Complex permittivity vs complex conductivity

by Dr. Colton, Physics 442 (last updated: 7 Jul 2025)

with thanks to Carson Tenney for help thinking through the issues related to complex $\tilde{\sigma}_f$ and $\tilde{\varepsilon}_r$

Introduction

As discussed in the "Complex wave number, index of refraction, and relative permittivity" handout, there is an intimate connection between exponentially decaying **E** and **B** fields as they penetrate into materials, and phase shifts between driving and response fields such as **E** and **P**. Those two items lead towards the following quantities being complex numbers: wave number $\tilde{k}$, susceptibility $\tilde{\chi}_e$, relative permittivity $\tilde{\varepsilon}_r$, and conductivity $\tilde{\sigma}$. This handout in particular looks more at the connection between the complex permittivity and complex conductivity.

As was mentioned in the last section of the end of that previous handout, since the polarization current is $\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$ and $\mathbf{P} = \varepsilon_0 \tilde{\chi}_e \mathbf{E}$, the assumption of a harmonic time dependence leads to

$$\mathbf{J}_p = -i\varepsilon_0 \tilde{\chi}_e \omega \mathbf{E} \tag{1}$$

which looks like an Ohm's law for insulators, with

$$\tilde{\sigma}_p = -i\varepsilon_0 \tilde{\chi}_e \omega \tag{2}$$

That is where we pick up the story.

Connection between complex $\tilde{\varepsilon}_r$ and $\tilde{\sigma}$ in insulators

Dividing $\tilde{\chi}_e$ into its real and imaginary components, we have:

$$\tilde{\sigma}_p = -i\omega\varepsilon_0 \left(\chi_{e,real} + i\chi_{e,imag}\right) \tag{3}$$

Moreover, this is ALL the conductivity an insulator is likely to have, since it won't have much if any free current-type conductivity, so we can just call $\tilde{\sigma}_p$ the complex $\tilde{\sigma}$.

Also, since $\tilde{\chi}_e = \tilde{\varepsilon}_r - 1$, we have

$$\left.\begin{aligned} \chi_{e,real} &= \varepsilon_{r,real} - 1 \\ \chi_{e,imag} &= \varepsilon_{r,imag} \end{aligned}\right\} \tag{4}$$

which, when substituted into (3) leads to

$$\tilde{\sigma} = -i\omega\varepsilon_0 \left(\varepsilon_{r,real} - 1 + i\varepsilon_{r,imag}\right)$$

$$\tilde{\sigma} = \omega\varepsilon_0 \varepsilon_{r,imag} - i\omega\varepsilon_0 \left(\varepsilon_{r,real} - 1\right)$$

so apparently the real part of $\tilde{\sigma}_p$ is related to the imaginary part of $\tilde{\varepsilon}_r$, and vice versa.

$$\boxed{\begin{aligned} \sigma_{real} &= \omega\varepsilon_0 \varepsilon_{r,imag} \\ \sigma_{imag} &= -\omega\varepsilon_0 \left(\varepsilon_{r,real} - 1\right) \end{aligned}} \tag{5}$$

Equation (5) gives the real and imaginary components of $\tilde{\sigma}$ in terms of $\varepsilon_{r,real}$ and $\varepsilon_{r,imag}$, but can easily be solved for the real and imaginary components of $\tilde{\varepsilon}_r$ in terms of σ_{real} and σ_{imag}:

$$\boxed{\begin{aligned}\varepsilon_{r,real} &= 1 - \frac{\sigma_{imag}}{\omega\varepsilon_0} \\ \varepsilon_{r,imag} &= \frac{\sigma_{real}}{\omega\varepsilon_0}\end{aligned}} \tag{6}$$

So if an insulator has even just a little bit of polarization current-related real conductivity, it will necessarily have an imaginary component to the dielectric constant $\tilde{\varepsilon}_r$. Equation (6) can also be expressed in the following way, which I have seen in many sources.

$$\boxed{\tilde{\varepsilon}_r = \varepsilon_{r,real} + i\frac{\sigma_{real}}{\omega\varepsilon_0}} \tag{7}$$

What about conductors?

Recall that one of the main topics of the "Complex wave number, index of refraction, and relative permittivity" handout was Griffiths (9.126) (5th edition), which when written with complex permittivity and complex conductivity becomes this:

$$\tilde{k}^2 = \mu\tilde{\varepsilon}\omega^2 + i\mu\tilde{\sigma}_f\omega \tag{8}$$

If we assume $\mu = \mu_0$, write $\tilde{\varepsilon}$ as $\varepsilon_0\tilde{\varepsilon}_r$, and μ_0 as $\frac{1}{\varepsilon_0 c^2}$, then (8) becomes:

$$\tilde{k}^2 = \left(\frac{1}{\varepsilon_0 c^2}\right)\varepsilon_0\tilde{\varepsilon}_r\,\omega^2 + i\left(\frac{1}{\varepsilon_0 c^2}\right)\tilde{\sigma}_f\omega$$

$$\tilde{k}^2 = \frac{\omega^2}{c^2}\left(\tilde{\varepsilon}_r + i\frac{\tilde{\sigma}_f}{\omega\varepsilon_0}\right) \tag{9}$$

Aha! Look how similar the term in parentheses is to (7) above. Technically the σ_{real} in (7) is just the polarization current component of the conductivity, so plugging (7) into (9) we have:

$$\tilde{k}^2 = \frac{\omega^2}{c^2}\left(\varepsilon_{r,real} + i\frac{\sigma_{p,real}}{\omega\varepsilon_0} + i\frac{\tilde{\sigma}_f}{\omega\varepsilon_0}\right) \tag{10}$$

By assuming that $\tilde{\sigma}_f$ for conductors is real, we then obtain:

$$\tilde{k}^2 = \frac{\omega^2}{c^2}\left(\varepsilon_{r,real} + i\frac{\sigma_{tot,real}}{\omega\varepsilon_0}\right) \tag{11}$$

And by analogy to (7) we can define for conductors, with σ now meaning the total conductivity:

$$\boxed{\tilde{\varepsilon}_r = \varepsilon_{r,real} + i\frac{\sigma_{real}}{\omega\varepsilon_0}} \quad \text{(for conductors}^*\text{)} \tag{12}$$

to obtain:

$$\boxed{\tilde{k}^2 = \frac{\omega^2}{c^2}\tilde{\varepsilon}_r} \qquad \text{(for conductors}^*\text{)} \tag{13}$$

$$\boxed{\tilde{\varepsilon}_r = \tilde{n}^2} \qquad \text{(for conductors}^*\text{)} \tag{14}$$

However, I have given Equations (12-14) three major asterisks because in my opinion there are two important caveats to these equations. First, is the assumption that $\tilde{\sigma}_f$ for conductors is real. That isn't necessarily the case, and if not, the imaginary part of $\tilde{\sigma}_f$ would be combined with $\varepsilon_{r,real}$ in (11). I think that's actually not too bad of a caveat, seems straightforward to deal with.

The second caveat is worse: by deciding to redefine $\tilde{\varepsilon}_r$ as per (12), we have actually lost the fundamental definition of $\tilde{\varepsilon}_r$ as describing the proportionality between **D** and **E** fields! Nevertheless, the way Equations (12)-(14) end up being identical to the corresponding equations for insulators—namely Equation (7) from this handout and Equations (13) and (10) from the "Complex wave number, index of refraction, and relative permittivity" handout—is so compelling that I believe everybody does this.

Concluding comments from Reitz, Milford, and Christy

Most sources just blindly present (12)-(14) without discussing or likely even thinking about the ramifications. The textbook *Foundations of Electromagnetic Theory* by Reitz, Milford, and Christy (RMC) is one of the few sources I have found which even mentions the similarities/differences of complex permittivity vs complex conductivity at all. Here is what they say on pages 498-499 (4^{th} edition), with a few small edits by me:

> We should also comment on a significant feature of this model [the Lorentz oscillator model for insulators]—that $\tilde{\varepsilon}_r$ is complex. Even though the model was set up for *bound* charges, the resulting complex $\tilde{\varepsilon}_r$ is characteristic of a *conducting* medium. There is a nonvanishing $\sigma_{real} = \varepsilon_{r,imag}\varepsilon_0\omega$, without the intentional introduction of a conduction current density J. In addition, with $\omega_0 = 0$ for *free* charges [see my Lorentz oscillator model handout], there is still a real relative permittivity component characteristic of a *dielectric* medium. This model automatically incorporates both real and imaginary parts of $\tilde{\varepsilon}_r$, corresponding to both the displacement current $\partial\mathbf{D}/\partial t$ and the conduction current **J** in Maxwell's $\nabla \times \mathbf{H}$ equation.
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> For each of the groups of oscillators—bound and free charges—we have to calculate **P** or **J** but not both, since they are equivalent expressions of the fact that the particle displacement has a component in phase with the E field and also a component 90° out of phase, or that the particle velocity has an out-of-phase component and also an in-phase component. For static fields, bound charges have *displacement* in phase with the field and free charges have *velocity* in phase with the field; but at high frequencies both bound and free charges can each have in-phase and out-of-phase components of both displacement and velocity. All of the discussion [of complex permittivity] could instead be formulated in terms of a complex conductivity and in certain contexts, it is more usual to do so.