Spring 2016
Physics 441
Final Exam
Dr. Colton, cell: 801-358-1970

barcode here

No time limit. Student calculators are allowed. One page of notes allowed (front & back). Books not allowed. FRONT AND BACK INSIDE COVERS OF TEXTBOOK SHOULD BE PROVIDED.

Name Solutions

Instructions: Please label & circle/box your answers. **Show your work,** where appropriate! And remember:

- **in any problems involving Gauss's Law, you should explicitly show your Gaussian surface.**
- **in any problems involving Ampere's Law, you should explicitly show your Amperian loop.**

There are 140 total points, with an additional 10 extra credit points possible on the last problem.

Some Legendre polynomials:
$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = 3/2\, x^2 - 1/2$
$P_3(x) = 5/2\, x^3 - 3/2\, x$

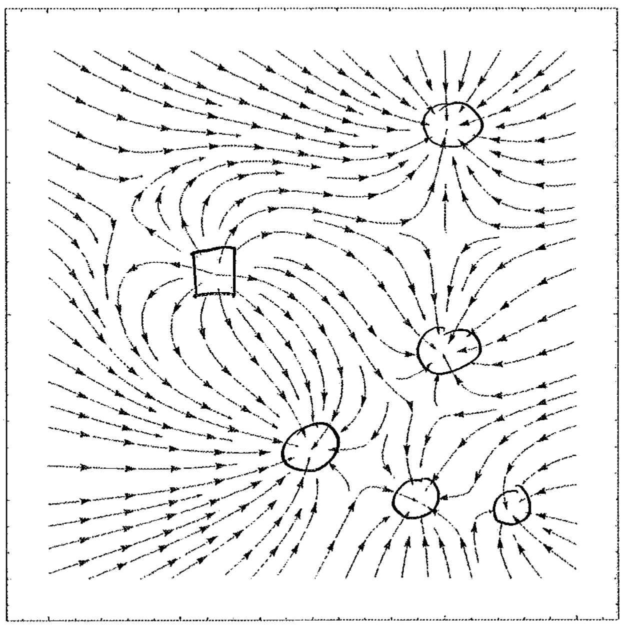
(18 pts) **Problem 1**: Multiple choice, 2 pts each. Circle the correct answer.

1.1. Various electric charges of different magnitudes (both positive and negative) are present in the x-y plane in a region of space. The figure shows a plot of the field lines in the region. How many charges are there?

(a) 3
(b) 4
(c) 5
(d) 6
(e) 7
(f) 8

○ = negative charges

□ = positive charge



1.2. A boundary exists between two linear dielectric materials, i.e. ϵ_r has two different values above and below the boundary. There are no free charges in the region considered. What must be continuous across the boundary?

✓ i) $E_{//}$ ii) $E_{\perp}$ iii) $D_{//}$ ✓ iv) $D_{\perp}$

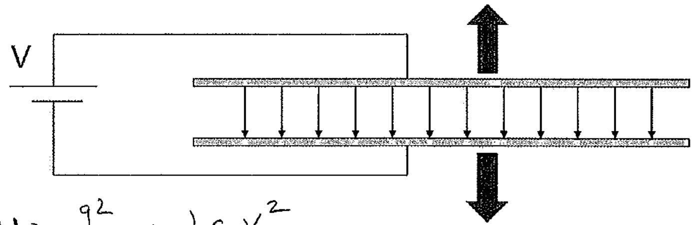
$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \rightarrow \boxed{E_{\parallel,1} = E_{\parallel,2}}$

$\nabla \cdot \vec{D} = \rho_{free} \rightarrow D_{\perp,1} - D_{\perp,2} = \sigma_{free}$

$\rightarrow \boxed{D_{\perp,1} = D_{\perp,2}}$ since no free charge

(a) (i) only
(b) (ii) only
(c) (iii) only
(d) (iv) only
(e) (i) and (ii)
(f) (i) and (iii)
(g) (i) and (iv)
(h) (ii) and (iii)
(i) (ii) and (iv)
(j) (iii) and (iv)

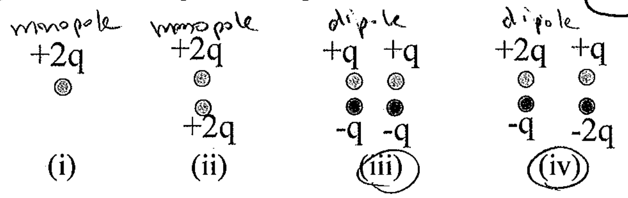
1.3. A parallel plate capacitor is attached to a battery which maintains a constant potential difference (voltage) between the capacitor plates. While the battery is attached, the plates are pulled straight apart from each other in the direction of the block arrows shown, increasing their separation. The electrostatic energy stored in the capacitor:
(a) increases
(b) decreases
(c) stays constant



$U = \frac{q^2}{2C} = \frac{1}{2}CV^2$ ← stays constant

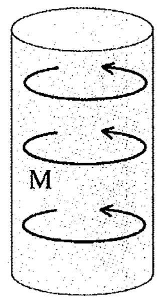
$C = \frac{\epsilon_0 A}{d}$ must be decreasing because d is increasing

1.4. Which of these charge distributions produce a potential which varies as $1/r^2$ when you get far away? → dipole!



(a) (i) only
(b) (ii) only
(c) (iii) only
(d) (iv) only
(e) (i) and (ii)
(f) (i) and (iii)
(g) (i) and (iv)
(h) (ii) and (iii)
(i) (ii) and (iv)
(j) (iii) and (iv)

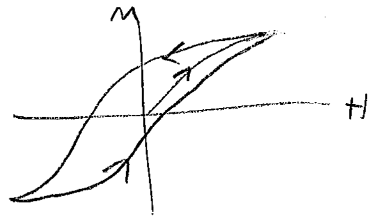
1.5. A solid cylinder has uniform magnetization **M** throughout the volume in the φ direction as shown. In which direction does the bound surface current flow on the "wrapper" (curved side)?
(a) There is no bound surface current on that surface.
(b) The current flows in the $\pm\hat{\boldsymbol{\phi}}$ direction.
(c) The current flows in the $\pm\hat{\mathbf{s}}$ direction.
(d) The current flows in the $\pm\hat{\mathbf{z}}$ direction.
(e) The direction is more complicated than answers (b), (c), or (d).



$\vec{K} = \vec{M} \times \hat{n}$
$= M\hat{\phi} \times \hat{s}$
$= M(-\hat{z})$

1.6. In class I said something like, "The magnetic susceptibility of a ferromagnetic is very large, but it's somewhat ill-defined." Why did I say it is ill-defined?

(i) The ratio of **M** to **H** depends on the current field.
(ii) The ratio of **M** to **H** depends on the history of what the field has been.
(iii) **M** and **H** do not have a linear relationship.



(a) (i) only
(b) (ii) only
(c) (iii) only
(d) (i) and (ii)
(e) (i) and (iii)
(f) (ii) and (iii)
(g) (i), (ii) and (iii)

1.7. Which of the following is/are true about mutual inductance between two conducting loops?

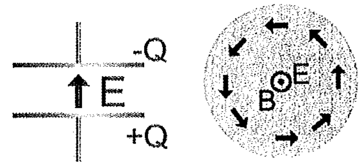
(i) Mutual inductance just depends on the sizes, shapes, orientation, and other geometrical quantities.
(ii) The mutual inductance of conductor 1 acting on conductor 2 is the same as the mutual inductance of 2 acting on 1.
(iii) Mutual inductance is a vector quantity.

(a) (i) only
(b) (ii) only
(c) (iii) only
(d) (i) and (ii)
(e) (i) and (iii)
(f) (ii) and (iii)
(g) (i), (ii) and (iii)

(i) and (ii) can be proven, and (iii) disproven from this eqn:

$$M = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}$$

1.8. The figures show a side and top view of a capacitor with charge Q and electric and magnetic fields **E** and **B** at time *t*. At this time the charge Q is:



(a) Increasing in time
(b) Decreasing in time.
(c) Constant in time.

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \underbrace{\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\vec{J}_D}$$

relationship between $\vec{B}$ and $\vec{J}_D$ given by right hand rule. $\vec{B}$ = CCW, then $\vec{J}_D$ (and hence $\frac{d\vec{E}}{dt}$) must be out of page, i.e. $\vec{E}$ = increasing

1.9. When a slab of dielectric material is placed in an electric field, it polarizes: positive bound charges move slightly in one direction and negative bound charges move slightly in the opposite direction. This movement of charge is a type of current. Where is this current taken into account in Maxwell's equations?

(i) In Ampere's law for **B**: through the μ_0**J** term
(ii) In Ampere's law for **B**: through the $\mu_0\epsilon_0 d$**E**$/dt$ term
(iii) In Ampere's law for **H**: through the $\mathbf{J}_{\text{free}}$ term
(iv) In Ampere's law for **H**: through the d**D**$/dt$ term

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

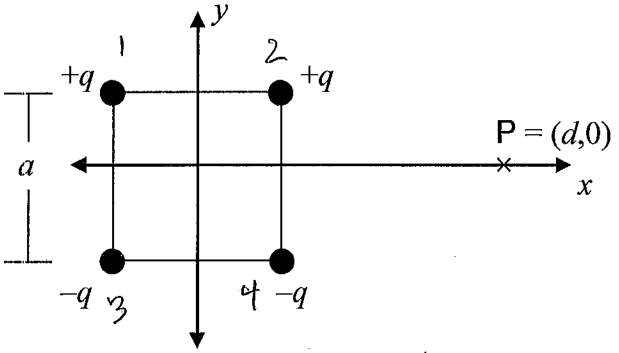
all currents, including polarization current

$$\nabla \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$$

includes $\frac{d\vec{P}}{dt}$ since $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

(a) (i) only
(b) (ii) only
(c) (iii) only
(d) (iv) only
(e) (i) and (iii)
(f) (i) and (iv)
(g) (ii) and (iii)
(h) (ii) and (iv)

(19 pts) **Problem 2**. Point charges of $+q$ and $-q$ are located at the corners of a square of side a, as shown. (a) Find the electric field at a point "P" located a distance d to the right of the center of the square. (b) Find the electric potential at the same point. (c) How can your answers to (a) and (b) be reconciled with the fact that $\mathbf{E} = -\nabla V$?



$\vec r = d\hat x$

From eg charge 1

$\vec r\,' = -\frac{a}{2}\hat x + \frac{a}{2}\hat y$

$\vec{\mathscr{r}} = \vec r - \vec r\,' = \left(d + \frac{a}{2}\right)\hat x - \frac{a}{2}\hat y$

$\mathscr{r} = \sqrt{(d + a/2)^2 + (a/2)^2}$

other charges done similarly in my head

(a) $\vec E = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i^3}\vec{\mathscr{r}}$

By symmetry the x components will cancel out, and + and − contributions will be equal

$$\vec E = \frac{1}{4\pi\epsilon_0}\left(\underbrace{\frac{q\left(-\frac{a}{2}\hat y\right)}{\sqrt{(d+a/2)^2+(a/2)^2}} \times 2}_{\text{charges } 1+3} + \underbrace{\frac{(q)\left(-\frac{a}{2}\hat y\right)}{\sqrt{(d+a/2)^2-(a/2)^2}} \times 2}_{\text{charges } 2+4}\right)$$

$$\boxed{\vec E = \frac{qa}{4\pi\epsilon_0}(-\hat y)\left(\frac{1}{\sqrt{(d+a/2)^2+(a/2)^2}} + \frac{1}{\sqrt{(d+a/2)^2-(a/2)^2}}\right)}$$

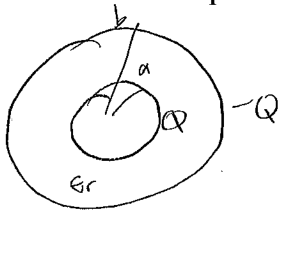
(b) $V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i}$

By symmetry V from 1 cancels V from 3, and
V from 2 cancels V from 4

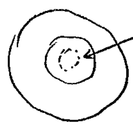
$\boxed{V = 0}$

(c) $\vec E = -\vec\nabla V$? Sure, that's still possible. Just because $V=0$ at one pt. doesn't mean that $\frac{dV}{dy} = 0$ at that point.
To use the equation you'd need to work out $V(y)$ for points around point P

(19 pts) **Problem 3**. Suppose you have a capacitor made out of two concentric spheres: a charge +Q exists on the inner conductor (radius a) and a charge of –Q exists on the outer conductor (radius b). In between the two conductors is a dielectric with relative permittivity ϵ_r. (a) Find the **E** and **D** fields everywhere. (b) Determine the total energy stored in the electric field by integrating **E · D** over all space. (c) Compare that with what the standard "energy of capacitor" formula from Phy 220: $U = Q^2/2C$. Explain why the two energies are the same or different. Recall that for this configuration $C = 4\pi\epsilon_0\epsilon_r(1/a - 1/b)^{-1}$ as we found in a worked problem done in class.



(a) $r < a$



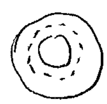
Gaussian surface

$\oint \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$

$E \cdot 4\pi r^2 = 0 \rightarrow \boxed{E = 0}$

similarly $\boxed{D = 0}$

$a < r < b$



$\oint \vec{D}\cdot d\vec{a} = q_{free,\,enc}$

$D\, 4\pi r^2 = Q \rightarrow \boxed{\vec{D} = \frac{Q}{4\pi r^2}\hat{r}}$

$\vec{D} = \epsilon_0\epsilon_r \vec{E} \rightarrow \boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0\epsilon_r r^2}\hat{r}}$

$r > b$

$\oint \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$

$E \cdot 4\pi r^2 = 0 \rightarrow \boxed{E = 0}$

similarly $\boxed{D = 0}$

(b) $U = \frac{1}{2}\int \vec{E}\cdot\vec{D}\, d\tau$ only need to worry about dielectric region, where E + D nonzero

$$= \frac{1}{2}\int_a^b \left(\frac{Q}{4\pi\epsilon_0\epsilon_r r^2}\hat{r}\right)\cdot\left(\frac{Q}{4\pi r^2}\hat{r}\right) 4\pi r^2\, dr$$

$$= \frac{Q^2}{2\cdot 4\pi\epsilon_0\epsilon_r}\underbrace{\int_a^b \frac{1}{r^2}\, dr}_{-\frac{1}{r}\Big|_a^b = \frac{1}{a} - \frac{1}{b}}$$

$$\boxed{U = \frac{Q^2}{8\pi\epsilon_0\epsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

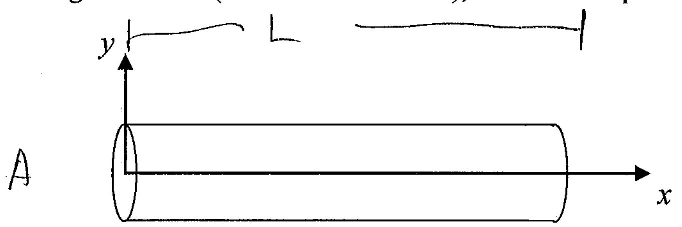
(c) compare to

$$U = \frac{Q^2}{2C} = \frac{Q^2}{2}\cdot\frac{1}{4\pi\epsilon_0\epsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right)$$

exactly the same!

This makes sense as the energy stored in a capacitor can be thought of as stored in the field. Refers to the same thing.

(17 pts) **Problem 4.** A cylinder is positioned on the x-axis, as shown. Its length is L and its cross-sectional area is A. The cylinder has a polarization that changes along its length: $\mathbf{P} = (ax^2 + b)\hat{\mathbf{x}}$. Find the bound charge densities (volume and surface), and show explicitly that the total bound charge adds up to zero.



$$\rho_b = -\nabla \cdot \vec{P}$$

$$= -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y}\right)$$

$$\boxed{\rho_b = -2ax}$$

→ amount of charge from this

$$Q = \int \rho \, d\tau$$

$$= \int_0^L (-2ax)\, A\, dx$$

$$= -2aA \underbrace{\int_0^L x\, dx}_{\frac{1}{2}L^2}$$

$$Q = -aAL^2$$

$$\sigma_b = \vec{P} \cdot \hat{n}$$

wrapper: $\boxed{\sigma_b = 0}$ for wrapper because $\vec{P} \perp \hat{n}$

right: $\sigma_b \big|_{x=L} = \vec{P} \cdot \hat{x} \big|_{x=L}$

$$\boxed{\sigma_b = aL^2 + b}$$

→ amount of charge from this

$$Q = \int \sigma \, da$$

$$= \sigma A$$

$$= (aL^2 + b)A$$

left: $\sigma_b \big|_{x=0} = \vec{P} \cdot (-\hat{x}) \big|_{x=0}$

$$\boxed{\sigma_b = -b}$$

→ amount of charge from this

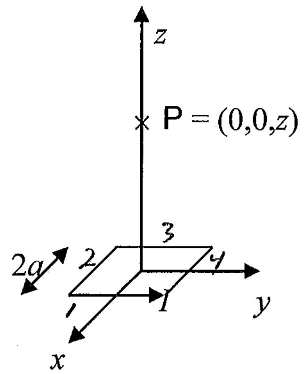
$$Q = \int \sigma \, da = \sigma A$$

$$= -bA$$

$$Q_{total} = -aAL^2 + (aL^2+b)A - bA$$

$$\boxed{Q_{total} = 0}$$ ✓

(20 pts) **Problem 5**. A current I travels around a square loop (side $2a$) in the x-y plane as shown. Using the Biot-Savart Law, set up an integral that you could use to determine the magnetic field at point P on the z-axis. Use symmetry to make the integral as simple as possible.



From symmetry: $\vec{B}$ will be in $\hat{z}$

all four sides will contribute equally

Considering side 1:

$$\vec{r} = z\hat{z}$$

$$\vec{r}\,' = a\hat{x} + y'\hat{y}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = -a\hat{x} - y'\hat{y} + z\hat{z}$$

$$\mathscr{r} = \sqrt{a^2 + y'^2 + z^2}$$

$$d\vec{l} = dy'\hat{y}$$

Biot-Savart

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$

zero from cross product; unimportant by symmetry

$$B_{tot} = 4\, B_{1z} = 4\frac{\mu_0}{4\pi} I \int_{-a}^{a} \frac{(dy'\hat{y}) \times (-a\hat{x} - \cancel{y'\hat{y}} + \cancel{z\hat{z}})}{(a^2 + y'^2 + z^2)^{3/2}}$$

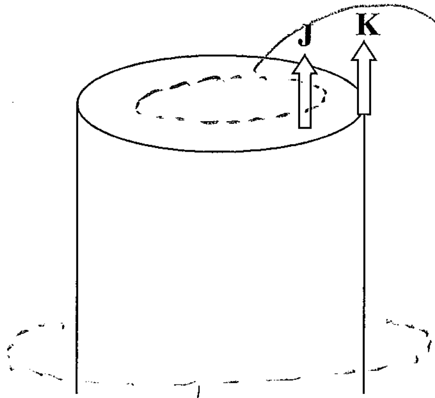
$$\boxed{\vec{B}_{tot} = \frac{\mu_0 I \hat{z}\, a}{\pi} \int_{-a}^{a} \frac{dy'}{(a^2 + y'^2 + z^2)^{3/2}}}$$

or even

$$= \frac{2\mu_0 I \hat{z}\, a}{\pi} \int_{0}^{a} \frac{dy'}{(a^2 + y'^2 + z^2)^{3/2}}$$

if you'd like to say left half of integral will give same as right half

(15 pts) **Problem 6**. An infinitely long solid cylinder of radius R carries volume and surface current densities as shown, $\mathbf{J} = \frac{k_1}{s}\hat{\mathbf{z}}$ and $\mathbf{K} = k_2\hat{\mathbf{z}}$. Find $\mathbf{B}$ everywhere due to these current densities.



inside Amperian loop

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\hookrightarrow I_{enc} = \int J\,da$$

$$= \int_0^s \left(\frac{k_1}{s}\right) 2\pi s\,ds$$

$$= \underline{2\pi k_1 s}$$

$$B\cdot \cancel{2\pi}\cancel{s} = \mu_0 \cancel{2\pi} k_1 \cancel{s}$$

$$\boxed{\vec{B} = \mu_0 k_1 \hat{\phi}} \qquad s < R$$

outside Amperian loop

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\hookrightarrow I_{enc} = \int J\,da + \int K\,dl$$

$$= \int_0^R \left(\frac{k_1}{s}\right) 2\pi s\,ds + (k_2)(2\pi R)$$

$$= 2\pi k_1 R + 2\pi k_2 R$$

$$= 2\pi R\,(k_1 + k_2)$$

$$B\cdot \cancel{2\pi} s = \mu_0 \cancel{2\pi} R\,(k_1 + k_2)$$

$$\boxed{\vec{B} = \mu_0 \frac{R}{s}(k_1 + k_2)\,\hat{\phi}}$$

(18 pts) **Problem 7**. A vector potential **A** exists, such that (in cylindrical coordinates):

$$\mathbf{A} = \begin{cases} k_1 s\hat{\boldsymbol{\phi}} & \text{for } s < R \\ k_2 s^2\hat{\boldsymbol{\phi}} & \text{for } s > R \end{cases}$$

(a) What's the relationship between k_1 and k_2? (b) Find the magnetic field corresponding to this **A**, for both s < R and $s > R$. (c) Is this **A** in the Coulomb gauge? How do you know? (d) Is there a surface current density at $s = R$? How do you know?

(a) $\vec{A}$ is continuous at R, so $k_1 \cancel{R} = k_2 R^{\cancel{2}} \rightarrow \boxed{k_1 = k_2 R}$

(b) $\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{s}\left(\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right)\hat{z}$

$= \frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi)\right)\hat{z}$ since all other terms $= 0$

$s < R$ $\quad = \frac{1}{s}\left(\underbrace{\frac{\partial}{\partial s}(s \cdot k_1 s)}_{2k_1 s}\right)\hat{z}$

$\boxed{\vec{B} = 2k_1\hat{z}}$ $s < R$

$s > R$ $\quad = \frac{1}{s}\left(\underbrace{\frac{\partial}{\partial s}(s \cdot k_2 s^2)}_{3k_2 s^2}\right)\hat{z}$

$\boxed{\vec{B} = 3k_2 s\,\hat{z}}$ $s > R$

(c) Coulomb gauge requires $\nabla \cdot \vec{A} = 0$

test: $\nabla \cdot A = \frac{1}{s}\frac{\partial}{\partial s}(s\cancel{A_s}) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \cancel{\frac{\partial A_z}{\partial z}}$

no s or z components,

$= \frac{1}{s}\frac{\partial A_\phi}{\partial \phi}$

no phi dependence

$= 0$

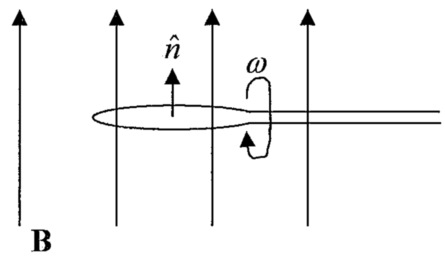
$\boxed{\text{Yes, it's Coulomb gauge}}$

(d) B is discontinuous at $s = R$

since $B_{\text{inside}} = 2k_1$ and $B_{\text{outside}} = 3k_2 R = 3k_1$

Therefore $\boxed{\text{there is a surface current}}$ (and magnitude $= \Delta B/\mu_0 = k_1/\mu_0$)

(14 pts) **Problem 8.** A hydroelectric generator is made by using water force to continuously rotate a loop of wire (radius R) through a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ produced by permanent magnets, at an angular frequency of ω rad/s. The normal to the surface of the loop changes continuously: $\hat{\mathbf{n}} = \hat{\mathbf{z}}\cos\omega t + \hat{\mathbf{x}}\sin\omega t$. (a) Calculate the magnetic flux passing through the loop, as a function of time. (b) Use Faraday's Law to calculate the AC voltage produced by the generator. (c) At $t = 0$, is the induced current clockwise or counter-clockwise (viewed from above)?



(a) $\Phi_B = \int \vec{B} \cdot d\vec{a}$

$= \int (B_0 \hat{z}) \cdot (\hat{z}\cos\omega t + \hat{x}\sin\omega t)\, da$

$= B_0 \cos\omega t \int da$

$$\boxed{\Phi_B = B_0 \cos\omega t\, \pi R^2}$$

(b) $\mathcal{E} = -\dfrac{d\Phi_B}{dt}$

$$\boxed{\mathcal{E} = +\omega B_0 \sin\omega t\, \pi R^2}$$

(two negatives cancel out)

(c) At $t=0$, area is max and decreasing.

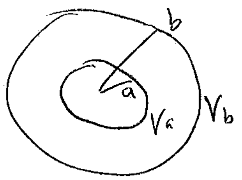
Therefore current must be $\boxed{\text{CCW}}$ by Lenz's law.

This is consistent with right hand rule and + sign of $\mathcal{E}$

(point thumb in $+\hat{z}$ direction, fingers curl CCW)

(10 pts) **Problem 9, extra credit.** **<u>No partial credit</u>, so you need to decide if this is worth your time.** Two concentric spheres of radii a and b are each held at a constant potential, V_a and V_b respectively. Determine $V(r)$ in the region between the two spheres, using the technique of separation of variables in spherical coordinates to obtain a solution which matches the boundary conditions. It's OK if you jump directly to the solutions for Laplaces's equation in spherical coordinates with no ϕ dependence as done in class, namely $R(r) = Ar^\ell + \frac{B}{r^{\ell+1}}$ and $\Theta(\theta) = C\,P_\ell(\cos\theta) + D\,Q_\ell(\cos\theta)$, where $Q_\ell(x)$ are the Legendre functions of the second kind (which are infinite at $x = \pm 1$) and the other symbols should be self-explanatory.

$\rightarrow$ D must $= 0$ since solution shouldn't blow up at $\theta = 0, 180°$



In region between, $\nabla^2 V = 0$

Spherical coords $\rightarrow$ $V = R(r)\,\Theta(\theta)$

$$\underline{V = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)}$$ general soln

bndy conds: 1) $V(r=a) = V_a$
2) $V(r=b) = V_b$
$\Big\}$ can't eliminate either A_ℓ or B_ℓ coefficients

BC 1) $$V_a = \sum_\ell \left(A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}}\right) P_\ell(\cos\theta)$$

Can multiply in a $P_0(\cos\theta)\ (=1)$, Then use orthogonality

$$V_a \underbrace{\int P_0(\cos\theta)\,P_{\ell'}(\cos\theta)\sin\theta\,d\theta}_{\substack{=0 \text{ unless } \ell'=0 \\ = \frac{2}{2\cdot 0+1} = 2 \text{ if } \ell'=0}} = \sum_\ell \left(A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}}\right) \underbrace{\int P_\ell(\cos\theta)\,P_{\ell'}(\cos\theta)\sin\theta\,d\theta}_{\substack{=0 \text{ unless } \ell=\ell' \\ = \frac{2}{2\ell'+1} \text{ if } \ell=\ell' \\ \rightarrow = 2 \text{ since } \ell'=0}}$$

$$\rightarrow V_a \cancel{2} = \left(A_0 a^0 + \frac{B_0}{a^1}\right)\cancel{2} \Rightarrow \underline{\underline{V_a = A_0 + \frac{B_0}{a}}}$$

BC 2) Similarly...

$$V_b \underbrace{\int P_0(\cos\theta)\,P_{\ell'}(\cos\theta)\sin\theta\,d\theta}_{\substack{=0 \text{ unless } \ell'=0 \\ = \frac{2}{2\cdot 0+1} = 2 \text{ if } \ell'=0}} = \sum_\ell \left(A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}}\right) \underbrace{\int P_\ell(\cos\theta)\,P_{\ell'}(\cos\theta)\sin\theta\,d\theta}_{\substack{=0 \text{ unless } \ell=\ell' \\ = \frac{2}{2\ell'+1} \text{ if } \ell=\ell' \\ = 2 \text{ since } \ell'=0}}$$

$$\rightarrow V_b \cancel{2} = \left(A_0 b^0 + \frac{B_0}{b^1}\right)\cancel{2} \rightarrow \underline{\underline{V_b = A_0 + \frac{B_0}{b}}}$$

cont on next page

2 eqns, 2 unknowns $\longrightarrow$ solve simultaneous

$$A_0 = V_a - B_0/a$$

$$\rightarrow \quad V_b = \left(V_a - B_0/a \right) + \frac{B_0}{b}$$

$$-\quad (V_b - V_a) = B_0 \left(\frac{1}{a} - \frac{1}{b} \right)$$

$$(V_a - V_b) = B_0 \left(\frac{b-a}{ab} \right)$$

$$B_0 = \frac{(V_a - V_b)\, ab}{b-a}$$

Then $$A_0 = V_a - \frac{1}{\cancel{a}} \left(\frac{(V_a - V_b)\, \cancel{a} b}{b-a} \right)$$

$$A_0 = \frac{V_a (\cancel{b} - a) - (\cancel{V_a} - V_b) b}{b-a}$$

$$A_0 = \frac{V_b b - V_a a}{b-a}$$

Final answer:

$$\boxed{V = \frac{V_b b - V_a a}{b-a} + \frac{(V_a - V_b)\, ab}{b-a} \frac{1}{r}}$$

Check when $r = a$, $V = V_a$ ✓

when $r = b$, $V = V_b$ ✓

also satisfies Laplace eqn

therefore (uniqueness) it must be correct answer :)