

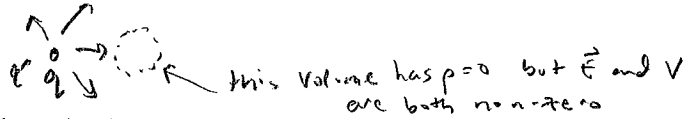
# Phys 441 Exam 2 Solns

(18 pts) **Problem 1:** Multiple choice, 2 pts each. Circle the correct answer.

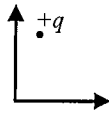
1.1. Poisson's equation tells us that  $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ . If the charge density throughout some volume is zero, what else must be true throughout that volume:

- (a)  $V = 0$
- (b)  $\mathbf{E} = 0$
- (c) Both  $V$  and  $\mathbf{E}$  must be zero
- (d) None of the above is necessarily true.

Counter example



1.2. A positive charge  $+q$  is close to two grounded conducting semi-infinite planes, as shown (the arrows are conductors which extend infinitely into and out of the page, as well as infinitely in the direction of the arrows).



Which of the following image configurations could be used to determine the electric potential near the charge  $q$ ?

(a)

(b)

(c)

(d)

(e) None of them.

the vertical symmetry makes  $V=0$  on this plane

the horizontal symmetry makes  $V=0$  on this plane

1.3. Which of the following are true statements about a function which is a solution to Laplace's equation in three dimensions?

- I. The function's value at a given point is the average of the surrounding values of a spherical surface centered at the point.
- II. Separation of variables always leads to the function involving products of sines and/or cosines, and exponentials. *Not true. Example: spherical coords give  $A r^n + B r^{-(n+1)}$  and  $P_n(\cos \theta)$  functions*
- III. There are no local maxima or minima; extrema of the function exist only at the boundaries.

- (a) I only
- (b) I and II only
- (c) I and III only
- (d) II and III only
- (e) I, II, and III

1.4. True/False: The following integral is zero due to the orthogonality of the Legendre polynomials:

$$\int_{-1}^1 P_3(x) P_5(x) dx$$

- (a) True
- (b) False

1.5. True/False: The following integral is zero due to the orthogonality of the Legendre polynomials:

$$\int_0^\pi P_3(\cos \theta) P_5(\cos \theta) d\theta$$

- (a) True  
 (b) False

↑  
 you'd need an additional  $\sin \theta$  here for it to be true

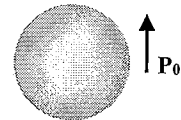
1.6. A spherical surface of radius  $R$  is somehow maintained at the following potential:  $V(\theta) = V_0(\cos^4 \theta + 1)$ . The formula for the potential inside the surface will involve:

- (a) An infinite sum of Legendre polynomials in  $\cos \theta$ . Since  $\cos^4 \theta + 1$  can be written as a finite sum of Legendre polynomials, this is like the problem we worked in class where you end up with a finite series.  
 (b) A finite sum of Legendre polynomials in  $\cos \theta$ . (Specifically  $\cos^4 \theta + 1 = \frac{8}{35} P_4 + \frac{4}{7} P_2 + \frac{6}{5} P_0$ , but you don't need to know that to answer this.)

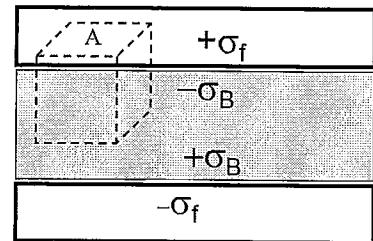
1.7. A sphere of radius  $R$  has a uniform polarization  $\mathbf{P} = P_0 \hat{z}$ . What is the total dipole moment of the sphere?

- (a) zero  
 (b)  $P_0$   
 (c)  $4\pi P_0$   
 (d)  $P_0 R^3$   
 (e)  $4\pi P_0 R^3/3$   
 (f) None of the above.

$$q = \int \mathbf{P} \cdot d\mathbf{T} = \mathbf{P} \times \text{Volume} = P_0 \cdot \frac{4}{3} \pi R^3$$



1.8. A parallel plate capacitor is charged up (charge  $Q$ , area  $A$ , charge density  $\sigma_f = Q/A$ ) with a dielectric present between the plates (dielectric constant  $\epsilon_r$ ). The dielectric polarizes, and bound charge densities  $+\sigma_B$  and  $-\sigma_B$  are produced as shown. What is the magnitude of the  $\mathbf{D}$  field in the dielectric? (Ignore fringing fields as usual.)



- (a)  $\sigma_B$   
 (b)  $2\sigma_B$   
 (c)  $\sigma_B/2$   
 (d)  $\sigma_f$   
 (e)  $2\sigma_f$   
 (f)  $\sigma_f/2$   
 (g)  $\sigma_f + \sigma_B$   
 (h)  $2(\sigma_f + \sigma_B)$   
 (i)  $(\sigma_f + \sigma_B)/2$

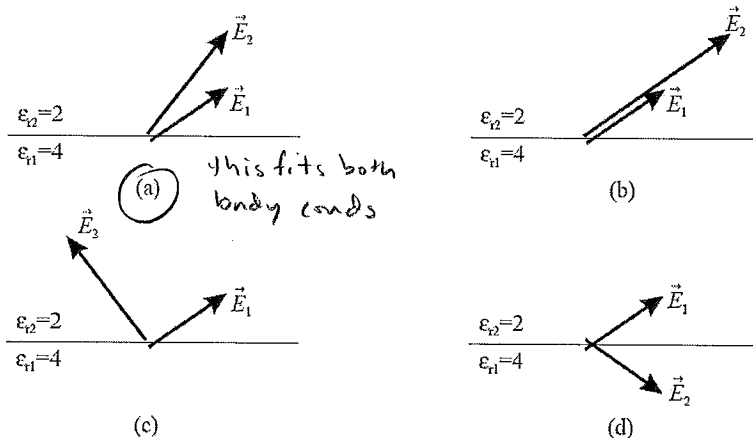
$$\oint \mathbf{D} \cdot d\mathbf{a} = q_{\text{free enc}}$$

$$D \cdot A = q_{\text{free}}$$

$$D \approx \frac{q_{\text{free}}}{A}$$

$$D = \sigma_f$$

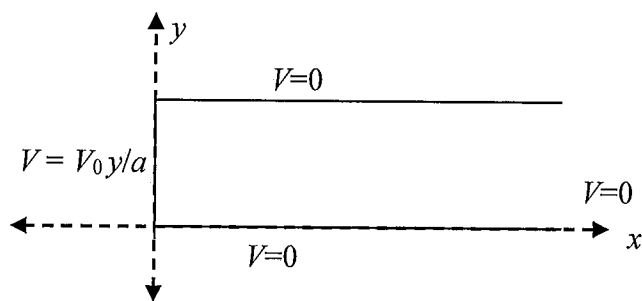
1.9. Assuming that there is no free surface charge on the boundary between the two dielectric media shown, which of the figures represents possible electric field intensity vectors on the two sides of the boundary?



(a) this fits both boundary conds

boundary conds  
 $D_{\perp \text{ above}} - D_{\perp \text{ below}} = \sigma_{\text{free}}$   
 $\rightarrow D_{\perp \text{ above}} = D_{\perp \text{ below}}$   
 or  $\frac{E_{\perp 1}}{\epsilon_1} = \frac{E_{\perp 2}}{\epsilon_2}$   
 $\rightarrow E_{\perp 2} = \frac{\epsilon_1}{\epsilon_2} E_{\perp 1}$   
 $E_{\parallel \text{ above}} = E_{\parallel \text{ below}}$

(18 pts) **Problem 2.** The figure below extends infinitely in the + and - z-directions (not shown), and in the + x-direction. The potential is held fixed along the sides as indicated: the upper and lower sides are held at 0, whereas the left-hand side changes linearly according to  $V_0 y/a$ . The goal of the problem is to find  $V(x,y)$  inside the boundary.



Boundary conditions

1.  $V = 0$  at  $y = 0$
2.  $V = 0$  at  $y = a$
3.  $V = 0$  at  $x = \infty$
4.  $V = V_0 y/a$  at  $x = 0$

I'll get you started solving this problem via separation of variables. First, you would write  $V(x,y) = X(x)Y(y)$ , and plug  $V(x,y)$  into Laplace's equation. That allows you to separate X and Y and (skipping some work) gives you sines/cosines and exponentials as your solutions to the X and Y equations.

BC 3 means that you must choose exponentials for X, and also that you must throw out the positive exponential as a potential solution. So the solution for X is:

$$X(x) = A e^{-kx}$$

The choice of exponentials for X means that you must get sines/cosines for Y. BC 1 means that the cosines will not work, so you are left with sines for Y:

$$Y(y) = B \sin ky$$

Now you continue...

(a) What restriction is put on  $k$ , and why?

boundary cond (2): To force  $Y(y)$  and hence  $V(x,y)$  to be zero at  $y=a$ , we

say  $B \sin ka = 0$

$$ka = n\pi \rightarrow k = \frac{n\pi}{a}$$

(b) Write down the general form on an infinite series solution for  $V$  with unknown coefficients.

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x} \sin \frac{n\pi}{a}y$$

(c) Use BC 4 and "Fourier's Trick" to determine the value of the coefficient of an arbitrary term of the infinite sum.

BC 4: 
$$V_0 \frac{y}{a} = \sum_{n=1}^{\infty} C_n e^{\underbrace{-\frac{n\pi}{a} \cdot 0}_{=1}} \sin \frac{n\pi}{a} \cdot y$$

Fourier's Trick: 
$$\underbrace{V_0 \frac{y}{a} \int_0^a y \sin \frac{m\pi}{a} y \, dy}_{= \frac{(-1)^{1+m} 2}{m\pi} \text{ from integral on pg 1}} = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^a \sin \frac{n\pi}{a} y \sin \frac{m\pi}{a} y \, dy}_{\substack{= 0 \text{ if } n \neq m \\ = \frac{1}{2} \cdot a \text{ if } n = m}}$$

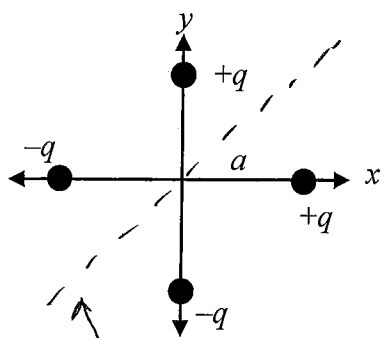
summation goes away, only the  $m^{\text{th}}$  term is left

$$\cancel{\frac{V_0}{a}} \cdot \frac{(-1)^{1+m} \cancel{a}^2}{m\pi} = C_m \cdot \frac{1}{2} \cancel{a}$$

$$C_m = \frac{2V_0}{m\pi} (-1)^{1+m}$$

think multipole expansion

(18 pts) **Problem 3:** Four charges exist each a distance  $a$  from the origin as shown below. Find the electric field at these two points (not shown), point 1 at  $(d, 0)$ , and point 2 at  $(0, d)$ , for  $d \gg a$  (i.e., very far away from where these charges are). (Just worry about the  $z = 0$  plane.)



No monopole

Yes dipole

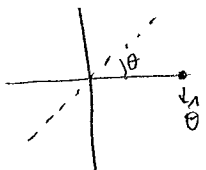
$$\begin{aligned}
 \vec{p} &= \sum_i q_i \vec{r}_i \\
 &= qa \hat{x} + qa \hat{y} - qa(-\hat{x}) - qa(-\hat{y}) \\
 &= 2qa (\hat{x} + \hat{y}) \quad (p = 2\sqrt{2}qa)
 \end{aligned}$$

dipole is along this axis, consider all angles, etc., relative to here

General dipole formula: 
$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos\theta \hat{r} + \sin\theta \hat{\theta})$$

point 1:  $\theta = 45^\circ$

$\hat{\theta} = -\hat{y}$   
 $r = d$   
 $\hat{r} = \hat{x}$

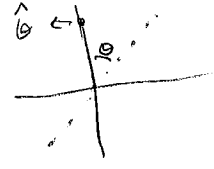


$$\vec{E} = \frac{2\sqrt{2}qa}{4\pi\epsilon_0 d^3} (2 \cos 45^\circ \hat{x} + \sin 45^\circ (-\hat{y}))$$

$$\vec{E} = \frac{qa}{2\pi\epsilon_0 d^3} (2\hat{x} - \hat{y})$$

point 2:

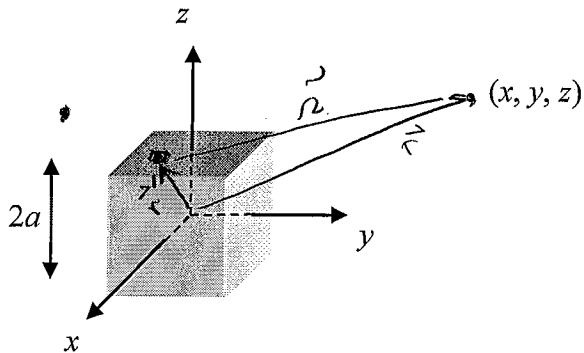
$\theta = 45^\circ$   
 $\hat{\theta} = -\hat{x}$   
 $r = d$   
 $\hat{r} = \hat{y}$



$$\vec{E} = \frac{2\sqrt{2}qa}{4\pi\epsilon_0 d^3} (2 \cos 45^\circ \hat{y} + \sin 45^\circ (-\hat{x}))$$

$$\vec{E} = \frac{qa}{2\pi\epsilon_0 d^3} (2\hat{y} - \hat{x})$$

(18 pts) **Problem 4.** A cube is centered at the origin, with the faces in the  $x$ ,  $y$ , and  $z$  axes as shown below. Each side has length of  $2a$ . It has a constant polarization in the  $z$ -direction,  $\vec{P} = P_0 \hat{z}$ . (a) Find  $\sigma_B$  (for all six surfaces) and  $\rho_B$ . (b) Use those bound charge densities to set up one or more integrals which could be used to find the potential,  $V$ , at some arbitrary point  $(x, y, z)$  by direct integration. The integral(s) should be in terms of only  $P_0$ ,  $\epsilon_0$ ,  $a$ ,  $x$ ,  $y$ ,  $z$ , and the variables of integration.



(a)  $\sigma_B = \vec{P} \cdot \hat{n}$

top:  $\hat{n} = \hat{z} \rightarrow \sigma_B = P_0$

bottom:  $\hat{n} = -\hat{z} \rightarrow \sigma_B = -P_0$

sides:  $\hat{n} \perp \hat{z} \rightarrow \sigma_B = 0$

$\rho_B = -\vec{\nabla} \cdot \vec{P} \rightarrow \rho_B = 0$  since  $\vec{P}$  is constant

(b)  $V = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma da}{r}$

For top surface,  $\vec{r}' = x\hat{x} + y\hat{y} + a\hat{z}$  as shown above

as usual  $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

so  $\vec{r} = \vec{r} - \vec{r}' = (x-x')\hat{x} + (y-y')\hat{y} + (z-a)\hat{z}$

and  $r = \sqrt{(x-x')^2 + (y-y')^2 + (z-a)^2}$

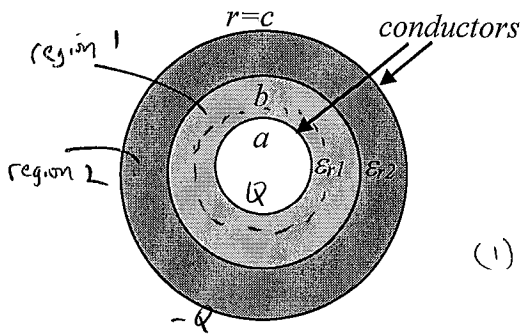
Bottom surface very similar, just opposite charge and  $z+a$  instead of  $z-a$

Combined:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \int_{-a}^a dx' \int_{-a}^a dy' \frac{P_0}{\sqrt{(x-x')^2 + (y-y')^2 + (z-a)^2}}$$

$$- \frac{1}{4\pi\epsilon_0} \int_{-a}^a dx' \int_{-a}^a dy' \frac{P_0}{\sqrt{(x-x')^2 + (y-y')^2 + (z+a)^2}}$$

(16 pts) **Problem 5.** Two spherical concentric conducting shells have radii  $a$  and  $c$ , as shown. Inside the conductors are two different dielectrics: from  $r = a$  to  $b$ , the first one has a dielectric constant of  $\epsilon_1$ ; from  $r = b$  to  $c$ , the second one has a dielectric constant of  $\epsilon_2$ . Determine the capacitance of this configuration.



Plan: (1) Find  $\vec{D}$  in between conductors  
 (2) Find  $\vec{E}$   
 (3) Find  $\Delta V = -\int \vec{E} \cdot d\vec{l}$   
 (4) then  $C = Q/\Delta V$

(1) Use Gauss surface shown,  $Q_{free enc} = Q$

$$\oint \vec{D} \cdot d\vec{a} = Q_{free enc}$$

$$D \cdot 4\pi r^2 = Q$$

$$D = \frac{Q}{4\pi r^2}$$

Get same result for region 2

(2)  $\vec{D} = \epsilon_0 \epsilon_r \vec{E}$ , so  $\vec{E} = \frac{\vec{D}}{\epsilon_0 \epsilon_r}$

region 1:  $\vec{E}_1 = \frac{Q}{4\pi r^2 \epsilon_0 \epsilon_1}$

region 2:  $\vec{E}_2 = \frac{Q}{4\pi r^2 \epsilon_0 \epsilon_2}$

(3)  $\Delta V = -\int \vec{E} \cdot d\vec{l} = -\int_c^b E_2 dr - \int_b^a E_1 dr$

$$= \frac{Q}{4\pi \epsilon_0 \epsilon_2} \int_c^b \frac{dr}{r^2} - \frac{Q}{4\pi \epsilon_0 \epsilon_1} \int_b^a \frac{dr}{r^2}$$

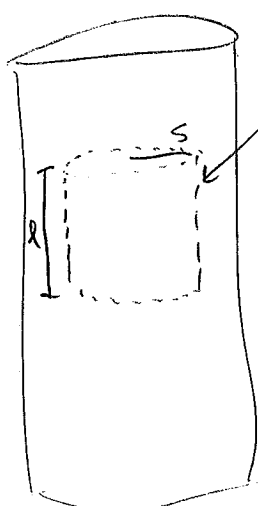
$$-\frac{1}{r} \Big|_c^b = -\frac{1}{b} + \frac{1}{c} \qquad -\frac{1}{r} \Big|_b^a = -\frac{1}{a} + \frac{1}{b}$$

$$\Delta V = \frac{Q}{4\pi \epsilon_0} \left[ \frac{1}{\epsilon_2} \left( \frac{1}{b} - \frac{1}{c} \right) + \frac{1}{\epsilon_1} \left( \frac{1}{a} - \frac{1}{b} \right) \right]$$

(4)  $C = Q/\Delta V$

$$C = 4\pi \epsilon_0 \left[ \frac{1}{\epsilon_2} \left( \frac{1}{b} - \frac{1}{c} \right) + \frac{1}{\epsilon_1} \left( \frac{1}{a} - \frac{1}{b} \right) \right]^{-1}$$

(12 pts) **Problem 6.** An infinite cylinder of radius  $R$  has a built-in polarization:  $\vec{P} = \frac{k}{s^2} e^{-s} \hat{s}$ . Use Gauss's Law for  $\vec{D}$  to obtain the electric field inside the cylinder ( $s < R$ ). As usual, be sure to explicitly draw your Gaussian surface.



Gaussian surface. By symmetry,  $\vec{D}$  and  $\vec{E}$  will be in  $\hat{s}$  direction only

$$\oint \vec{D} \cdot d\vec{a} = q_{\text{free enc}}$$

$$\underbrace{\int_{\text{top}}}_{\text{zero, no flux}} + \underbrace{\int_{\text{bottom}}}_{\text{zero, no flux}} + \underbrace{\int_{\text{wrapper}}}_{D \cdot 2\pi s l} = 0$$

$$0 \cdot 2\pi s l = 0$$

$$\underline{\underline{D = 0}}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\rightarrow \vec{E} = \frac{1}{\epsilon_0} (\vec{D} - \vec{P})$$

$$\vec{E} = -\frac{\vec{P}}{\epsilon_0}$$

$$\vec{E} = \frac{-k}{\epsilon_0 s^2} e^{-s} \hat{s}$$