Summer 2024
Physics 442 Exam 1
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No time limit. Student calculators are allowed. Notes not allowed. Books not allowed.

Name ____________________________________

Instructions: Please label & circle/box your answers. **Show your work**, where appropriate.

Griffiths front and back covers

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl: $\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$
$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Special case derivatives:
(similar things true for $\mathscr{r}$)

$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta(\mathbf{r})$ $\quad$ $\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla\cdot\mathbf{D} = \rho_f \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \quad \mathbf{D} = \epsilon\mathbf{E} \\ \mathbf{M} = \chi_m\mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$

Momentum: $\mathbf{P} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c}q^2a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85\times10^{-12}\,\mathrm{C^2/Nm^2}$ (permittivity of free space)

$\mu_0 = 4\pi\times10^{-7}\,\mathrm{N/A^2}$ (permeability of free space)

$c = 3.00\times10^{8}\,\mathrm{m/s}$ (speed of light)

$e = 1.60\times10^{-19}\,\mathrm{C}$ (charge of the electron)

$m = 9.11\times10^{-31}\,\mathrm{kg}$ (mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Potentially useful mathematical information

$$\int_{-L/2}^{L/2}\sin\left(\frac{\pi z}{L}\right)z\,dz = \frac{2L^2}{\pi^2}$$

$$\int_{-L/2}^{L/2}\cos\left(\frac{\pi z}{L}\right)z\,dz = 0$$

$$\int_{-L/2}^{L/2}\sin\left(\frac{\pi z}{L}\right)z^2\,dz = 0$$

$$\int_{-L/2}^{L/2}\cos\left(\frac{\pi z}{L}\right)z^2\,dz = \frac{L^3(\pi^2-8)}{2\pi^3}$$

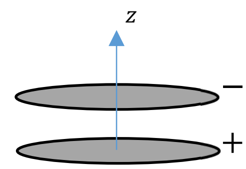
$$\int_0^1 J_0(u_{0m}r)J_0(u_{0n}r)r\,dr = \begin{cases} 0 & \text{if } n\neq m \\ \frac{1}{2}\left(J_1(u_{0m})\right)^2 & \text{if } n = m \end{cases}$$

$$\int_0^1 J_\alpha(u_{\alpha m}r)J_\alpha(u_{\alpha n}r)r\,dr = \begin{cases} 0 & \text{if } n\neq m \\ \frac{1}{2}\left(J_{\alpha+1}(u_{\alpha m})\right)^2 & \text{if } n = m \end{cases}$$

Some zeroes of the first few Bessel functions and the Bessel function derivatives, to five sig figs.

m	J_0	J_1	J_2	J_0'	J_1'	J_2'
1	2.4048	3.8317	5.1357	3.8317	1.8412	3.0542
2	5.5201	7.0156	8.4172	7.0156	5.3314	6.7061
3	8.6537	10.173	11.620	10.173	8.5363	9.9695
4	11.792	13.324	14.796	13.324	11.706	13.170

(10 pts) **Problem 1**. The axis of a circular parallel-plate capacitor is oriented in the z direction. The capacitor is charged up in the direction indicated. You can ignore edge effects.



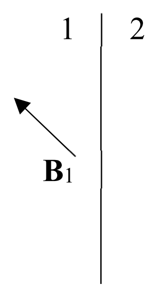
(a) What is the direction of the electric field inside the capacitor?

(b) What is the direction of the induced magnetic field as the capacitor is charging, and why?

(c) What is the direction of the Poynting vector? What does this indicate?

(10 pts) **Problem 2**. The figure represents a boundary between two linear isotropic materials, with $\epsilon_{r2} = 2\epsilon_{r1}$ and $\mu_{r2} = 2\mu_{r1}$. There is no free charge nor free current on the boundary surface, although there potentially could be bound charge or bound current.

A magnetic field vector in region 1 just to the left of the boundary is shown. What boundary conditions are most relevant to connect the parallel and perpendicular components of $\mathbf{B}_1$ and $\mathbf{B}_2$? Explicitly write them out below and specify which of Maxwell equations they come from. Then draw in a $\mathbf{B}_2$ vector that is consistent with the given $\mathbf{B}_1$ and those boundary conditions, using $\mathbf{B}_1$ as given for the scale.

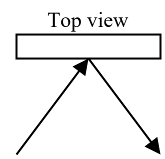


(12 pts) **Problem 3**. A circular loop of wire with radius b and current I in the $\hat{\boldsymbol{\phi}}$ direction lies in the *x-y* plane, centered on the origin. A much smaller loop with radius a, that is $a \ll b$, also in the *x-y* plane, is centered on the larger loop. Determine the mutual inductance of the two loops.

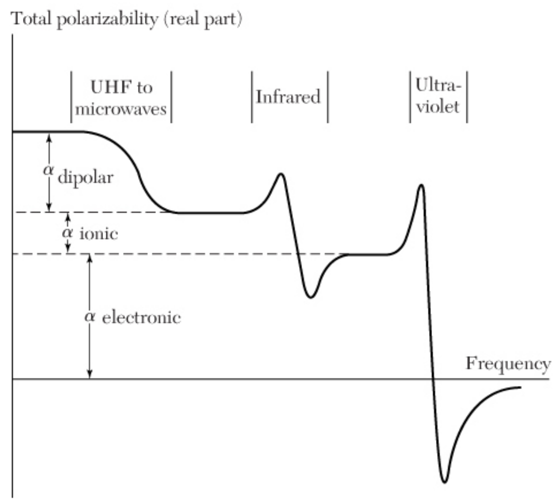
Hint: in Physics 441 we used the Biot-Savart law to find the magnetic field a distance z along the axis of a current loop to be $\mathbf{B} = \frac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}}\hat{\mathbf{z}}$. Also, close to the z-axis, the magnetic field will not change much in the lateral direction.

(16 pts) **Problem 4**. Estimate how much momentum is present in the sunlight in my office on a typical day, in units of $kg \cdot m/s$. Hint: as you should have learned in the solar sail HW problem, sunlight at the orbit of the Earth has an intensity of 1362 W/m^2. You can ignore the seasonal tilting of the earth on its axis but keep in mind that the light in my office is not nearly as bright as the light outside. Be specific about all of the estimates or approximations that you use, and all reasonable approximations will be given full credit as long as your reasoning and equations are correct.

(16 pts) **Problem 5.** A laser beam traveling 2 inches above the surface of an optical table reflects off of a glass slide as shown. Describe what you would need to know to make a good theoretical prediction of the fraction of power reflected. Specify the equations you would use to do the calculation and how you would use them. Be reasonably complete.



(20 pts) **Problem 6**. As a reminder, the Lorentz model applied to multiple regimes looks like the figure on the right from Kittel, *Introduction to Solid State Physics*. The figure is a schematic of the real part of the polarizability; the real part of the dielectric constant ε_r looks qualitatively the same.



The "Lyddane-Sachs-Teller" (or "LST") relation for insulators in the infrared/ionic regime is the following:

$$\frac{\omega_{LO}^2}{\omega_{TO}^2} = \frac{\varepsilon_0}{\varepsilon_\infty}$$

In this region the motion of the ions is what partially screens the external electric field; coordinated ionic motion is called "phonons". The symbols in the LST relation have the following meanings:

- ω_{TO} is the frequency of "transverse optical" phonons, which corresponds to what we called ω_0, the resonant frequency of the ions in response to the transverse electromagnetic waves.
- ω_{LO} is the frequency at which $\varepsilon = 0$. Because the ions cannot respond to transverse E&M waves with longitudinal motion, this is also called the "longitudinal optical" phonon frequency and there is no dielectric screening at this frequency.
- ε_0 and ε_∞ are the low and high frequency limiting values of the dielectric constant ε_r for this regime.

(a) Mark on the figure above what frequencies correspond to the values of ε_0 and ε_∞.

(b) Use the Lorentz model in the limit of no damping to derive the LST relation. Hint 1: remember that the "1" in the Lorentz model equation given on the formula sheet is not always a 1. Hint2: as an intermediate step, it's helpful to figure out what ω_p^2 is in terms of ω_{LO}, ω_{TO}, and ε_∞.

(16 pts) **Problem 7**. A "resonant cavity" can be made from a waveguide by chopping off the z-direction to a certain length and putting metal caps on the top and bottom. This causes sinusoidal standing waves to form in the z-direction (characterized by a third mode number, ℓ), which in turn makes only discrete frequencies possible rather than all frequencies above a cutoff. Derive an equation for the discrete resonant frequencies of the $\text{TE}_{\alpha m \ell}$ mode in a cylindrical cavity, as a function of α, m, and ℓ, and then use your equation to calculate the resonant frequency in Hz for the TE_{011} mode of a cavity with 3 cm height and 1 cm radius. I used such a cavity as a post-doc at the Naval Research Lab for electron spin resonance experiments. *Hint*: when sinusoidal standing waves form in the z-direction, the usual traveling wave wavenumber k becomes $k_z = \frac{\ell\pi}{height}$. The first two mode numbers of the cavity are the same as for waveguides (cylindrical, in this case).