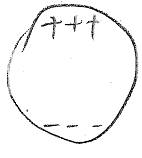
Problem

Given an insulating spherical shell,
$\sigma = \sigma_0 \cos\theta$



What is $V(r,\theta)$? (inside and out)

BC 1: at $r=0$, V is not infinite

BC 2: at $r=R$, $V_{inside} = V_{outside}$

BC 3: at $r=R$, $E_{\perp outside} - E_{\perp inside} = \frac{\sigma}{\epsilon_0}$

BC 4: at $r=\infty$, V is not infinite

Method: use SOV in spherical coordinates

$$V = R(r)\,\Theta(\theta)$$

Solutions are $R = A r^{\ell} + \frac{B}{r^{\ell+1}}$

$$\Theta = P_\ell(\cos\theta)$$

and linear combinations (throw out the $Q_\ell(\cos\theta)$ solutions)

inside BC 1 $\rightarrow$ $B=0$ for inside potential

Form of solution is $\boxed{V_{inside}(r,\theta) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)}$

outside BC 4 $\rightarrow$ $A=0$ for outside potential

Form of solution is $\boxed{V_{outside}(r,\theta) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)}$

connecting inside and outside use BC 2 and BC 3

BC 2 $\quad V_{in}\Big|_{r=R} = V_{out}\Big|_{r=R}$

$$\sum_\ell A_\ell R^\ell P_\ell = \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell$$

equate coeffs of the same P_ℓ

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}}$$

(hold that thought)

BC 3 $$-\frac{\partial V_{out}}{\partial r}\Big|_{r=R} + \frac{\partial V_{in}}{\partial r}\Big|_{r=R} = \frac{\sigma}{\epsilon_0} \rightarrow \begin{matrix} = \sigma_0 \cos\theta \\ = \sigma_0 P_1 \end{matrix}$$

$$-\sum_\ell B_\ell (-\ell-1) r^{-\ell-2} P_\ell \Big|_{r=R} + \sum_\ell A_\ell \ell r^{\ell-1} P_\ell \Big|_{r=R} = \frac{\sigma_0 P_1}{\epsilon_0}$$

$$+\sum_\ell \left(B_\ell (\ell+1) \frac{1}{R^{\ell+2}} + A_\ell \ell R^{\ell-1} \right) P_\ell = \frac{\sigma_0}{\epsilon_0} P_1$$

Equate coeffs → only $\ell = 1$ term survives!
all other A_ℓ and $B_\ell = 0$

$$B_1 (1+1) \frac{1}{R^{1+2}} + A_1 (1) R^{1-1} = \frac{\sigma_0}{\epsilon_0}$$

$$2\frac{B_1}{R^3} + A_1 = \frac{\sigma_0}{\epsilon_0}$$

Combine with upper eqn (with $\ell = 1$ specifically) to get
2 eqns, 2 unknowns → solve for A_1 and B_1

upper eqn: $A_1 R = \frac{B_1}{R^2} \rightarrow B_1 = A_1 R^3$

plug into lower eqn: $2\frac{(A_1 \cancel{R^3})}{\cancel{R^3}} + A_1 = \frac{\sigma_0}{\epsilon_0}$

$$3A_1 = \frac{\sigma_0}{\epsilon_0}$$

$$\boxed{A_1 = \frac{\sigma_0}{3\epsilon_0}}$$

then $\boxed{B_1 = \frac{\sigma_0}{3\epsilon_0} R^3}$

We're done! all other A_ℓ and B_ℓ terms are zero.

Just plug into boxed eqns on pg 1

and use $P_1(\cos\theta) = \cos\theta$

$$\boxed{V_{inside}(r,\theta) = \frac{\sigma_0}{3\epsilon_0} r\cos\theta}$$

$$\boxed{V_{outside}(r,\theta) = \frac{\sigma_0}{3\epsilon_0} \frac{R^3}{r^2} \cos\theta}$$

Compare to answers of previous problem, very similar!

and hopefully reasons for differences are clear.