 0$

$0 = 0$ ✓ yes it works!

Guess $\left.\begin{aligned} S &= 1 \\ S' &= 0 \\ S'' &= 0 \end{aligned}\right\}$ $s^2(0) + s(0) \stackrel{?}{=} 0$

$0 = 0$ ✓ yes it works!

$$\boxed{S = \begin{cases} \ln s \\ 1 \end{cases} \text{ or linear combs}}$$

$S = C_0 \ln s + D_0$ is general solution

(subscripts indicate $m = 0$)

5) cont'd

(c)

(2) When $m \neq 0$

Guess $S = s^m$

$$\left.\begin{aligned} S' &= m s^{m-1} \\ S'' &= m(m-1) s^{m-2} \end{aligned}\right\} \quad s^2\left(m(m-1)s^{m-2}\right) + s\left(m s^{m-1}\right) \stackrel{?}{=} m^2\left(s^m\right)$$

$$(m^2 - \cancel{m}) s^m + \cancel{m s^m} \stackrel{?}{=} m^2 s^m$$

$$m^2 s^m = m^2 s^m \checkmark \text{ It works!}$$

Guess $S = s^{-m}$

$$\left.\begin{aligned} S' &= -m s^{-m-1} \\ S'' &= -m(-m-1) s^{-m-2} \\ &= m(m+1) s^{-m-2} \end{aligned}\right\} \quad s^2\left(m(m+1)s^{-m-2}\right) + s\left(-m s^{-m-1}\right) \stackrel{?}{=} m^2\left(s^{-m}\right)$$

$$(m^2 + \cancel{m}) s^{-m} - \cancel{m s^{-m}} \stackrel{?}{=} m^2 s^{-m}$$

$$m^2 s^{-m} = m^2 s^{-m} \checkmark$$

It works!

$$\boxed{S = \begin{cases} s^m \\ s^{-m} \end{cases} \text{ or linear combs}}$$

$S = C_m s^m + D_m s^{-m}$ is general solution

(d) $V = (m=0 \text{ term}) + \left(\sum m \neq 0 \text{ terms}\right)$

$$= S\Phi\,(m=0) + \sum_{m\neq 0} S\Phi$$

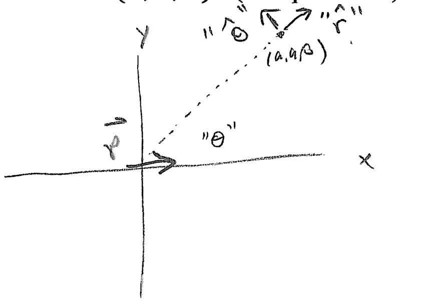
$$= \left(C_0 \ln s + D_0\right)\left(\cancel{A \sin 0} + B\cos 0\right) + \sum_{m\neq 0}\left(C_m s^m + D_m s^{-m}\right)\left(A\sin m\phi + B\cos m\phi\right)$$

can lump in with C_0 + D_0

$$\boxed{V(s,\phi) = C_0 \ln s + D_0 + \sum_{m\neq 0}\left(C_m s^m + D_m s^{-m}\right)\left(A \sin m\phi + B\cos m\phi\right)}$$

(could lump either A or B in with C_m and D_m if desired)

(12 pts) **Problem 6.** An electric dipole at the origin is pointing in the positive x direction: $\mathbf{p} = p_0\hat{\mathbf{x}}$. What are the electric potential and electric field at an arbitrary point far away from the origin along the line $\boldsymbol{y} = \boldsymbol{x}$, located at $(\boldsymbol{a}, \boldsymbol{a}, \mathbf{0})$? ($a$ is positive.) Note that the point is a distance of $\boldsymbol{a}\sqrt{\mathbf{2}}$ from the origin.



$$r = a\sqrt{2}$$
$$\theta = 45^\circ$$
$$\hat{r} = \frac{\hat{x} + \hat{y}}{\sqrt{2}}$$
$$\hat{\theta} = \frac{-\hat{x} + \hat{y}}{\sqrt{2}}$$

dipole potential: $$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

$$= \frac{p_0 \cos 45^\circ}{4\pi\epsilon_0 (a\sqrt{2})^2} = \frac{p_0 \frac{1}{\sqrt{2}}}{4\pi\epsilon_0\, 2a^2} = \boxed{\frac{p_0}{8\sqrt{2}\,\pi\epsilon_0 a^2}}$$

dipole field: $$\vec{E} = \frac{p}{4\pi\epsilon_0}\frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

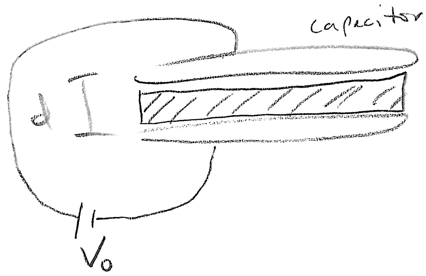
$$= \frac{p_0}{4\pi\epsilon_0}\frac{1}{a^3\, 2\sqrt{2}}\left(2\frac{1}{\sqrt{2}}\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{-\hat{x}+\hat{y}}{\sqrt{2}}\right)\right)$$

$$= \frac{p_0}{16\sqrt{2}\,\pi\epsilon_0}\frac{1}{a^3}\left(2\hat{x} + 2\hat{y} - \hat{x} + \hat{y}\right)$$

$$\boxed{\vec{E} = \frac{p_0}{16\sqrt{2}\,\pi\epsilon_0}\frac{1}{a^3}\left(\hat{x} + 3\hat{y}\right)}$$

(12 pts) **Problem 7**. A linear dielectric material with relative permittivity ϵ_r is placed inside a parallel plate capacitor (capacitance C, separation distance d) that is maintained at a fixed voltage V_0 by a battery. (a) Does the battery pump charge onto the capacitor or remove charge from the capacitor when this is done? Explain. (b) What is the electric field inside the material, assuming the material completely fills the capacitor and you can neglect edge effects?

capacitor C



dielectric put in place ϵ_r

(a) $Q = C V_0$ for capacitor

C before dielectric $= C$

C after dielectric $= \epsilon_r C$

Since $\epsilon_r > 1$, $\boxed{Q_{after} > Q_{before}}$ $\boxed{\text{charge is pumped onto the cap.}}$

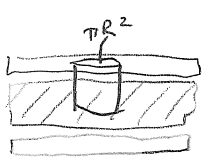
(b) constant electric field → constant V gradient

$$|\vec{E}| = \nabla V = \frac{V_0}{d}$$

$$\boxed{E = \frac{V_0}{d}}$$ in the $\hat{z}$ or $-\hat{z}$ direction, depending on how battery is oriented

There are other ways to do this, but that is by far the easiest method.

Could e.g. do Gauss's Law for $\vec{D}$ and get same result.



$$\oint \vec{D} \cdot d\vec{a} = q_{free\ enc}$$

$$\underbrace{S_{top}}_{=0\ (\text{no } D)} + \underbrace{S_{bottom}}_{= D\pi R^2} + \underbrace{S_{wrapper}}_{=0\ (\text{no flux})} = \left(\frac{Q}{A}\right)\pi R^2$$

$$D \cancel{\pi R^2} = \frac{Q}{A} \cancel{\pi R^2}$$

$$\epsilon_0 \epsilon_r E = \frac{Q}{A}$$

$$E = \frac{Q}{\epsilon_0 \epsilon_r A}$$

$$Q = C_{final} V_0 = \left(\frac{\epsilon_r \epsilon_0 A}{d}\right) V_0$$

So $$E = \frac{\epsilon_r \epsilon_0 A/d \; V_0}{\epsilon_0 \epsilon_r A}$$

$$E = \frac{V_0}{d}$$ ✓ It works!

(14 pts) **Problem 8**. A magnetic field in a certain region of space is given by $\mathbf{B} = 5y\hat{\mathbf{x}} + 2y^3\hat{\mathbf{z}}$, with all variables being in standard SI units. (a) What are the units of the numbers 5 and 2? (b) What current distribution $\mathbf{J}$ would give rise to such a field? (c) A small magnetic dipole with dipole moment $\mathbf{m} = (0.2\hat{\mathbf{y}} + 0.1\hat{\mathbf{z}}) \times 10^{-9}\mathrm{A}\cdot\mathrm{m}^2$ is placed in the field at $y = 1$ m. What is the torque on the dipole? Does the torque try to rotate the dipole moment to be aligned with $\mathbf{B}$, anti-aligned with $\mathbf{B}$, or other? Explain.

(a) units: $T = (5\_\_)\, m \rightarrow$ 5 has units of T/m

$T = (2\_\_)\, m^3 \rightarrow$ 2 has units of T/m^3

(b) Use Ampere's Law to get $\vec{J}$ from $\vec{B}$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \rightarrow \vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B}$$

$$\vec{J} = \frac{1}{\mu_0} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 5y & 0 & 2y^3 \end{vmatrix} = \frac{1}{\mu_0} \left[\hat{x}(2\cdot 3y^2) - \hat{y}(0) + \hat{z}(-5\cdot 1) \right]$$

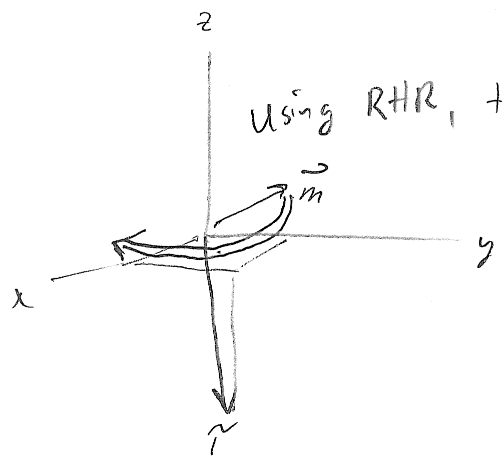
$$\boxed{\vec{J} = \frac{1}{\mu_0}\left(6y^2\hat{x} - 5\hat{z}\right)}$$

(c) $\vec{\tau} = \vec{m} \times \vec{B}$ ($\vec{B}$ at $y=1$)

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 5 & 0 & 2 \end{vmatrix} \cdot 10^{-9}$$

$$\boxed{\tau = (.4\hat{x} + .5\hat{y} - 1\hat{z}) \cdot 10^{-9}\ \mathrm{Nm}}$$

The torque tries to align the dipole moment with the field



Using RHR, torque causes $\vec{m}$ to rotate in direction indicated, swinging it around towards the $(5\hat{x} + 2\hat{z})$ direction

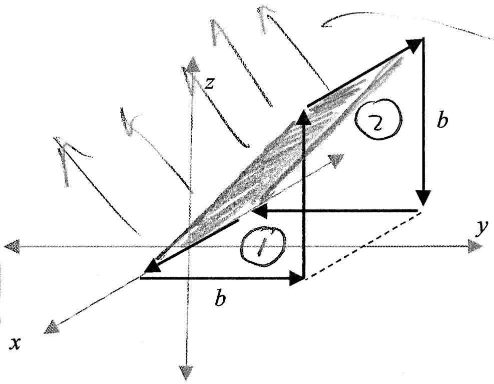
(14 pts) **Problem 9**. As we discussed in class, the magnetic dipole moment of a current loop is $\mathbf{m} = I\mathbf{a}$, where $\mathbf{a}$ is the "vector area" of the loop, $\mathbf{a} = \int d\mathbf{a}$. For planar loops the vector area is just equal to the area, with its direction given by the right hand rule relative to the direction of current flow. In class I said that I believed for *non-planar* loops, $\mathbf{a}$ turns out to be the maximal area of the loop as you view it from arbitrary directions, with the direction of $\mathbf{a}$ given by the direction that maximizes the viewable area (and the right hand rule). I still believe this is accurate, although I haven't found a reference that puts it quite like that.

(a) Prove that my claim works for the case of a rectangular loop bent at 90 degrees to form two square sections as shown, each with side length b. What value of $\mathbf{a}$ do you get from $\int d\mathbf{a}$ (you can add together the $\int d\mathbf{a}$ from the two rectangles, each of which is planar)? And what do you get from my "viewable area" description?

$$\int d\vec{a} = \int_1 d\vec{a} + \int_2 d\vec{a}$$

$$= b^2\hat{z} + b^2(-\hat{y})$$

$$\boxed{\vec{a} = b^2\sqrt{2}\left(\frac{-\hat{y}+\hat{z}}{\sqrt{2}}\right)}$$

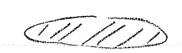
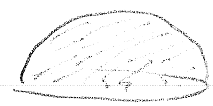


max viewable area from this direction, $= \frac{-\hat{y}+\hat{z}}{\sqrt{2}}$

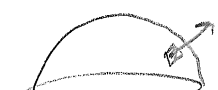
Area of shaded rectangle is $(b\sqrt{2})(b)$

$= b^2\sqrt{2}$ ✓ yes, they match!

(b) As an interesting corollary of this description, it doesn't actually matter which surface you use to do the $\int d\mathbf{a}$ integral as long as that surface is bounded by the loop. As a demonstration of this, prove that you get the same vector area for a circular loop in the x-y plane (radius R, current in $\hat{\boldsymbol{\phi}}$ direction) by integrating $\int d\mathbf{a}$ over (1) a planar disk (the answer is trivial, you don't need to do anything, it's just $\pi R^2\hat{\mathbf{z}}$, and (2) a hemispherical shell bounded by the loop (radius R, extending into the positive z-direction). In other words, I'm asking you to integrate $\int d\mathbf{a}$ for a hemispherical shell and show it equals $\pi R^2\hat{\mathbf{z}}$.

compare $\int d\vec{a}$ for  vs 

$d\vec{a} = \pi R^2 \hat{z}$



arbitrary $d\vec{a}$ in spherical coords $= R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$$\text{hemisphere} \int d\vec{a} = \int_0^{2\pi}\int_0^{\pi/2} R^2 \sin\theta \, d\theta \, d\phi \, (\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z})$$

$$= R^2 \int_0^{2\pi}\int_0^{\pi/2} (\hat{x}\underbrace{\sin^2\theta\cos\phi}_{=0 \text{ from } \phi \text{ integral}} + \hat{y}\sin^2\theta\sin\phi + \hat{z}\underbrace{\sin\theta\cos\theta}_{\phi \text{ integral gives } 2\pi})\, d\theta\, d\phi$$

a) cont.

$$\text{hemisphere } d\vec{a} = R^2 \cdot 2\pi \hat{z} \underbrace{\int_0^{\pi/2} \sin\theta \cos\theta \, d\theta}_{= 1/2 \text{ from Mathematica}}$$

$$\boxed{d\vec{a} = R^2 \cdot \pi \cdot \hat{z}}$$ ✓

Yes, they match!

Note: everyone sketched field lines, which is not what I meant. So I made this problem only worth 5 pts → 2.5 for correct B field and 2.5 for recognizing $H \neq 0$ since the B field is nonzero outside the magnet

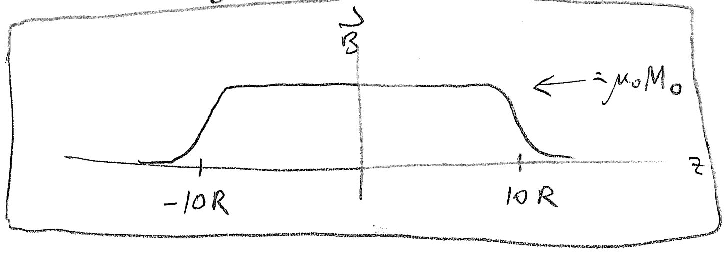
looks like finite solenoid

(~~10~~ 5 pts) **Problem 10**. (a) Sketch the **B** field along the axis of a finite cylindrical bar magnet where its length is much longer than its radius, say $L \approx 20\ R$. A constant magnetization points along the axis of the cylinder. In your sketch include both the inside and the outside. You don't have to work out exact equations, just a general (but fairly accurate) sketch is fine. (b) On a separate graph, sketch the **H** field for the same situation.

What I intended:

Exactly done in a class handout. Looks like:

(a)

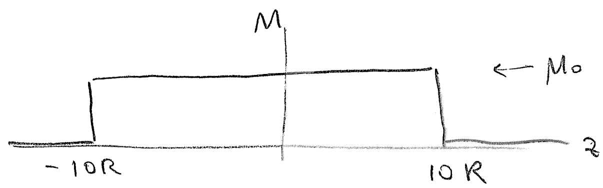


($\vec{B}$ is in $\hat{z}$ direction)

(b) $\vec{H} = \dfrac{\vec{B}}{\mu_0} - \vec{M}$

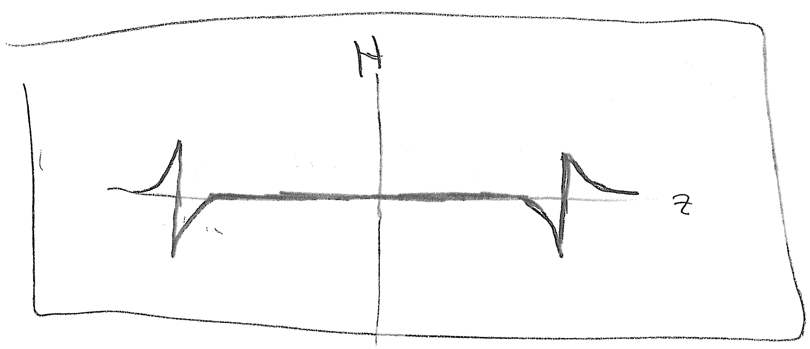
$\vec{M} = M_0 \hat{z}$, but only for $-10R < z < 10R$

So, basically we are subtracting off this



from the upper graph (divided by μ_0)

The result is:



In particular, note that the H-field is not zero!

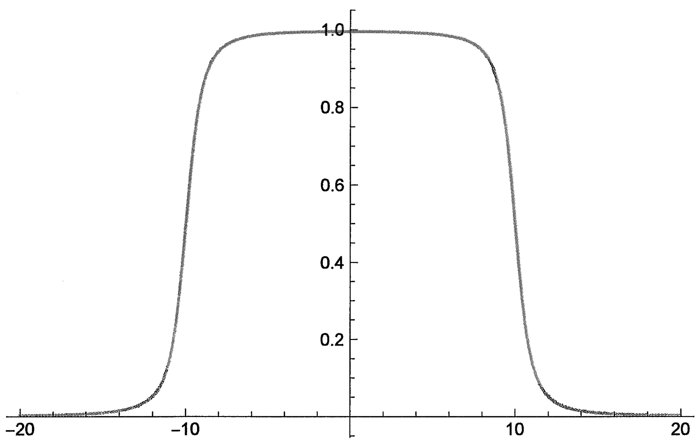
Done exactly with Mathematica on next page

■ B and H Field of a magnetized cylinder (on axis)

In[1]:=
```
(* exact formula from previous handout, setting mu0 = 1 and R = 1 *)
b[z_, L_] = 1/2 ((z + L/2) / Sqrt[1 + (z + L/2)^2] - (z - L/2) / Sqrt[1 + (z - L/2)^2]);
```

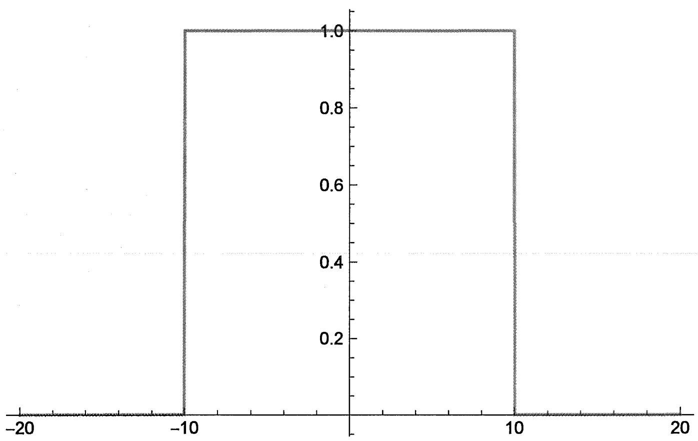
In[2]:=
```
Plot[b[z, 20], {z, -20, 20}, PlotRange → All] (* length = 20 x radius *)
```

Out[2]=



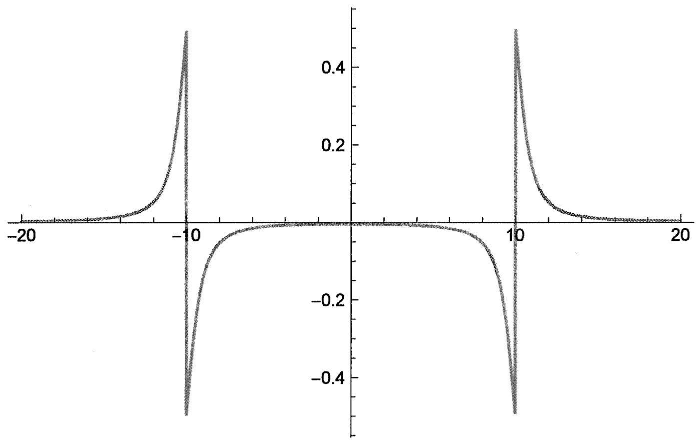
In[3]:=
```
m[z_, L_] = UnitBox[z / L];
Plot[m[z, 20], {z, -20, 20}]
```

Out[4]=



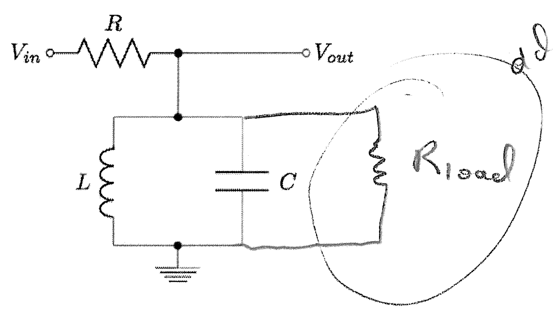
In[5]:=
```
h[z_, L_] = b[z, L] - m[z, L];
Plot[h[z, 20], {z, -20, 20}, PlotRange → All]
```

Out[6]=



(16 pts) **Problem 11**. I mentioned in class and/or on the homework that the voltage divider equation was obtained under the assumption that no current "leaks out", which is really only the case for when the output of the voltage divider is hooked up to a circuit with a large input impedance. Since all of our filter examples were obtained using the voltage divider equation, that means the filter transfer functions that we plotted are only accurate when the filters are hooked up to circuits with large input impedances. The point of this problem is to explore that. We are going to analyze connecting the band pass filter to a circuit with a varying input impedance.

Start with the band pass filter picture,



add this resistor to the circuit

and add *another* resistor, let's call it R_{load}, going from V_{out} to ground. That represents the input impedance of the load circuit, i.e. whatever circuit you are hooking the filter up to. Notice that R_{load} is in parallel with L and C.

Use these numbers: $V_{in} = 1\text{ V}\cos\omega t$, $R = 100\text{ k}\Omega$, $L = 10\text{ mH}$, and $C = 10\text{ nF}$ (same RLC values as in the band pass filter example in the Circuits 2 handout).

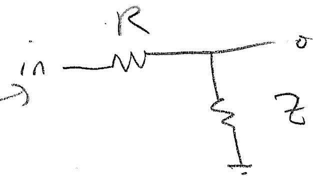
The voltage of R_{load} is the voltage delivered to the load circuit. Use Mathematica or similar program to plot the magnitude and phase (on separate graphs) of this voltage as a function of ω, for R_{load} equal to:

(a) 10 kΩ
(b) 100 kΩ
(c) 1000 kΩ

Force your magnitude plots to all go from 0 to 1 V on the y-axis so you can easily see the difference between the situations. The plots for part (c) should look fairly similar to the transfer function plots in the handout, which essentially assumed an infinite R_{load}.

Can either say $\tilde{V}_{load} = \tilde{V}_{in} - \tilde{I}_{tot} R$

or $\tilde{V}_{load} = \tilde{V}_{Z_2}$



$Z_2 = Z_L // Z_C // R_{load}$

Both give same answer, but I'll do it this way so it's basically a voltage divider again

$$\tilde{V}_{Z_2} = \tilde{V}_{in} \frac{Z_2}{R + Z_2}$$ } plots done for given values with Mathematica

✓

($V_{in} = 1\angle 0°$ which is nice; so answer is just the transfer function. ☺)

11) cont

```
In[1]:= R = 100 000;
L = 0.01;
c = 10*^-9;
z2[w_, Rload_] = (1/Rload + 1/(I w L) + 1/(-I/(w c)))^-1
h[w_, Rload_] := z2[w, Rload]/(R + z2[w, Rload])
```

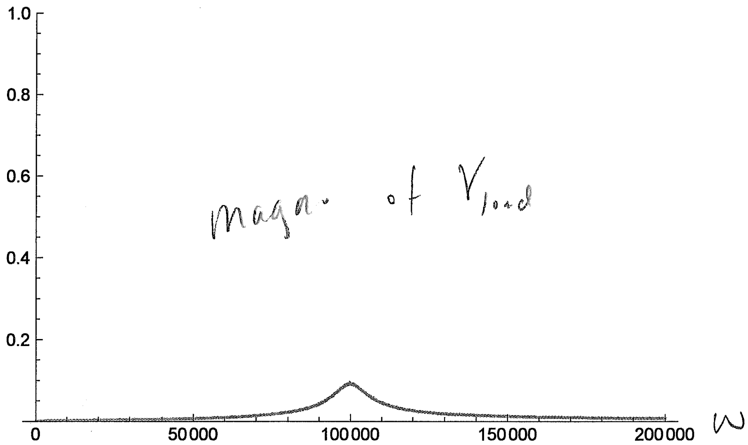
Out[4]= $$\frac{1}{\frac{1}{\text{Rload}} - \frac{0.+100.\,i}{w} + \frac{i\,w}{100\,000\,000}}$$

(a)

$R_{load} = 10k\,\Omega$

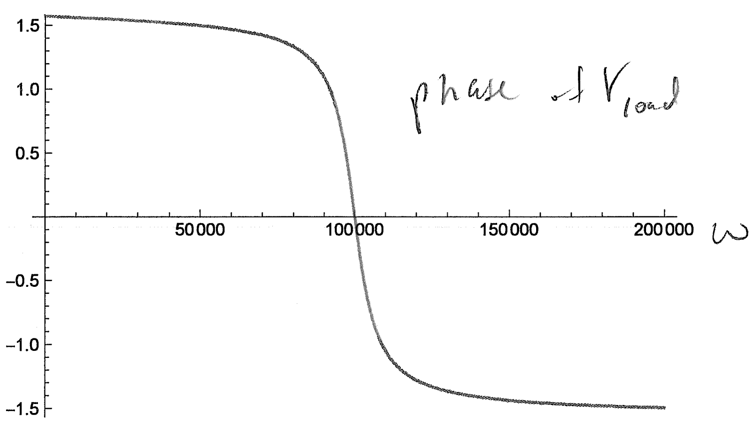
```
In[6]:= Plot[Abs[h[w, 10 000]], {w, 0, 200 000}, PlotRange → {0, 1}]
Plot[Arg[h[w, 10 000]], {w, 0, 200 000}, PlotRange → {-Pi/2, Pi/2}]
```

Out[6]=



Output voltage is greatly supressed when R_{load} is small!

Out[7]=



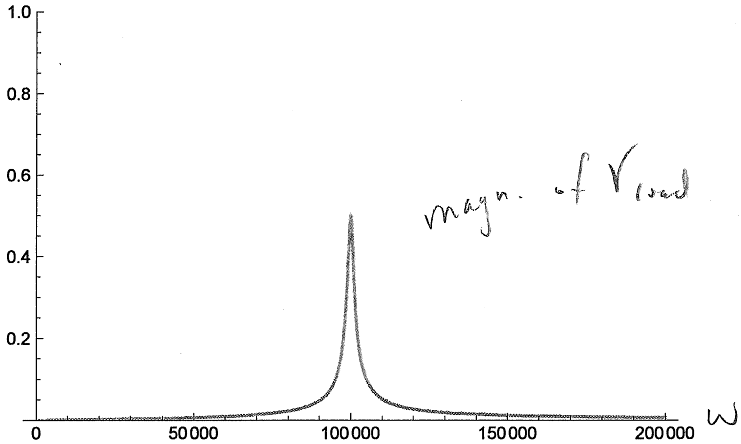
11) cont

(b)

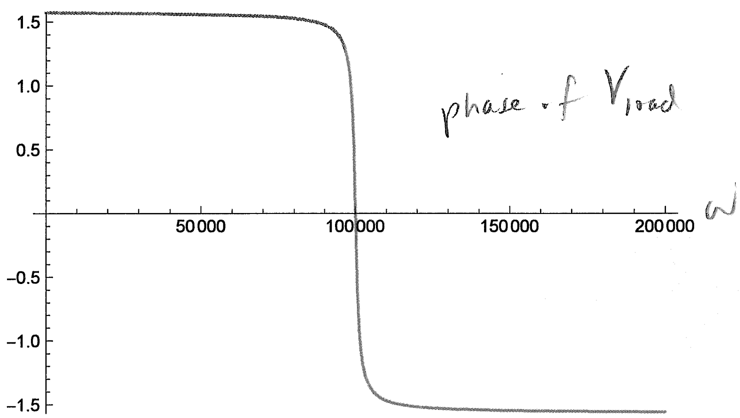
In[8]:=
```
Plot[Abs[h[w, 100000]], {w, 0, 200000}, PlotRange → {0, 1}]
Plot[Arg[h[w, 100000]], {w, 0, 200000}, PlotRange → {-Pi/2, Pi/2}]
```

$R_{load} = 100k\Omega$

Out[8]=



Out[9]=



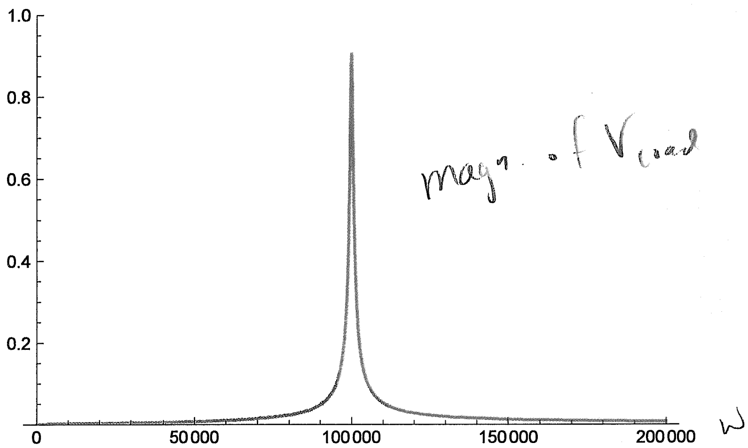
11) cont

(c)

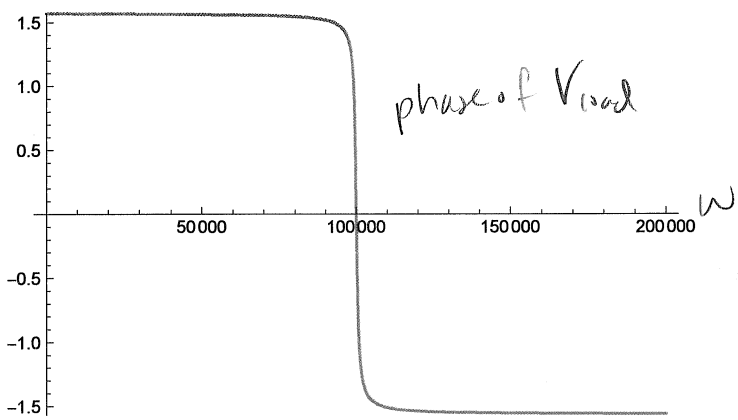
In[10]:=
```
Plot[Abs[h[w, 1000000]], {w, 0, 200000}, PlotRange → {0, 1}]
Plot[Arg[h[w, 1000000]], {w, 0, 200000}, PlotRange → {-Pi/2, Pi/2}]
```

$R_{load} = 1000 k\Omega$

Out[10]=



Out[11]=



with a really large R_{load}, output voltage looks very similar to our previously derived plots of band pass filter transfer function!

(6 pts) **Problem 12**. Extra credit. Two points for each equation, no partial credit (i.e. 0, 2, 4, or 6 points are possible). In class on the last day I showed you that the potential and field from a charge distribution ρ can be written as convolutions:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}d\tau' \quad \rightarrow V(\mathbf{r}) = V_0 \otimes \rho$$
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}d\tau' \quad \rightarrow \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \otimes \rho$$

Here V_0 and $\mathbf{E}_0$ are the potential and field of a unit point charge located at the origin.

I also told you without proving it (due to lack of time) that the Biot-Savart law giving the magnetic field from a current density $\mathbf{J}$ could be written as a cross-product type convolution like this:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}d\tau' \quad \rightarrow \mathbf{B}(\mathbf{r}) = -\epsilon_0\mu_0\mathbf{E}_0 \otimes (\times\,\mathbf{J})$$

You can double check that yourself to make sure you understand what's going on.

This problem: figure out how to write Griffiths equations 5.65, 4.9, and 6.11 (3rd edition: Eqns 5.63, 4.9, 6.11) as convolutions. Those are the equations for the vector potential of a current density $\mathbf{J}$ in the Coulomb gauge, the scalar potential of a polarized object, and the vector potential of a magnetized object, respectively.

Eqn 5.65 $\quad \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}d\tau'$

From intuition using the previous examples as a guide, I'll guess

$$A = C\,\frac{1}{4\pi\epsilon_0 r} \otimes \vec{J} = C\,V_0 \otimes \vec{J}$$

But what is C?

$$C\cdot\frac{1}{4\pi\epsilon_0 r} \otimes \vec{J} = \vec{J} \otimes C\cdot\frac{1}{4\pi\epsilon_0 r}$$
$$= \frac{C}{4\pi\epsilon_0}\int \vec{J}(\vec{r}\,')\frac{1}{\mathscr{r}}d\tau'$$

which is correct if $\frac{C}{4\pi\epsilon_0} = \frac{\mu_0}{4\pi} \rightarrow C = \epsilon_0\mu_0$

So $\boxed{\vec{A} = \epsilon_0\mu_0\, V_0 \otimes \vec{J}}$ or $\epsilon_0\mu_0\,\vec{J}\otimes V_0$ if you prefer

could write $\frac{1}{c^2}$ for $\epsilon_0\mu_0$

12) cont

Eqn 4.9 $\quad V_{\substack{\text{polarized}\\ \text{object}}} = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}')\cdot\hat{\imath}}{\imath^2}\,d\tau'$

Guess $V = C\,\frac{1}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \circledast (\cdot\vec{P}) = C\,\vec{E}_0 \circledast (\cdot\vec{P})$

but what is C?

$$C\,\frac{1}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\circledast(\cdot\vec{P}) = \frac{C}{4\pi\epsilon_0}\,\vec{P}\circledast\left(\cdot\frac{\hat{r}}{r^2}\right)$$

$$= \frac{C}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}')\cdot\hat{\imath}}{\imath^2}\,d\tau'$$

which is correct if $C=1$!

$$\boxed{V_{\text{pol object}} = \vec{E}_0 \circledast (\cdot\vec{P})}$$

or $\vec{P}\circledast(\cdot\vec{E}_0)$ if you prefer

Eqn 6.11 $\quad \vec{A}_{\substack{\text{magnetized}\\ \text{object}}} = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}')\times\hat{\imath}}{\imath^2}\,d\tau'$

Guess $\vec{A} = C\,\frac{1}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\circledast(\times\vec{M}) = C\,\vec{E}_0\circledast(\times\vec{M})$

what is C?

$$C\,\frac{1}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\circledast(\times\vec{M}) = -\frac{C}{4\pi\epsilon_0}\,\vec{M}\circledast\left(\times\frac{\hat{r}}{r^2}\right)$$

negative since conv, crossproduct is anti-symm.

$$= -\frac{C}{4\pi\epsilon_0}\int \frac{\vec{M}(\vec{r}')\times\hat{\imath}}{\imath^2}\,d\tau'$$

which is correct if $-\frac{C}{4\pi\epsilon_0} = \frac{\mu_0}{4\pi} \rightarrow C = -\epsilon_0\mu_0$

$$\boxed{\vec{A}_{\text{magn. object}} = -\epsilon_0\mu_0\,\vec{E}_0\circledast(\times\vec{M})}$$

or $+\epsilon_0\mu_0\vec{M}\circledast(\times\vec{E}_0)$ if you prefer

could write $\frac{1}{c^2}$ for $\epsilon_0\mu_0$