Legendre Polynomials, by Dr Colton
Physics 441

The Legendre polynomials, $P_\ell(x)$ are a series of polynomials of order ℓ, $A + Bx + Cx^2 + \ldots + Zx^\ell$, that:

(a) come up often, especially in partial differential equations
(b) have interesting properties
(c) are well understood and have been studied for centuries

Here they are:

LegendreP[ℓ,x] is built into *Mathematica*, just like Sin[x]

```
In[1]:= P0[x_] = LegendreP[0, x]
Out[1]= 1
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$P_0(x)$

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In[2]:= P1[x_] = LegendreP[1, x]
Out[2]= x
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$P_1(x)$

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In[3]:= P2[x_] = LegendreP[2, x]
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Out[3]= $\frac{1}{2}\left(-1+3x^2\right)$

$P_2(x)$

Sometimes the polynomials are written in terms of $x = \cos\theta$, e.g.

$$P_2(\cos\theta) = -\frac{1}{2} + \frac{3\cos^2\theta}{2}$$

```
In[4]:= P3[x_] = LegendreP[3, x]
```

Out[4]= $\frac{1}{2}\left(-3x+5x^3\right)$

$P_3(x)$

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In[5]:= P4[x_] = LegendreP[4, x]
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Out[5]= $\frac{1}{8}\left(3-30x^2+35x^4\right)$

$P_4(x)$

```
In[6]:= P5[x_] = LegendreP[5, x]
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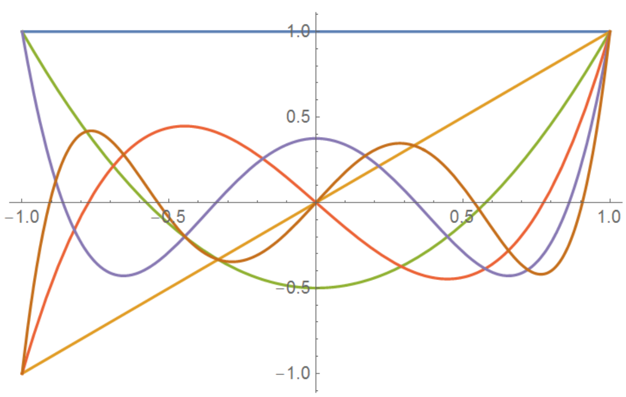
Out[6]= $\frac{1}{8}\left(15x-70x^3+63x^5\right)$

$P_5(x)$

Etc.

Plots:

```
In[7]:= Plot[{P0[x], P1[x], P2[x], P3[x], P4[x], P5[x]}, {x, -1, 1}]
```

Out[7]=


The subscript ℓ is the order of the polynomial = highest power of x.

The functions alternate even/odd.

A given polynomial has either only odd or only even powers of x.

They all go to either 1 or –1 on the boundary when plotted in the range from $-1 \le x < 1$.

Very important facts

- Can be computed via the "Rodriguez formula": $P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$ or via summation $P_\ell(x) = 2^\ell \cdot \sum_{k=0}^{\ell} x^k \binom{\ell}{k}\binom{(\ell+k-1)/2}{\ell}$ where $\binom{\ell}{k}$ is the binomial coefficient "ℓ choose k".

- Orthogonality: $\int_{-1}^{1} P_\ell(x)\, P_m(x) dx = \begin{cases} 0 & \text{if } \ell \neq m \\ \frac{2}{2\ell+1} & \text{if } \ell = m \end{cases}$

    - If $x = \cos\theta$, then $dx = -\sin\theta d\theta$; use negative sign to switch limits then we have:
      $\int_0^\pi P_\ell(\cos\theta)\, P_m(\cos\theta) \sin\theta\, d\theta = \begin{cases} 0 & \text{if } \ell \neq m \\ \frac{2}{2\ell+1} & \text{if } \ell = m \end{cases}$

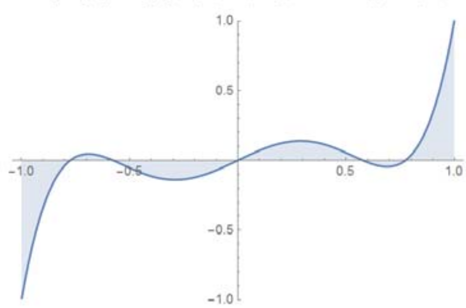
- Differential equation: $\frac{d}{dx}\left((1-\mathrm{x}^2)\frac{df}{dx}\right) + \ell(\ell+1)f = 0$ has solution $f = P_\ell(x)$, or linear combinations, $f = \sum_{\ell=0}^{\infty} A_\ell\, P_\ell(\mathrm{x})$
    - If $x = \cos\theta$, then equation is $\frac{d}{d\theta}\left(\sin\theta \frac{df}{d\theta}\right) = -\ell(\ell+1)\sin\theta\, f$, solution is $f = P_\ell(\cos\theta)$.

- There are a second set of solutions to that differential equation, called $Q_n(x)$, the "Legendre functions of the second kind"; these diverge at $x = \pm 1$ and are therefore often not used.

Orthogonality, depicted

```
In[8]:= Plot[P2[x] * P3[x], {x, -1, 1}, PlotRange → {-1, 1}, Filling → Axis]
```

Out[8]=

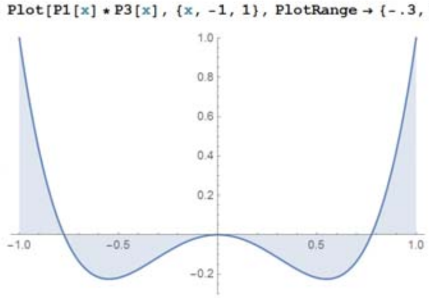



```
In[9]:= Integrate[P2[x] * P3[x], {x, -1, 1}]
Out[9]= 0
```

$\int_{-1}^{1} P_2(x)P_3(x)dx = ?$
The product of an even and an odd function is odd, so this will clearly integrate to zero.

```
In[10]:= Plot[P1[x] * P3[x], {x, -1, 1}, PlotRange → {-.3, 1}, Filling → Axis]
```

Out[10]=




```
In[11]:= Integrate[P1[x] * P3[x], {x, -1, 1}]
Out[11]= 0
```

$\int_{-1}^{1} P_1(x)P_3(x)dx = ?$
Here the product is an even function, and so isn't so obvious that it integrates to zero. But you can see the positive areas still cancel out the negative areas.

$\int_{-1}^{1} P_n(x)P_m(x)dx$ always equals 0 if $n \neq m$!

Comparison with sines & cosines:

	Sines/Cosines	**Bessel functions**
1.	Two oscillatory functions: $\sin(x)$ and $\cos(x)$. Sometimes one of them is not used, due to the symmetry of the problem.	Two functions for each ℓ: $P_\ell(x)$ and $Q_\ell(x)$. Typically Q_ℓ is not used because it's infinite at the boundaries.
2.	You typically determine the value of $\sin(x)$ or $\cos(x)$ for arbitrary x by using a calculator or computer program.	You typically determine the value of $P_\ell(x)$ for arbitrary x by using a calculator or computer program.
3.	$\sin(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$ $\cos(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$	$P_\ell(x)=\frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$, or $P_\ell(x)=2^\ell\cdot\sum_{k=0}^{\ell} x^k\binom{\ell}{k}\binom{(\ell+k-1)/2}{\ell}$
4.	$\sin(m\pi x)$ = odd for all m. $\cos(m\pi x)$ = even for all m.	$P_\ell(x)$ = odd for odd ℓ. $P_\ell(x)$ = even for even ℓ.
5.	At $x = 1$, $\sin(m\pi x) = 0$ for all m. At $x = 1$, $\cos(m\pi x) = \pm 1$ for all m.	At $x = 1$, $P_\ell(x) = 1$ for all ℓ. At $x = -1$, $P_\ell(x) = \pm 1$ for all ℓ.
6.	The differential equation satisfied by $f = \sin(m\pi x)$ is $f''+(m\pi)^2 f=0$	The differential equation satisfied by $f = P_\ell(x)$ is $\frac{d}{dx}\left((1-\mathrm{x}^2)\frac{df}{dx}\right)+\ell(\ell+1)f=0$
7.	$\sin(n\pi x)$ is orthogonal to $\sin(m\pi x)$ on the interval (0,1): $\int_0^1 \sin(n\pi x)\sin(m\pi x)dx=\begin{cases}0, if\ n\neq m\\ \frac{1}{2}, if\ n=m\end{cases}$	$P_\ell(x)$ is orthogonal to $P_m(x)$ on the interval (–1,1): $\int_{-1}^{1}P_\ell(x)\,P_m(x)dx=\begin{cases}0 & \text{if } \ell\neq m\\ \frac{2}{2\ell+1} & \text{if } \ell=m\end{cases}$ $P_\ell(\cos\theta)$ is orthogonal to $P_m(\cos\theta)$ on the interval (0,π), with respect to a "weighting" of $\sin\theta$: $\int_0^{\pi}P_\ell(\cos\theta)\,P_m(\cos\theta)\sin\theta\,d\theta=\begin{cases}0 & \text{if } \ell\neq m\\ \frac{2}{2\ell+1} & \text{if } \ell=m\end{cases}$