

Gaussian Units

by Dr. Colton, Physics 441 (last updated: Spring 2021)

SI Units Review

As “everyone knows”, the standard SI units are meters, kilograms, seconds, newtons,¹ joules, etc. Certain of those, such as meters, kilograms, and seconds, are considered to be more fundamental and are called the “base units” (see https://en.wikipedia.org/wiki/SI_base_unit), whereas others are derived from the base units. For example force is defined as mass \times acceleration through Newton’s Second Law, so a newton is a $\text{kg}\cdot\text{m}/\text{s}^2$.

The base units were originally related to physically tangible things. A meter was originally defined as a ten millionth of the distance from the equator to the North Pole. A kilogram was originally defined in terms of the mass of a reference chunk of platinum kept in France (chosen to be equal to the mass of a $10\text{ cm} \times 10\text{ cm} \times 10\text{ cm}$ cube of water, so that water would have a density of $1\text{ g}/\text{cm}^3$). Both meters and kilograms have since been replaced by more accurate definitions as knowledge has increased and measuring ability has improved.

Now, let’s put ourselves in the positions of the early scientists trying to define the base units of electricity and magnetism. They—or, let’s say, “We”—have discovered that two charges exert a force on each other that depends on the amount of the charges and the distance between them, according to Coulomb’s law:

$$F \sim \frac{q_1 q_2}{r^2}$$

We say that charges produce an “electric field” which allows them to produce a force at a distance.

We also have discovered that moving charges form currents, $I = \frac{dq}{dt}$, which produce a new type of field called “magnetic” which exerts a force on other moving charges. We call the combination of this new force with the original electric field force the Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (\text{Lorentz force, SI units})$$

Also we have found that two nearby parallel currents also exert a force on each other, according to what I will call the “Ampere force law”:²

$$F \sim \frac{I_1 I_2}{r}$$

But we don’t yet have precise definitions of

- coulomb units
 - and therefore also not of volts or electric fields, since (in SI units) we have:
[electric field] = N/C , and volts = [electric field] \times meters

¹ I’ll use the convention that *names* of units should always be written as lower case even when named after people, although the *symbols* for the units (m, kg, s, N, J, etc.) can be either upper or lower case.

² I have not actually ever heard this equation ever called by that name before but it seemed appropriate as an analog to Coulomb’s law. Exercise for the reader: prove that two parallel wires of length L , separated by distance r where $r \ll L$, will exert an attractive force on each other which in SI units is given by $F = \left(\frac{\mu_0 L}{2\pi}\right) \frac{I_1 I_2}{r}$.

- and therefore also not of teslas since (in SI units) the units of magnetic field and electric field are related through the Lorentz force equation (the two terms in the parentheses must have the same units), which leads to: [magnetic field] = [electric field]/[velocity].
- the proportionality constant from Coulomb's law, which involves ϵ_0
- ampere units
 - which we could define if we had a definition of coulombs, as ampere = coulomb/second
- the proportionality constant from the Ampere force law, which involves μ_0

Now we are trying to figure out a way of defining those units and establishing values for those proportionality constants. So, how do you do that?

Attempt 1. Maybe you try to use Coulomb's law, which you have now written in a more precise form as the following:

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \quad (\text{SI units})$$

So you set up an experiment and measure the force between two balls of charge.

Problem. Without having a definition of a coulomb, there's no way to measure the value of ϵ_0 . It could be literally anything! It depends completely on how you define the coulomb. And similarly, without having a definition of ϵ_0 , there's no way to measure what a coulomb is. We're stymied.

Attempt 2. Well, if you could define an ampere, then you could use the ampere = coulomb/second relationship to define a coulomb. So maybe you next try to use the Ampere force law between wires, which as per the footnote on the last page is now written in a more precise form as the following:

$$F = \left(\frac{\mu_0 L}{2\pi}\right) \frac{I_1 I_2}{r} \quad (\text{SI units})$$

So you set up an experiment and measure the force between two long, straight wires.

Problem. Without having a definition of an ampere, there's no way to measure the value of μ_0 . It could be literally anything! It depends completely on how you define the ampere. And similarly, without having a definition of μ_0 , there's no way to measure what an ampere is. We're stymied.

The Vicious Cycle

The fundamental issue is that there are seven intertwined quantities:

- coulomb
- ampere
- electric field units
- volt
- tesla
- ϵ_0
- μ_0

If we knew what a coulomb is, we could define the units of amperes ($= C/s$), electric field ($= N/C$), volts ($= [\text{electric field}] \times [\text{distance}]$), and teslas ($= [\text{electric field}]/[\text{velocity}]$). We could measure ϵ_0 from Coulomb's law and μ_0 from the Ampere force law. Everything would be established.

If we knew what an ampere is, we could similarly establish all six other quantities.

If we knew what the unit of electric field is, we could establish all six other quantities.

If we knew what a volt is, we could establish all six other quantities.

If we knew what a tesla is, we could establish all six other quantities.

If we knew what ϵ_0 was, we could establish all six other quantities.

If we knew what μ_0 is, we could establish all six other quantities.

But without knowing any of the seven, there's no way to break free of the vicious cycle. It's like a crime drama where there are seven guilty subjects who can only be convicted once one of them flips.

How to Break the Cycle – the SI Solution

The SI solution to breaking the vicious cycle is to make μ_0 a defined quantity. Specifically, people chose $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$, which you have seen before but probably wondered about. The 4π is there for historical reasons, and the 10^{-7} is to make an ampere a pretty usable quantity for typical currents.

This allows the six other guilty subjects to be flipped. This definition of μ_0 combined with the Ampere force law gives a way of measuring an ampere,³ which gives a coulomb, which gives all of the other quantities. It results in an experimentally measured value of $\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/\text{Nm}^2$. One trade-off is that this choice for μ_0 results in a coulomb being an extremely large value of charge.

How to Break the Cycle – the Gaussian Solution

Viewed in the context of the above discussion, it should be clear that there are at least six other ways that we could break the cycle: we could define any of the values of the other six quantities through a different way (coulomb, ampere, electric field, volt, tesla, ϵ_0). For example, we could define a coulomb to be the amount of charge deposited on a metal ball by running a certain size Van de Graaff generator a certain length of time. That might not be a *great* definition, because it would likely depend on the humidity, the materials used, the machining tolerances, and so forth, but it could be done.

The Gaussian solution to breaking the cycle is to make ϵ_0 a defined quantity, specifically:

$$\epsilon_0 = \frac{1}{4\pi} \quad (\text{Gaussian units})$$

In fact, the symbol " ϵ_0 " isn't even ever written out in any equations that involve Gaussian units—you'll only see factors of 4π instead. This choice of a defined quantity will make the Gaussian unit of charge (which is NOT a coulomb anymore; see below) pretty usable for typical charges. It allows the six other guilty subjects to be flipped. It results in an experimentally measured value of μ_0 which is given below. One trade-off is that this choice for ϵ_0 results in the Gaussian unit for current being an extremely small amount of current.

³ Force between wires is not actually the way an ampere is defined, as it would be too imprecise. But the point is that it *could* be defined that way.

Another choice made in the selection of Gaussian units is to use centimeters, grams, and seconds as the base units (CGS) instead of meters, kilograms, and seconds like the SI system (MKS). That's an independent choice, unrelated to fixing ϵ_0 , and we'll need to keep that in mind for all Gaussian unit equations that have length and mass in them. The choice to use CGS also results in dynes and ergs as the derived units for force and energy, respectively; $1 \text{ dyne} = 1 \text{ g}\cdot\text{cm}/\text{s}^2 = 10^{-5} \text{ N}$, and $1 \text{ erg} = 1 \text{ g}\cdot\text{cm}^2/\text{s}^2 = 10^{-7} \text{ J}$.

The Consequences of the Gaussian Solution

One main consequence of the Gaussian solution, and in fact the reason why ϵ_0 is defined as 4π is to simplify Coulomb's law. Replacing ϵ_0 with $\frac{1}{4\pi}$, it becomes:

$$\boxed{F = \frac{q_1 q_2}{r^2}} \quad (\text{Coulomb's law, Gaussian units})$$

Remember that force must be measured in dynes and distance in centimeters. Analyzing that equation for units quickly results in $\text{dyne} = [\text{charge}]^2/\text{cm}^2$, which becomes $[\text{charge}] = \text{cm}\cdot\text{dyne}^{1/2}$. That unit of charge is given its own name—actually several names, there is not general agreement!—and is called either “statcoulombs”, “franklins”, or “electrostatic unit of charge (esu)”. The three unit names are interchangeable, with in my opinion the latter being slightly more common so that's what I'll use. The relationship to coulombs is this:

$$1 \text{ esu} = 3.3356 \times 10^{-10} \text{ C} \quad (\text{Gaussian unit of charge})$$

Current is defined as esu/second, and is not given its own special unit name.

As mentioned above, another consequence of Gaussian units is that μ_0 becomes an experimentally measured quantity. Specifically, it is:

$$\mu_0 = 1.398 \times 10^{-20} \text{ s}^2/\text{cm}^2 \quad (\text{Gaussian units, but not used})$$

However, as with ϵ_0 , the symbol “ μ_0 ” is never written out in any equations that involve Gaussian units. As you've probably seen somewhere before, and is discussed in Griffiths Chapter 9, Maxwell's Equations lead to the prediction of electromagnetic waves traveling at $1/\sqrt{\epsilon_0\mu_0}$, which precisely matches the experimental speed of light, c . Since ϵ_0 is defined as $1/4\pi$, we can write $c = 1/\sqrt{\mu_0/4\pi}$, which leads to:

$$\boxed{\mu_0 = \frac{4\pi}{c^2}} \quad (\text{Gaussian units})$$

The unit of electric field is defined by the equation $\mathbf{F} = q\mathbf{E}$, so $[\text{electric field}] = \text{dyne/esu}$. This is given its own special name, namely “gauss”:

$$\text{gauss} = \text{dyne/esu} \quad (\text{Gaussian unit of electric field})$$

But wait! There's more!

As mentioned above, the SI units of B were defined in terms of the units of E through the Lorentz force law, which resulted in tesla = (newton/coulomb)/(meter/second).

But why write down the law like that? Just because the magnetic force depends on the cross product of the particle's velocity with the magnetic field doesn't mean we *have* to write magnetic force as $\mathbf{F}_{\text{mag}} = q\mathbf{v} \times \mathbf{B}$. Some clever person (I suspect Lorentz himself but couldn't verify that) recognized that we could just as easily write down the force law like this:

$$\mathbf{F} = q \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \quad (\text{Lorentz force, Gaussian units})$$

It still contains the essential physics. What does that do to the units of \mathbf{B} ? Notice that inside the parentheses the units of v and c will exactly cancel out! Therefore in Gaussian units, \mathbf{B} and \mathbf{E} must have the same units! Both the electric and the magnetic fields have units of gauss. And it can be shown (but I won't) that magnetic field in gauss relates to the SI magnetic field in tesla in the following way:

$$1 \text{ gauss} = 10^{-4} \text{ tesla} \quad (\text{Gaussian unit of magnetic— and also electric—field})$$

Three Translation Rules

There are a few other subtleties which I'm skipping for now (in particular when dealing with materials), but the following three "translation rules" should work about 90% of the time when you need to convert from SI to Gaussian units. The first one is by definition, the second is a consequence of the speed of light being $1/\sqrt{\epsilon_0\mu_0}$, and the third one is a consequence of the modification of the Lorentz force law:

1. If you see an ϵ_0 , replace it with $1/4\pi$
2. If you see a μ_0 , replace it with $4\pi/c^2$
3. If you see a B , replace it with B/c

Maxwell's Equations in both SI and Gaussian Units

Using those rules, we can convert Maxwell's equations from SI to their Gaussian form:

<u>SI</u>	<u>Gaussian</u>
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\nabla \cdot \mathbf{E} = 4\pi\rho$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\nabla \times \frac{\mathbf{B}}{c} = \frac{4\pi}{c^2} \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$
	$\rightarrow \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$

Other Equations

You can do the same translation process with other equations. As just one final example, let's consider the energy stored in the fields.

SI

$$U_{electric} = \frac{\epsilon_0}{2} \int E^2 d\tau$$

$$U_{magnetic} = \frac{1}{2\mu_0} \int B^2 d\tau$$

Gaussian

$$U_{electric} = \frac{1/4\pi}{2} \int E^2 d\tau$$

$$\rightarrow \boxed{U_{electric} = \frac{1}{8\pi} \int E^2 d\tau}$$

$$U_{magnetic} = \frac{1}{2(4\pi/c^2)} \int \left(\frac{B}{c}\right)^2 d\tau$$

$$U_{magnetic} = \frac{c^2}{8\pi} \int \frac{B^2}{c^2} d\tau$$

$$\rightarrow \boxed{U_{magnetic} = \frac{1}{8\pi} \int B^2 d\tau}$$

The two fields contribute equally to the stored electromagnetic energy! Given the same volume of space, a one gauss electric field holds the exact same amount of energy as a one gauss magnetic field. This satisfying fact is completely lost when the equations are viewed in SI units.

In Summary...

To sum up, here are a few reasons many people still use Gaussian units in research and graduate classes, even though SI units have completely taken over at the high school and undergraduate level.

1. Coulomb's law is simplified; there's no $1/(4\pi\epsilon_0)$ proportionality constant.
2. There's no need for the really strange defined constant of $\mu_0 = 4\pi \times 10^{-7}$ N/A² (which must surely confuse all Physics 220 students when they first see it, as there's no quick explanation).
3. There are fewer letters in the equations— ϵ_0 and μ_0 do not show up anywhere!
4. The fact that electromagnetic waves travel at c is built directly into the equations.
5. Electric field and magnetic field have the same units.
6. Some similarities between \mathbf{E} and \mathbf{B} are made even more explicit (e.g. in the stored energy equations).