Quadrupole Moment Potential and Tensor

by Dr. Colton, Physics 441 (last updated: Fall 2022)

Introduction

Each of the terms in the multipole expansion, Griffiths Eq (3.95), can be separated into two pieces: one that describes the source (which does not depend on **r**), and one that describes how that source produces a potential (which does depend on **r**). For example, for the monopole term, $n = 0$, we have:

$$Q = \int \rho(\mathbf{r}')d\tau' \tag{1}$$

$$V_{mono} = \frac{1}{4\pi\epsilon_0 r} Q \tag{2}$$

Here ρ is the charge density function and $\mathbf{r}'$ represents the locations where the charges are (which varies over the integral). Q, which is the net charge, can also be called the "monopole moment". All of the information about the charge distribution is contained in the calculation for (1). All of the information about the field point is contained in the calculation for (2).

Similarly, for the dipole term $n = 1$, we have:

$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau' \tag{3}$$

$$V_{dip} = \frac{1}{4\pi\epsilon_0 r^2} \mathbf{p} \cdot \hat{\mathbf{r}} \tag{4}$$

Here **p** is called the "dipole moment". Again, all of the information about the charge distribution is contained in the calculation for (3) and all of the information about the field point is contained in the calculation for (4).

Notice that whereas the monopole moment involved a single number, the dipole moment, being a vector, actually involves a collection of three numbers: (p_x, p_y, p_z). Also, notice how the dipole term falls off faster than the monopole term by an additional power of r: $1/r^2$ compared to $1/r$. Those trends continue: for the quadrupole term we will see that (a) the quadrupole moment will involve a collection of even more numbers than the dipole moment; specifically, it will involve nine numbers assembled in what we call a tensor (basically a matrix), and (b) the quadrupole potential will fall off by an additional power of r, namely $1/r^3$.

Derivation

Setting $n = 2$ in Griffiths Eq (3.95) and recognizing that $P_2(\cos\alpha) = \frac{3}{2}\cos^2\alpha - \frac{1}{2}$, we obtain the quadrupole term:

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} \int r'^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) \rho(\mathbf{r}')d\tau' \tag{5}$$

Here, α is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

Working with the two terms inside the integral, we have:

First term

$$\frac{3}{2}r'^2\cos^2\alpha = \frac{3}{2}(r'\cos\alpha)^2 \tag{6}$$

Note that $r'\cos\alpha = \hat{\mathbf{r}}\cdot\mathbf{r}'$ which can be written as a summation, $\sum_i \hat{r}_i \cdot r_i'$. In other words, $\hat{r}_i$ for $i = x, y, z$ represents Cartesian components of the unit vector $\hat{\mathbf{r}}$.

Because we have two different $r'\cos\alpha$ terms multiplied together, we'll have two summations. We can write the second one as being over j.

$$\frac{3}{2}r'^2\cos^2\alpha = \frac{3}{2}\sum_{i,j}\hat{r}_i r_i' \hat{r}_j r_j' \tag{7}$$

Second term

$$\frac{1}{2}r'^2 = \frac{1}{2}r'^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) \tag{8}$$

This is true since $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}$ is just equal to one. We can then write $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}$ as a summation, $\sum_i \hat{r}_i \cdot \hat{r}_i$, and turn it into a double summation using the Kronecker delta function δ_{ij}: $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \sum_{i,j}\hat{r}_i\hat{r}_j\delta_{ij}$. Thus we have

$$\frac{1}{2}r'^2 = \sum_{i,j}\frac{1}{2}r'^2\hat{r}_i\hat{r}_j\delta_{ij} \tag{9}$$

Piecing together

Eq (5) then turns into the following:

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}\sum_{i,j}\int\left(\frac{3}{2}\hat{r}_i r_i'\hat{r}_j r_j' - \frac{1}{2}r'^2\hat{r}_i\hat{r}_j\delta_{ij}\right)\rho(\mathbf{r}')d\tau' \tag{10}$$

where I've interchanged the order of the integral and the summation. Next I'll pull out $\hat{r}_i\hat{r}_j$ from each terms and put in front of the integral because the integral is over the primed coordinates:

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}\sum_{i,j}\hat{r}_i\hat{r}_j\left\{\int\left(\frac{3}{2}r_i' r_j' - \frac{1}{2}r'^2\delta_{ij}\right)\rho(\mathbf{r}')d\tau'\right\} \tag{11}$$

The stuff in the curly braces no longer has any $\mathbf{r}$ dependence! That was exactly our goal—we have separated out the source information. Eq (11) can now be written as two separate equations.

Quadrupole moment tensor

$$\boxed{Q_{ij} = \int\left(\frac{3}{2}r_i' r_j' - \frac{1}{2}r'^2\delta_{ij}\right)\rho(\mathbf{r}')d\tau'} \tag{12}$$

The stuff in the curly braces from Eq 11 is called the "quadrupole moment tensor", and given the unfortunate symbol Q_{ij} (Q for "quadrupole") which is not to be confused with Q the net charge. Q_{ij} is property of only the charge distribution and does not depend on the field point at all. It is a collection of nine numbers (i and j each can be x, y, z) which can be written in matrix form as a 3×3 array if you like, analogous to how **p** can be written as a 3-element vector. Moreover, the formula in Eq (12) is symmetric in reversing i and j, so there are only six independent matrix elements. In matrix form it can be written like this:

$$\mathbf{Q_{ij}} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{xy} & Q_{yy} & Q_{yz} \\ Q_{xz} & Q_{yz} & Q_{zz} \end{pmatrix} \qquad (13)$$

Quadrupole potential

$$\boxed{V_{quad}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} \sum_{i,j} \hat{r}_i \hat{r}_j \, Q_{ij}} \qquad (14)$$

Once the quadrupole moment tensor is known, the potential can be determined via Eq (14). Notice the $1/r^3$ dependence. Recall that $\hat{r}_i$ represents the three Cartesian components of the unit vector pointing to the field point, $\hat{\mathbf{r}}$.

The similarities between the quadrupole potential equation and the dipole potential equation can be made even more striking by rewriting the dot product in Eq (4) in terms of a summation:

$$V_{dip} = \frac{1}{4\pi\epsilon_0 r^2} \sum_i \; \hat{r}_i \, p_i \qquad (15)$$

Disclaimer: Eqs (12) and (14) are given in the 4th edition of Griffiths problem 3.52. By contrast, the 3rd edition (in problem 3.45) defines Q_{ij} without the factor of 1/2 in the two terms, choosing instead to include an extra 1/2 in the equation for V_{quad}. A check of the internet reveals some continued disagreement on this, but most sources actually seem to favor the 3rd edition version of the equations. However, I personally prefer the 4th edition equations so that's what I've used in this handout.

Extensions

The next term of the multipole expansion will give rise to an "octopole" potential and an octopole tensor O_{ijk} (O for octopole) which will contain 27 terms in a 3×3×3 array of numbers. Recognizing the patterns in Eqs (12) and (14), we can write them as the following:

$$O_{ijk} = \int \left(\tfrac{5}{2} r'_i r'_j r'_k - \tfrac{3}{2}[stuff]\right) \rho(\mathbf{r}') d\tau' \qquad (16)$$

$$V_{oct}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^4} \sum_{i,j,k} \hat{r}_i \hat{r}_j \hat{r}_k \, O_{ijk} \qquad (17)$$

For the sake of completeness the "stuff" in Eq (16) is the following: $r'^2\left(r'_i \delta_{jk} + r'_j \delta_{ik} + r'_k \delta_{ij}\right)$, but you are likely to never need to know that.

Similarly, there will be a "hexadecapole" term, described by a 81 element tensor H_{ijkl} which will contain remnants of the Legendre polynomial P_3, along with a $V_{hexadec}$ equation which varies as $1/r^5$ and involves an 81 term summation over i,j,k, and l.