Summer 2024
Physics 442 Final Exam
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No time limit. Student calculators are allowed. Notes not allowed. Books not allowed.

Name Solutions

Instructions: Please label & circle/box your answers. **Show your work**, where appropriate.

Griffiths front and back covers

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Curl:
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

Special case derivatives:
(similar things true for $\mathscr{r}$)

$$\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta(\mathbf{r}) \qquad \nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r})$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^8\ \mathrm{m/s}$	(speed of light)
$e = 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Potentially useful mathematical information

quadratic formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\int_0^a \frac{s\,ds}{\sqrt{z^2+s^2}} = \sqrt{a^2+z^2} - z \quad \text{for } a > 0, z > 0$$

$$\int_0^a \frac{ds}{\sqrt{z^2+s^2}} = \frac{1}{2}\ln\left(1 + \frac{2a^2 + 2a\sqrt{a^2+z^2}}{z^2}\right) \quad \text{for } a > 0, z > 0$$

$$\int_{-L/2}^{L/2} \sin\left(\frac{\pi z}{L}\right) z\,dz = \frac{2L^2}{\pi^2}$$

$$\int_{-L/2}^{L/2} \cos\left(\frac{\pi z}{L}\right) z\,dz = 0$$

$$\int_{-L/2}^{L/2} \sin\left(\frac{\pi z}{L}\right) z^2\,dz = 0$$

$$\int_{-L/2}^{L/2} \cos\left(\frac{\pi z}{L}\right) z^2\,dz = \frac{L^3(\pi^2-8)}{2\pi^3}$$

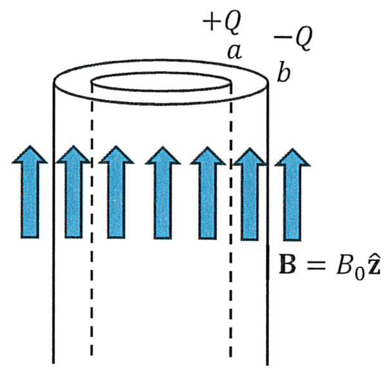
$$\int_0^1 J_0(u_{0m}r)J_0(u_{0n}r)\,r\,dr = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2}\left(J_1(u_{0m})\right)^2 & \text{if } n = m \end{cases}$$

$$\int_0^1 J_\alpha(u_{\alpha m}r)J_\alpha(u_{\alpha n}r)\,r\,dr = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2}\left(J_{\alpha+1}(u_{\alpha m})\right)^2 & \text{if } n = m \end{cases}$$

Some zeroes of the first few Bessel functions and the Bessel function derivatives, to five sig figs.

m	J_0	J_1	J_2	J_0'	J_1'	J_2'
1	2.4048	3.8317	5.1357	3.8317	1.8412	3.0542
2	5.5201	7.0156	8.4172	7.0156	5.3314	6.7061
3	8.6537	10.173	11.620	10.173	8.5363	9.9695
4	11.792	13.324	14.796	13.324	11.706	13.170

(24 pts) **Problem 1**. Two charged, concentric insulating cylindrical shells are placed in a constant magnetic field pointing along the cylindrical axis, $\mathbf{B} = B_0\hat{\mathbf{z}}$. The inner shell has radius a and charge Q. The outer shell has radius b and charge $-Q$. You can assume them to be nearly infinite, i.e. their length ℓ is much larger than a and b.



From Gauss's law it is straightforward to show that the electric field between the shells is $\mathbf{E} = \frac{Q}{2\pi\varepsilon_0 \ell}\frac{1}{s}\,\hat{\mathbf{s}}$. (You don't have to do this.)

(a) Determine the momentum density of the stored momentum in the region of space between the shells, as a function of s.

$(s, \phi, z) \rightarrow \hat{s} \times \hat{z} = -\hat{\phi}$

$$\frac{\vec{mom}}{vol} = \epsilon_0 \vec{E} \times \vec{B}$$

$$= \cancel{\epsilon_0} \frac{Q}{2\pi \cancel{\epsilon_0} \ell} \frac{1}{s} \hat{s} \times B_0 \hat{z}$$

$$\boxed{\frac{\vec{mom}}{vol} = \frac{-Q B_0}{2\pi \ell s} \hat{\phi}}$$

(b) In addition to there being stored momentum in the fields, there is stored *angular* momentum. The angular momentum density about the z-axis at a point $\mathbf{r} = s\hat{\mathbf{s}}$ is $\frac{\textbf{ang. mom}}{volume} = s\hat{\mathbf{s}} \times \frac{\textbf{momentum}}{volume}$. Determine the total stored angular momentum by integrating the angular momentum density over the volume.

$$\frac{\vec{ang\ mom}}{vol} = \cancel{s}\hat{s} \times \frac{-QB_0}{2\pi \ell \cancel{s}} \hat{\phi} \qquad s \times \phi = z$$

$$\frac{\vec{ang\ mom}}{vol} = -\frac{QB_0}{2\pi \ell} \hat{z}$$

$$\vec{ang\ mom} = \int \left(-\frac{QB_0}{2\pi \ell}\hat{z}\right) d\tau$$

constant in space, so integral just gives volume

$$= -\frac{QB_0}{2\cancel{\pi}\cancel{\ell}} \hat{z} \left(\cancel{\pi} b^2 - \cancel{\pi} a^2\right)\cancel{\ell}$$

$$\boxed{\vec{ang\ mom} = +\frac{QB_0}{2}(a^2 - b^2)\hat{z}}$$

(c) Now the magnetic field is ramped down to zero in a time T, according to: $\mathbf{B} = B_0\left(1 - \frac{t}{T}\right)\hat{\mathbf{z}}$. The changing $\mathbf{B}$ field will induce an $\mathbf{E}$ field, which will cause a force on each of the shells. Determine the amount of force on each shell.

$$\underbrace{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$\hookrightarrow$ "$\vec{J}_{Faraday}$" $= -B_0\left(-\frac{1}{T}\right)\hat{z} = \frac{B_0}{T}\hat{z}$

$J_{Faraday}$ produces an $\vec{E}$ field like regular $\vec{J}$ produces $\vec{B}$ field, in $\hat{\phi}$ direction

$\oint \vec{E}\cdot d\vec{\ell} = I_{enc} \longrightarrow \int \vec{J}_{Faraday}\cdot d\vec{a} = J\cdot \pi s^2$ since J is constant in space

$I_{enc} = \frac{B_0}{T}\pi s^2$

$E\, 2\cancel{\pi}\cancel{s} = \frac{B_0}{T}\cancel{\pi} s^{\cancel{2}}$

$$\boxed{\vec{E} = \frac{B_0 s}{2T}\hat{\phi}}$$

$\vec{F} = q\vec{E} \longrightarrow \boxed{\vec{F}_{inside\ shell} = \frac{B_0 Q a}{2T}\hat{\phi}}$

$$\boxed{\vec{F}_{outside\ shell} = -\frac{B_0 Q b}{2T}\hat{\phi}}$$

(d) In addition to there being a force, because the force is in the $\hat{\boldsymbol{\phi}}$ direction there is also a torque on each shell. The amount of torque on a shell about the z-axis is $\textbf{torque} = \mathbf{s} \times \mathbf{F}$, where $\mathbf{s}$ is the distance to the shell. Torque exerted on an object over time leads to a gain in angular momentum according to: $\textbf{ang. mom.} = \int \textbf{torque}\; dt$. Determine the amount of angular momentum gained by each shell, and compare these answers to your answer for part (b).

inside

$\vec{\tau} = \frac{B_0 Q a}{2T}(a\hat{s})\times\hat{\phi}$

$\vec{\tau} = \frac{B_0 Q a^2}{2T}\hat{z}$

$\overrightarrow{ang\ mom} = \int \tau\, dt = \tau \times T$ since τ is constant in time

$$\boxed{\overrightarrow{ang\ mom} = \frac{B_0 Q a^2}{2}\hat{z}}$$

outside

$\tau = -\frac{B_0 Q b}{2T}(b\hat{s})\times\hat{\phi}$

$\tau = -\frac{B_0 Q b^2}{2T}\hat{z}$

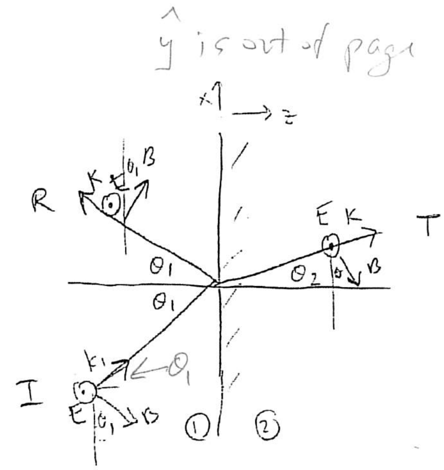
$\overrightarrow{ang\ mom} = \int \tau\, dt = \tau \times T$

$$\boxed{\overrightarrow{ang\ mom} = -\frac{B_0 Q b^2}{2}\hat{z}}$$

The sum of these two equals the initial stored ang. momentum!

So ang. mom. is conserved.

(12 pts) **Problem 2**. Here is a diagram which I used to solve the Fresnel equations for s-polarization.



(a) Why are the **B** vectors for the I and T rays pointing downward whereas the **B** vector for the R ray points upward?

$\vec{E}$ was arbitrarily chosen to be out of the page. $\vec{E} \times \vec{B}$ needs to be in the direction of $\vec{k}$, so that sets $\vec{B}$ in the directions indicated.

(b) Write down the appropriate equations for generic **E** and **B** waves **for the incident wave only**, in complex exponential form. Assume the wave speed in region 1 is known to be v_1. Include the direction of travel through the wavevector, include the polarization as its own vector, and relate the B-field amplitude to the E-field amplitude.

$$\vec{E}_{1,I} = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \hat{n} \quad \text{in general} \qquad \frac{\omega}{k} = v$$

$$\boxed{\vec{E}_{1,I} = \tilde{E}_0 e^{i(k_1 \sin\theta_1 x + k_1 \cos\theta_1 z - \omega t)} \hat{y}} \quad \text{where } \boxed{k_1 = \frac{\omega}{v_1}}$$

$$\vec{B} \text{ magnitude} = \frac{E_0}{v_1} \quad \left(\text{or} = \frac{k_1}{\omega} E_0\right)$$

$$\boxed{\vec{B}_{1,I} = \frac{1}{v_1}\tilde{E}_0 e^{i(k_1 \sin\theta_1 x + k_1 \cos\theta_1 z - \omega t)} (-\cos\theta_1 \hat{x} + \sin\theta_1 \hat{z})}$$

(c) For this situation, what boundary condition would be the best choice to relate the **electric field** of the I, R, and T waves, and why?

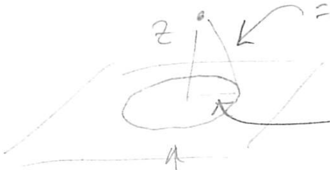
$$E_{1\parallel} = E_{2\parallel}$$

$$\boxed{\vec{E}_{1I} + \vec{E}_{1R} = \vec{E}_{1T}}$$

This is much better than the $E_{1\perp} - E_{2\perp} = \sigma/\epsilon_0$ boundary cond because you don't know what the bound charges are.

(18 pts) **Problem 3**. A sheet of current in the x-y plane having $\mathbf{K} = K_0\hat{\mathbf{y}}$ abruptly switches on at $t = 0$.

(a) Find V and $\mathbf{A}$ in the Lorentz gauge for a point above the plane at $\mathbf{r} = z\hat{\mathbf{z}}$. *Hint*: this is very similar to the problem done in class where current in a wire switched on abruptly at $t = 0$. There are some integrals on page 2 of the exam.

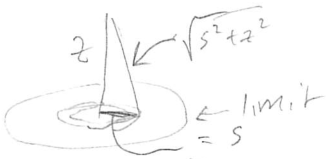


observer only "sees" current inside this circle.

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(t_r)}{\mathscr{r}}\,d\tau'$$

$$\boxed{V = 0} \text{ since no } \rho$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(t_r)}{\mathscr{r}}\,d\tau' = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(t_r)\,da'}{\mathscr{r}}$$



during integral $da' = 2\pi s'ds'$

$$\vec{A} = \frac{\mu_0}{\cancel{4\pi}_{\,2}} K_0\hat{y}\underbrace{\int_0^{\sqrt{c^2t^2-z^2}} \frac{\cancel{2\pi} s'ds'}{\sqrt{s^2+z^2}}}_{= \sqrt{(c^2t^2 - z^2) + z^2} - z}$$

from page 2 integral

$$= ct - z$$

$$\boxed{\vec{A} = \frac{\mu_0}{2} K_0 (ct - z)\,\hat{y}}$$

→ use "field from potentials" formulas, $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$

(b) How would you find **E** and **B** at **r**? You don't have to do so, although you can if you'd like; just explain what the best method would be.

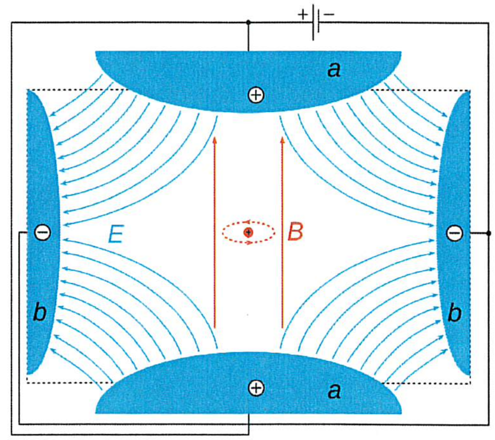
$$\vec{E} = -\cancel{\nabla V} - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A} = -\frac{\partial A_y}{\partial z}\hat{x} \text{ (only non zero term)}$$

$$\boxed{\vec{E} = -\frac{\mu_0}{2} K_0 c\,\hat{y}}$$

$$\boxed{\vec{B} = \frac{\mu_0}{2} K_0\,\hat{x}}$$

(16 pts) **Problem 4**. A Penning trap is a device used to store charged particles, first built in 1959 and part of the 1989 Nobel Prize in Physics. A strong constant magnetic field in the z-direction keeps the particles from escaping in the x- and y-directions (they tend to revolve in circles around the B-field lines), and a quadrupole electric field keeps the particles from escaping in the z-direction. The trapped particles can be cooled by letting the particles radiate away their kinetic energy as they move in circles with centripetal acceleration. The characteristic time scale of cooling can be obtained by calculating $\tau = \frac{E}{dE/dt}$ where E is the kinetic energy of the particles and dE/dt is the rate of energy loss due to radiation. Calculate τ for electrons in a 3 T magnetic field. You can assume this is nonrelativistic and that the electrons are moving in perfect circles of radius R (not actually the case; their actual trajectories are more complicated due to the complicated electric field pattern). *Hint*: for perfect circles, the net force is from the magnetic field only, the centripetal acceleration is $a_c = v^2/R$, and the radius $R = mv/(qB)$.



Penning trap. Image modified from Wikipedia.

$E = \frac{1}{2}mv^2$

$\frac{dE}{dt} = \frac{\mu_0 q^2 a^2}{6\pi c}$ from Larmor

$= \frac{\mu_0 q^2}{6\pi c} \frac{v^4}{R^2}$

$= \frac{\mu_0 q^2}{6\pi c} \frac{v^4}{m^2 v^2/(q^2 B^2)}$

$\frac{dE}{dt} = \frac{\mu_0 q^4 v^2 B^2}{6\pi c m^2}$

So $\tau = \frac{E}{\left(\frac{dE}{dt}\right)} = \frac{\frac{1}{2}m\cancel{v^2}}{\frac{\mu_0 q^4 \cancel{v^2} B^2}{6\pi c m^2}}$

$\boxed{\tau = \frac{3\pi c m^3}{\mu_0 q^4 B^2}}$

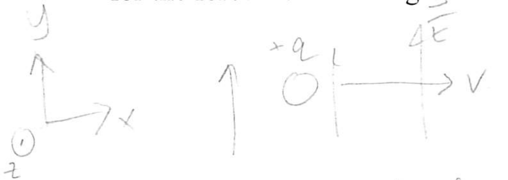
plugging in numbers,

$\tau = \frac{3\pi \cdot 3 \cdot 10^8 (9.11 \cdot 10^{-31})^3}{(4\pi \cdot 10^{-7})(1.6 \cdot 10^{-19})^4 (3)^2}$

$\boxed{\tau = 0.288 \text{ sec}}$

(18 pts) **Problem 5**. A "velocity selector" is a configuration of **E** and **B** fields where a charge with only the proper velocity will pass through a region of space undeflected. Orient your axes such that $\boldsymbol{x}$ = to the right, $\boldsymbol{y}$ = up, and $\boldsymbol{z}$ = out of page. Let a positive charge q come flying in from left to right at a speed v into a region of space where $\mathbf{E} = E_0\hat{\mathbf{y}}$.

(a) In terms of the given quantities, what magnitude and direction must the magnetic field have, in order for the force from the magnetic field to exactly cancel the force from the electric field?



will get deflected up, from $\vec{E}$

mag. field must produce a downward force

$\vec{F} = q\vec{v}\times\vec{B}$ → since $\vec{v}$ is in $\hat{x}$, $\vec{B}$ must be in $\hat{z}$

$q\vec{E} + q\vec{v}\times\vec{B} = 0$ → $E = vB$

$$\boxed{\vec{B} = \frac{E_0}{v}\hat{z}}$$

(b) From the point of view of the *charge*, what are the electric and magnetic fields? What is the net force from the electric and magnetic fields in this reference frame?

Field transformation eqns:

$E_x' = E_x \rightarrow \boxed{E_x' = 0}$

$E_y' = \gamma(E_y - vB_z) \rightarrow E_y' = \gamma\left(E_0 - v\frac{E_0}{v}\right)$

$\boxed{E_y' = 0}$

$E_z' = \gamma(\cancel{E_z} + v\cancel{B_y}) \rightarrow \boxed{E_z' = 0}$

$B_x' = B_x \rightarrow \boxed{B_x' = 0}$

$B_y' = \gamma\left(\cancel{B_y} + \frac{v}{c^2}\cancel{E_z}\right) \rightarrow \boxed{B_y' = 0}$

$B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \rightarrow \boxed{B_z' = \gamma\left(\frac{E_0}{v} - \frac{v}{c^2}E_0\right)}$ → can simplify $= \gamma\frac{E_0}{v}\underbrace{\left(1 - \frac{v^2}{c^2}\right)}_{\frac{1}{\gamma^2}}$

$$\boxed{B_z' = \frac{E_0}{\gamma v}}$$

$\vec{F} = q\vec{v}\times\vec{B}$

$\boxed{\vec{F} = 0}$ → since $v = 0$ in this frame!

(12 pts) **Problem 6.** In class I introduced a rank 2 tensor which is the 4-vector analog of the Maxwell stress tensor we discussed in Chapter 8. Its symbol is $T^{\mu\nu}$, and the specific components are given in the formula sheet. Determine what important equation from Chapter 8 most closely relates to the $\nu = 0$ part of this 4-vector equation:

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} = 0,$$

and specifically, under what conditions will that equation be true? What does the equation mean, physically speaking? *Hint*: there is an implied summation over μ.

Summation over $\mu \rightarrow$ $\frac{\partial T^{0\nu}}{\partial x^0} + \frac{\partial T^{1\nu}}{\partial x^1} + \frac{\partial T^{2\nu}}{\partial x^2} + \frac{\partial T^{3\nu}}{\partial x^3} = 0$

for $\nu = 0$: $\frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{10}}{\partial x^1} + \frac{\partial T^{20}}{\partial x^2} + \frac{\partial T^{30}}{\partial x^3} = 0$

$$\frac{\partial u}{\partial (ct)} + \underbrace{\frac{\partial S_x/c}{\partial x} + \frac{\partial S_y/c}{\partial y} + \frac{\partial S_z/c}{\partial z}}_{\frac{1}{c}\nabla\cdot\vec{S}} = 0$$

$$\frac{1}{c}\frac{\partial u}{\partial t} + \frac{1}{c}\nabla\cdot\vec{S} = 0$$

$$\boxed{\nabla\cdot\vec{S} = -\frac{\partial u}{\partial t}}$$

This is conservation of energy!

→ True if no work done on charges, e.g. if no $\vec{J}$.

→ Energy lost in the fields gets transported out of the region of space by the Poynting vector.