Lorentz Oscillator Model

by Dr. Colton, Physics 442/471/581 (last updated: 7 Jul 2025)

Introduction

The Lorentz oscillator model, also known as the Drude-Lorentz oscillator model, involves modeling an electron as a driven damped harmonic oscillator. It is extremely important, as can be seen by it being used in these textbooks: Physics 442 – Griffiths, *Introduction to Electrodynamics*; Physics 471 – Peatross and Ware, *Physics of Light and Optics*; Physics 581 – Kittel, *Introduction to Solid State Physics*, and Stokes, *Solid State Physics for Advanced Undergraduate Students*; and also a graduate school textbook on semiconductor physics I used – Yu and Cardona, *Fundamentals of Semiconductors*.

In this model the electron is bound to a nucleus with a force that can be represented as a spring force, spring constant C. The driving force is the oscillating electric field. There is also a damping force whose source is not specified but is present so that the oscillations don't go infinite when the driving force is at the resonant frequency. Also, although the model is for insulators, by removing the spring force so the free electrons are not bound to atoms we can also make it apply to conductors. The method of this model is as follows:

1) Obtain an equation of motion for the electron, $\tilde{x}(t)$, using complex numbers to represent phase shifts as usual.
2) Since the nucleus basically doesn't move at all, the oscillating dipole moment of the electron-nuclear pair of charges will be $q\tilde{x}(t)$.
3) If we then assume there are N electrons per volume, all with that same response, then the oscillating polarization $\tilde{P}$ (which is dipole moment density) will be $\tilde{P} = Nq\tilde{x}(t)$.
4) We'll find that this is proportional to the driving electric field, then we can read off the proportionality constant as $\varepsilon_0\tilde{\chi}_e$ to obtain an equation for $\tilde{\chi}_e$, and from that $\tilde{\varepsilon}_r$ and $\tilde{n}$.

Driven, damped harmonic oscillator

There are three forces on the electron in this model. First is the driving force from the electric field, where $F_{driving} = qE = qE_0\cos(-\omega t)$. I'm using $\cos(-\omega t)$ here instead of $\cos(+\omega t)$ so that it matches time dependence of a standard traveling EM wave, namely $\cos(kx - \omega t)$. Next is the restoring force from the spring; we can put the spring force in terms of the resonant frequency ω_0 instead of spring constant C, so $F_{spring} = -Cx = -m\omega_0^2 x$. Finally is the damping force, which is proportional to the velocity and is described by damping coefficient γ (units of γ chosen such that force $= \gamma mv$). Then Newton's Second Law becomes:

$$F_{driving} + F_{spring} + F_{damping} = m\ddot{x} \tag{1}$$

$$qE_0\cos(-\omega t) - m\omega_0^2 x - \gamma m\dot{x} = m\ddot{x} \tag{2}$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{qE_0}{m}\cos(-\omega t) \tag{3}$$

That is the equation of motion we need to solve. It's simpler to solve if you use complex numbers to represent the oscillations. Then we have:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{qE_0}{m}e^{-i\omega t} \tag{4}$$

To solve (4) we'll use one of the physicist's favorite tricks, which is to guess the answer and plug it into the equation. Here I want to guess a sinusoidal oscillation with added phase shift.

Guess $x = x_0 \cos(-\omega t + \phi)$ as solution $\rightarrow$ $\tilde{x} = x_0 e^{i\phi} e^{-i\omega t}$ in complex notation
$\tilde{x} = \tilde{x}_0 e^{-i\omega t}$ (the phase ϕ is lumped in with complex $\tilde{x}_0$)

The time derivatives bring down factors of $-i\omega$, so we have:

$$(-i\omega)^2 \tilde{x}_0 e^{-i\omega t} + \gamma(-i\omega)\tilde{x}_0 e^{-i\omega t} + \omega_0^2 \tilde{x}_0 e^{-i\omega t} = \frac{qE_0}{m} e^{-i\omega t} \tag{5}$$

We can cancel the $e^{-i\omega t}$ factors, then this remains:

$$\tilde{x}_0(-\omega^2 - i\omega\gamma + \omega_0^2) = \frac{qE_0}{m} \tag{6}$$

$$\tilde{x}_0 = \frac{qE_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{7}$$

That is the solution to the complex amplitude of the electron's oscillating motion. The fact that it is complex just means there is a time delay (phase shift) between the driving electric field and the response of the electron. Since $\tilde{E}(t) = E_0 e^{-i\omega t}$, we can add back into the time dependence to find:

$$\tilde{x}(t) = \frac{q}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \tilde{E}(t) \tag{8}$$

We can separate the amplitude into real and imaginary parts, which turn out to be:

$$\left.\begin{aligned} \mathrm{Re}\{\tilde{x}_0\} &= \frac{qE_0}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \\ \mathrm{Im}\{\tilde{x}_0\} &= \frac{qE_0}{m} \frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \end{aligned}\right\} \tag{9}$$

The complex phase of the amplitude is the phase shift ϕ.

Susceptibility and Permittivity

As described in the Introduction, the complex polarization $\tilde{P} = Nq\tilde{x}(t)$, and is therefore:

$$\tilde{P}(t) = Nq \frac{q/m}{\omega_0^2 - \omega^2 - i\omega\gamma} \tilde{E}(t) \tag{10}$$

Also as mentioned in the Introduction the proportionality constant between $\tilde{P}$ and $\tilde{E}$ is $\varepsilon_0 \tilde{\chi}_e$, so we can just read off the complex susceptibility as being the stuff multiplying $\tilde{E}$, divided by ε_0:

$$\tilde{\chi}_e = \frac{Nq^2}{m\varepsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{11}$$

The quantity $\frac{Nq^2}{m\epsilon_0}$ has units of frequency squared, and its square root is called the *plasma frequency* ω_p, because it happens to also be the frequency at which a plasma will naturally oscillate if the positive and negative charges in the plasma are offset from each other.

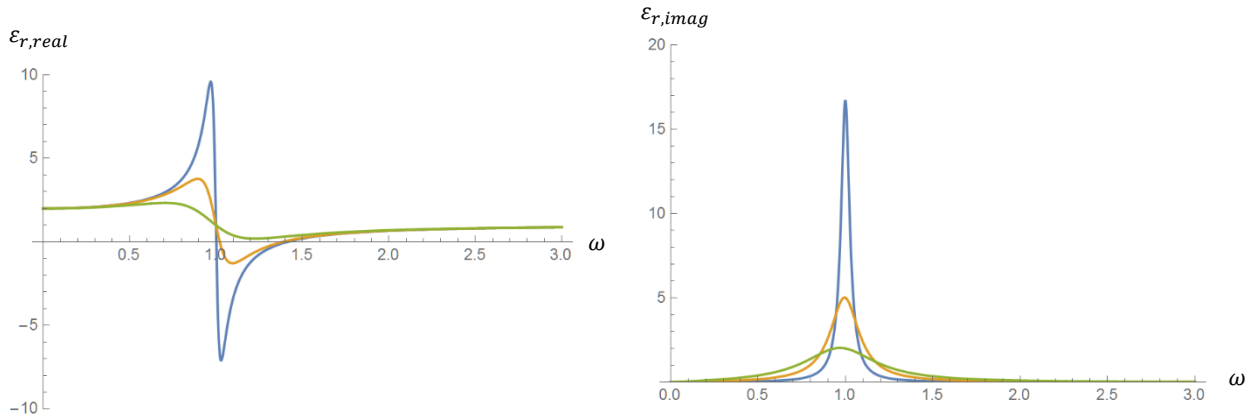

Figure 1. Plots of $\varepsilon_{r,real}$ and $\varepsilon_{r,imag}$ for a material with an electronic resonance, plotted with $\omega_p = 1$ and $\omega_0 = 1$, for three different damping values: $\gamma = 0.06, 0.2, 0.5$. (The taller peaks are those with less damping.)

$$\omega_p = \sqrt{\frac{Nq^2}{m\epsilon_0}} \tag{12}$$

That allows us to write the complex susceptibility in a nice compact form,

$$\tilde{\chi}_e = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{13}$$

The susceptibility and the permittivity (aka dielectric constant) are related via $\tilde{\varepsilon}_r = 1 + \tilde{\chi}_e$, so we have:

$$\boxed{\tilde{\varepsilon}_r = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}} \tag{14}$$

This is the main result of the Lorentz oscillator model: a specific equation and functional form for the frequency dependence of the relative permittivity of insulators in AC electric fields. The rest of this handout just describes minor extensions to that equation, and then applies the equation to a few specific situations.

Note that even in this fairly straightforward model of *insulators* we arrive at a *complex* permittivity.

Plots of Lorentz oscillator model permittivity, index of refraction, and reflectivity

Figure 1 shows the real and imaginary parts of $\tilde{\varepsilon}_r$ as a function of ω and Figure 2 shows the real and imaginary parts of $\tilde{n} = \sqrt{\tilde{\varepsilon}_r}$, both plotted with $\omega_p = 1$ and $\omega_0 = 1$ for simplicity, and for three values of the damping constant: $\gamma = 0.06$, 0.2, and 0.5. As the damping decreases, the peaks get narrower and taller.

Close to the resonance there is a large imaginary component to both $\tilde{\varepsilon}_r$ and $\tilde{n}$, which means a lot of absorption, but far from the resonance both values are nearly all real. It is only in the neighborhood of the resonance (when ω is close to ω_0) that the complex nature of the permittivity is important. Also notice how n_{real} and n_{imag} are always positive, so $\tilde{n}$ stays in the first quadrant, as it must, whereas $\varepsilon_{r,real}$ can be negative so $\tilde{\varepsilon}_r$ can be in the second quadrant.

Figure 3 plots the reflectivity at normal incidence at an air-material interface, which is given by:

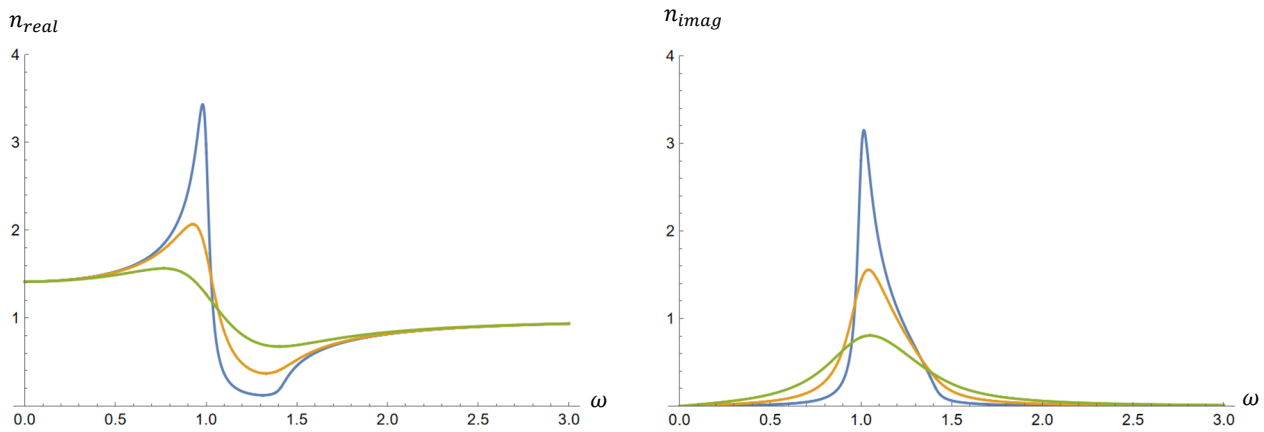

Figure 2. Plots of n_{real} and n_{imag} for a material with an electronic resonance, plotted with $\omega_p = 1$ and $\omega_0 = 1$, for three different damping values: $\gamma = 0.06, 0.2, 0.5$. (The taller peaks are those with less damping.)

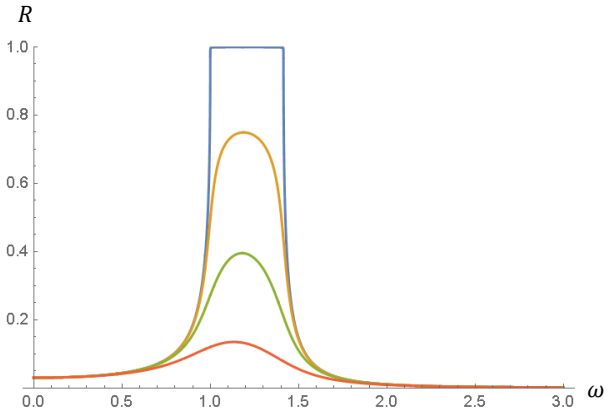

Figure 3. Plot of R for the electronic (UV) resonance in an insulator, with $\omega_p = 1$ and ω_0=1, and four different damping values: $\gamma = 0$, 0.06, 0.2, and 0.5. (The taller peaks are those with less damping.)

$$R = \left|\frac{1-\tilde{n}}{1+\tilde{n}}\right|^2 \tag{15}$$

Insulators become very reflective for frequencies where $\tilde{\varepsilon}_r$ goes negative and hence $\tilde{n}$ goes imaginary, which it can be shown is the range between ω_0 and $\sqrt{\omega_0^2 + \omega_p^2}$. Damping rounds out the corners and decreases the maximum reflectivity.

Low and high frequency limits

As can be seen in the permittivity plots, $\tilde{\varepsilon}_r$ goes to a specific real value at low frequencies, and to a different specific real value at high frequencies. I will call those values $\varepsilon(0)$ and ε_∞, respectively, and the values can be obtained by using $\omega = 0$ and $\omega = \infty$ in the Lorentz oscillator model permittivity equation.

Low frequency value:

$$\varepsilon(0) = 1 + \frac{\omega_p^2}{\omega_0^2 - 0^2 - i0\gamma} = 1 + \frac{\omega_p^2}{\omega_0^2} \tag{16}$$

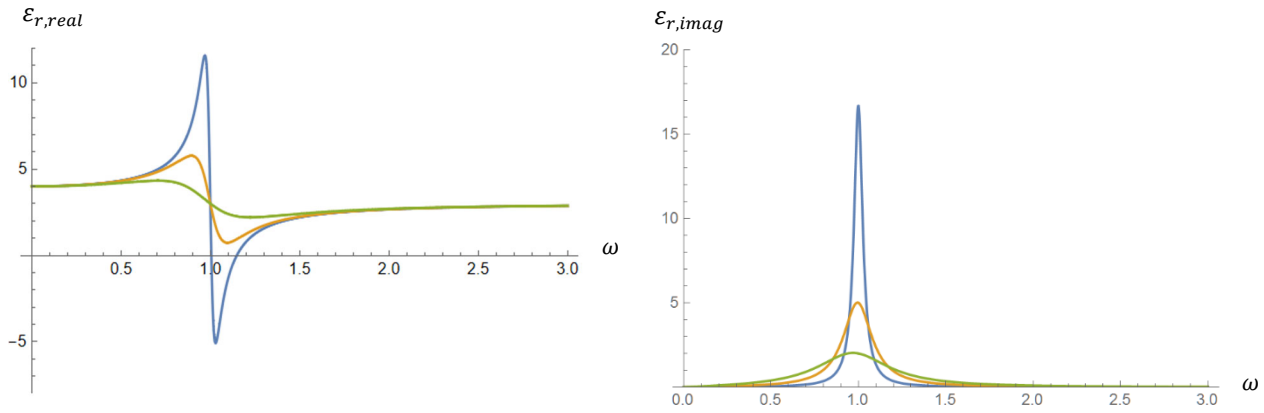

Figure 4. Plots of $\varepsilon_{r,real}$ and $\varepsilon_{r,imag}$ for a material with $\varepsilon_\infty = 3$, plotted with $\omega_p = 1$ and $\omega_0 = 1$, for three different damping values: $\gamma = 0.06, 0.2, 0.5$. (The taller peaks are those with less damping.)

High frequency value:

$$\varepsilon_\infty = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma} = 1 \tag{17}$$

Modification 1: Changing ε_∞

For situations where you are truly at infinitely high frequencies, the relative permittivity really does need to go to 1, because nothing can respond quickly enough to produce dielectric screening. However, there are many situations where some additional screening might exist at frequencies just above the resonance that you are looking at. In those cases, the high frequency value will NOT be 1, but will be something else. To take that into account, we can just manually change the "1" in the complex permittivity equation to ε_∞:

$$\boxed{\tilde{\varepsilon}_r = \varepsilon_\infty + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}} \tag{18}$$

That changes the low frequency value to $\varepsilon(0) = \varepsilon_\infty + \frac{\omega_p^2}{\omega_0^2}$ and the high frequency limit (of course) to ε_∞.

It also affects the plots. Figures 4, 5, and 6, are the same as Figures 1, 2, and 3, just redone for the case of $\varepsilon_\infty = 3$ instead of 1. Notice how shifting $\tilde{\varepsilon}_r$ up by 2 reduces the range where $\tilde{\varepsilon}_{r,real}$ goes negative, in turn reducing the range where $\tilde{n}$ is purely imaginary and the material is highly reflective. Specifically, the region of high reflectivity becomes the range between ω_0 and $\sqrt{\omega_0^2 + \frac{\omega_p^2}{\varepsilon_\infty}}$.

Modification 2: Multiple oscillators, and "oscillator strength"

One situation where you have an ε_∞ value which is not 1, is if you have multiple resonances but you want to focus on the lower one. To allow for multiple resonances we can introduce a summation. Let's assume for now that the multiple resonances arise because you have electrons bound to different atoms; each atom could have its own response. Here ω_{pi}, ω_{0i} and γ_i are the plasma frequency, resonant frequency, and damping coefficient, respectively, for the i^{th} type of electron.

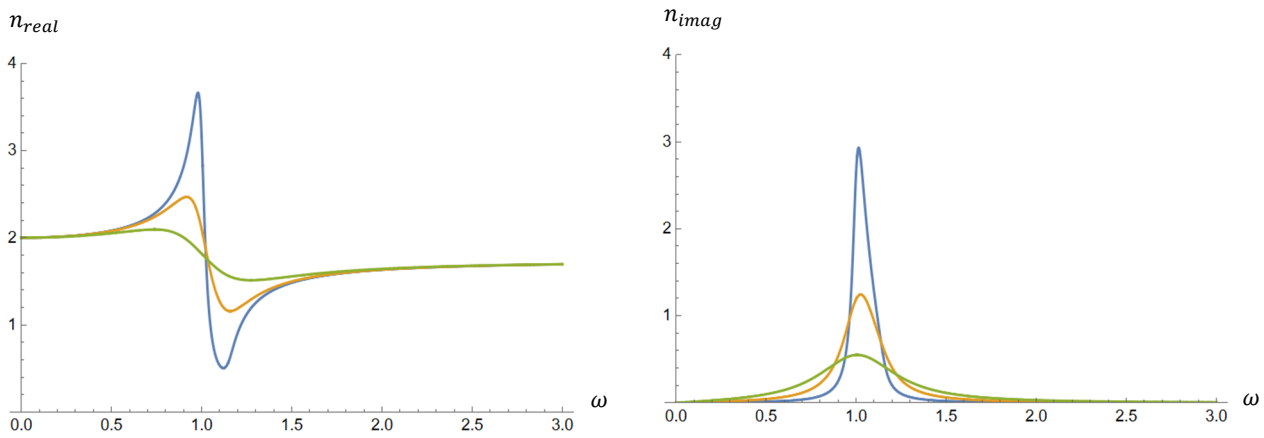

Figure 5. Plots of n_{real} and n_{imag} for a material with $\varepsilon_\infty = 3$, plotted with $\omega_p = 1$ and $\omega_0 = 1$, for three different damping values: $\gamma = 0.06, 0.2, 0.5$. (The taller peaks are those with less damping.)

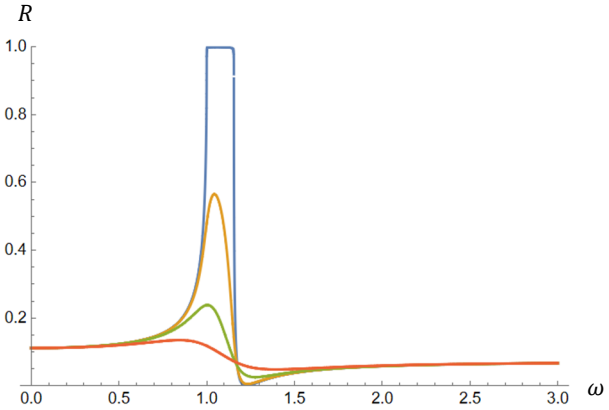

Figure 6. Plot of R for a material with $\varepsilon_\infty = 3$, plotted with $\omega_p = 1$ and ω_0=1, and four different damping values: $\gamma = 0$, 0.06, 0.2, and 0.5. (The taller peaks are those with less damping.)

$$\tilde{\varepsilon}_r = \varepsilon_\infty + \sum_i \frac{\omega_{pi}^2}{\omega_{0i}^2 - \omega^2 - i\omega\gamma_i} \tag{19}$$

By introducing the *oscillator strength*, $f_i = N_i/N_{tot}$, which describes the fraction of electrons of type i, we can change the equation into this:

$$\boxed{\tilde{\varepsilon}_r = \varepsilon_\infty + \omega_p^2 \sum_i \frac{f_i}{\omega_{0i}^2 - \omega^2 - i\omega\gamma_i}} \tag{20}$$

The plasma frequency in (20) is now defined by the total number of oscillating electrons per volume, $\omega_p = \sqrt{\frac{N_{tot}q^2}{m\epsilon_0}}$, so when you multiply $\omega_p^2 f_i$, you get $N_{tot} f_i$ as the N of each term, which is the number per volume of that species. If there is no dielectric response above the highest frequency, then set $\varepsilon_\infty = 1$ again.

Figure 7 depicts the permittivity for a situation with two resonances. Specifically, I've chosen $\varepsilon_\infty = 1$ and $\omega_p = 3$; then the left resonance has $\omega_0 = 1$, $\gamma = 0.06$, and $f = 0.3$; the right resonance has $\omega_0 = 4$, $\gamma = 0.04$, and $f = 0.7$. Notice how in particular the right resonance's $\varepsilon(0)$, which equals $1 + \omega_p^2/\omega_0^2 = 1.5625$, becomes the left resonance's ε_∞.

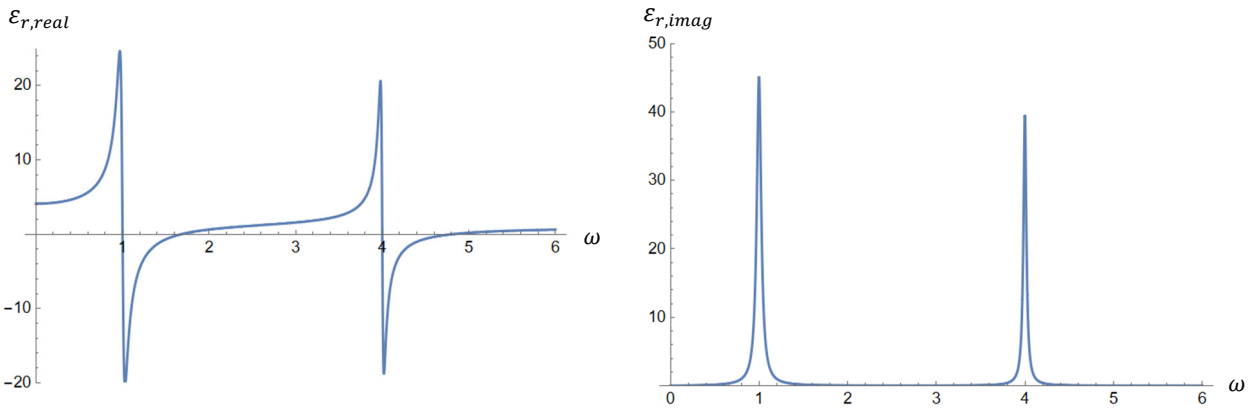

Figure 7. Plots of $\varepsilon_{r,real}$ and $\varepsilon_{r,imag}$ for a material with two resonances, using these parameters: $\varepsilon_\infty = 1$, $\omega_p = 3$; $\omega_{01} = 1$, $\gamma_1 = 0.06$, and $f_1 = 0.3$; and $\omega_{02} = 4$, $\gamma_2 = 0.04$, and $f_2 = 0.7$.

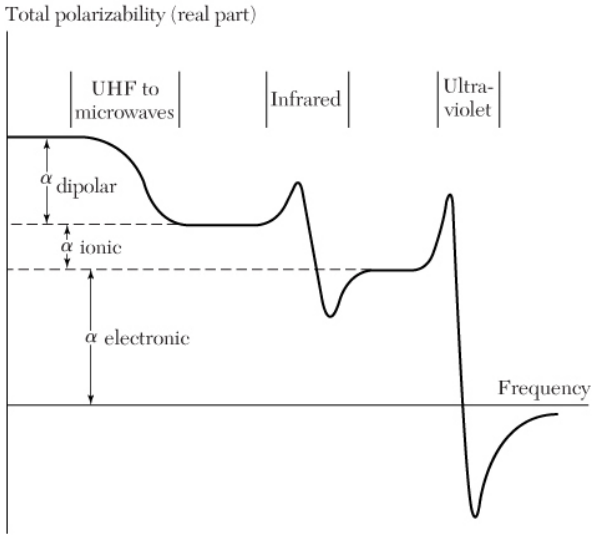

Figure 8. Schematic of the frequency dependence of the several contributions to the polarizability. From Kittel, Fig. 16-8.

Schematic plots like Figure 8 are used to depict how the susceptibility or polarizability of atoms in solids changes with frequency. Aside from what happens at low frequencies, labeled as a dipolar effect, there is a lot of similarity with the two-resonance plot of Fig. 7 (left). In Fig. (8) one resonance is labeled "electronic" and occurs in the ultraviolet from the dielectric response of the electrons; the other is labeled "ionic" and occurs in the infrared frequencies due to the dielectric response of ions in the lattice. The ionic model is identical to what was presented above for electrons, except the dipoles are formed by the combined motion of the positive and negative ions instead of the motion of the electrons. Because the ions are much heavier than electrons, the resonant frequencies in the ionic regime are much lower.

Applied to glasses

The electronic resonance itself is typically at UV frequencies. The shape of $\tilde{n}_{real}$ at frequencies below the resonance (i.e. visible wavelengths) explains the normal dispersion found in glasses. See for example Figure 9, which is a plot of $\tilde{n}_{real}$ vs. λ for several common glasses; compare against the shape of $\tilde{n}_{real}$ below resonance from Fig. 2 (plotted vs. ω).

Applied to gases

Griffiths applies (20) to a gas of molecules, but defines the quantities slightly differently. Instead of N being the total number of oscillating electrons and f_i being the fraction of electrons of type i, he uses N as the total

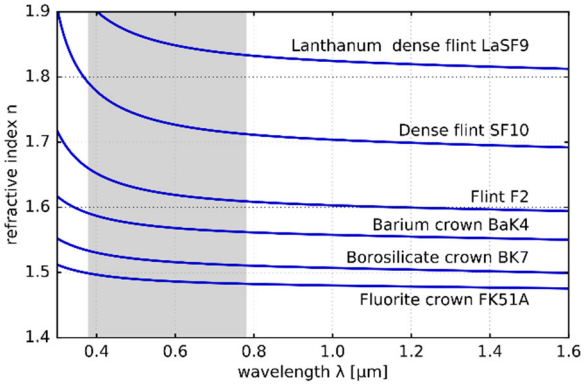

Figure 9. Real part of index of refraction plotted vs wavelength for several glasses. These frequencies are just below the electronic resonance ω_0. From Wikipedia, *Dispersion (optics).*

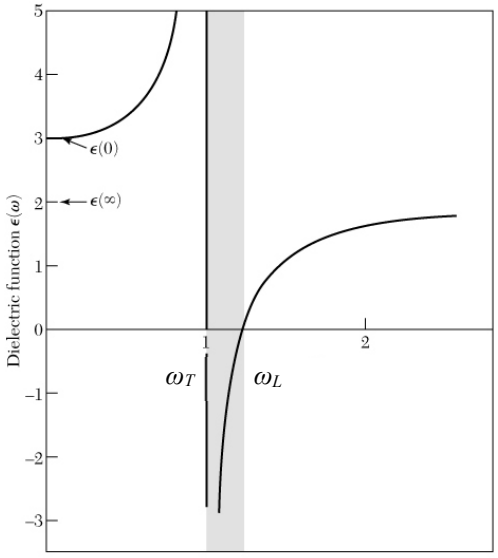

Figure 10. The real part of the relative permittivity near the resonant frequency of the oscillating ions in a solid, using the Lorentz model with no damping ($\gamma = 0$). Kittel has chosen $\varepsilon_\infty = 2$, $\omega_0 = 1$, and ω_p such that $\varepsilon(0) = 3$.. From Kittel, Fig. 14.13a.

number of *molecules* an d f_i being the *number of oscillating electrons of type i within each molecule*. It amounts to the same thing, though, since the number of molecules × number of oscillating electrons of type i within each molecule is the same as the number of oscillating electrons × fraction of electrons of type i.

Additionally, since gases are dilute, N will be small, and the terms in the summation will be much less than one. Therefore the index of refraction can be approximated by $\tilde{n} = \sqrt{\tilde{\varepsilon}_r} = \sqrt{1 + \text{small}} \approx 1 + \frac{1}{2}(\text{small})$. Griffiths uses that approximation to obtain Cauchy's formula, Equation (9.176) (5th edition):

$$n \approx 1 + A\left(1 + \frac{B}{\lambda^2}\right) \tag{21}$$

Applied to ions

As mentioned above, the Lorentz oscillator model can be applied to ionic materials. In such materials, the atoms are charged and their interactions with neighboring atoms cause them to be anchored to their lattice spots as with a spring. The relevant charge q is the ionic charge, which in general could be different than the charge of an electron, e. In partially ionic materials the charge can even be fractional.

Figure 10 depicts the Lorentz model applied to such oscillating ions, plotting the real part of the relative permittivity near the resonant frequency of the oscillating ions. Kittel has chosen $\gamma = 0$, $\varepsilon_\infty = 2$, $\omega_0 = 1$, and ω_p such that $\varepsilon(0) = 3$. The permittivity goes negative at ω_0, which is labeled ω_T because it represents a transverse oscillation of the ions in response to the electric field. The frequency at which the permittivity goes positive is defined in the text as ω_L because it relates to a longitudinal oscillation of the ions, at which there is no dielectric response because the electric field itself is a transverse oscillation. Electromagnetic waves with frequencies in the shaded region, between ω_T and ω_L, will not propagate in the medium but instead will have 100% reflectivity (if no damping).

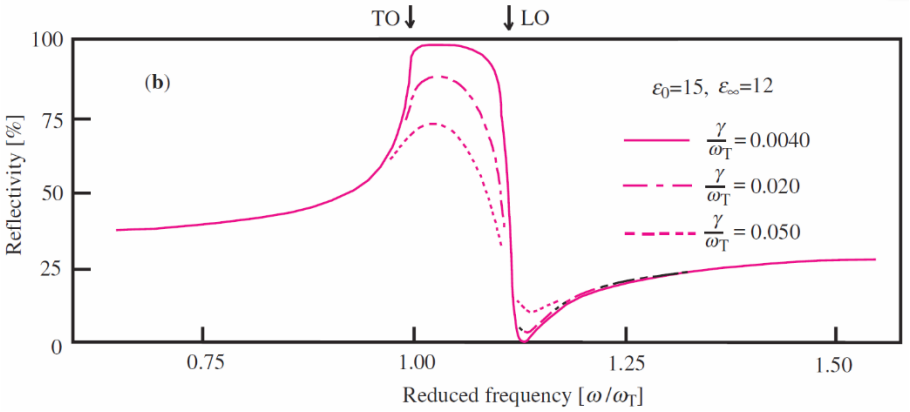

Figure 11. The infrared reflectivity from ionic oscillations, calculated from the Lorentz model for a few different values of damping. T and L stand for the transverse and longitudinal oscillations, the O stands for "optical phonon" modes. From Yu & Cardona, Fig. 6.31(b).

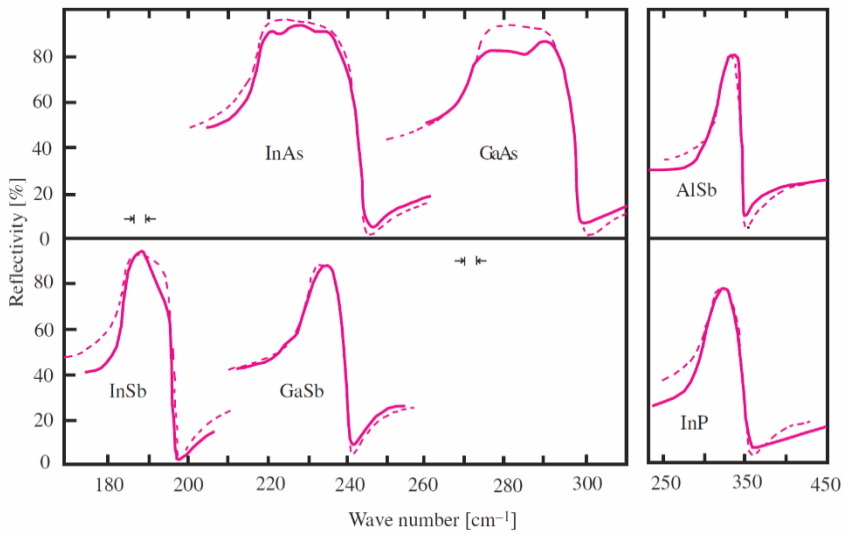

Figure 12. Actual experimental reflectivity data (solid curves), and theoretical fits from the Lorentz model (dashed curves) using ω_T, ω_L, and γ as fitting parameters. From Yu & Cardona, Fig. 6.32.

A well-known semiconductor physics textbook by Yu and Cardona plots the results of this model applied to ions, along with some actual experimental data for real materials. These are presented in Figures 11 and 12.

Important note on units: As you can see, the *x*-axis of Fig. 12 is labeled "Wave number [cm^{-1}]". CAUTION: that is not what we've been calling the wave number! The wave number k, as we've been using it, would be labeled as rad/m or rad/cm. By contrast, when you see experimental data that is labeled "cm^{-1}", particularly with optical data like this, they always mean $1/\lambda$ instead of $2\pi/\lambda$. I believe in the "olden days" k was originally defined as $1/\lambda$, and this has persisted in some things even today. If you see a feature on a graph like this at, say, 185 cm^{-1}, you can convert it to regular wavelength like this:

$$185 \text{ cm}^{-1} = 18500 \text{ m}^{-1} \rightarrow \text{ take inverse, } \lambda = 5.405\text{e-}5 \text{ meters} \approx 54 \text{ μm}$$

Applied to metals

The fun doesn't stop! In metals the valence electrons are not anchored to their nuclei (no springs) but instead are free to move around the material. However, they still respond to the electric field and experience damping in the same way as the electrons in insulators for which we derived the Lorentz oscillator model above. Can we just set $\omega_0 = 0$ to account for no restoring forces? Yes we can! Setting $\omega_0 = 0$ in the Lorentz oscillator model (which, with no restoring forces is sometimes called the Drude model) results in:

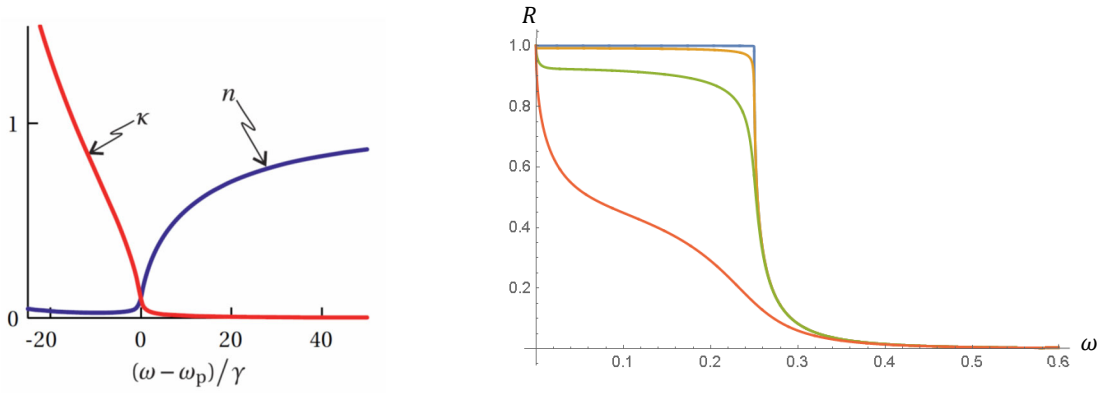

Figure 13. (left) Real and imaginary parts of the index of refraction for a conductor with $\varepsilon_\infty = 1$ and $\omega_p = 50\gamma$. From Peatross and Ware, Fig. 2.7. (right) Plot of R for a metal with $\varepsilon_\infty = 1$, $\omega_p = 0.25$ and four different damping values: $\gamma = 0,\ 0.001,\ 0.01,\ 0.1$. (The taller peaks are those with less damping.) Given this ω_p, the plot on the left corresponds to $\gamma = 0.005$, between the yellow and green curves.

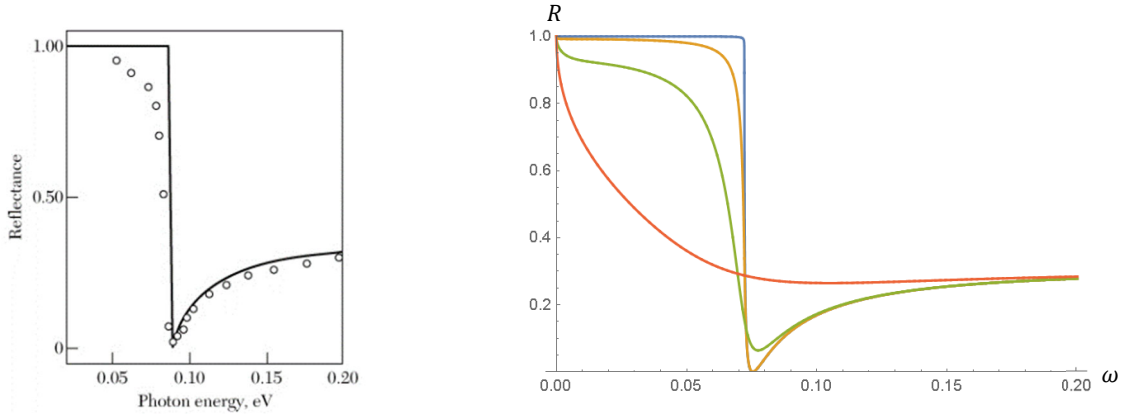

Figure 14. (left) Experimental reflectivity of InSb (empty points), fitted with the Lorentz model with no damping (solid line). From Kittel, Fig. 14.3. (right) Plot of R with $\varepsilon_\infty = 12$, $\omega_p = 0.25$, and four different damping values: $\gamma = 0,\ 0.001, 0.01, 0.1$. (The taller peaks are those with less damping.) This plot makes it look like Kittel should have used some damping in the fit on the left to smooth out the sharp features in his solid line.

$$\tilde{\varepsilon}_r = \varepsilon_\infty - \frac{\omega_p^2}{\omega^2 + i\omega\gamma} \tag{22}$$

Figure 13 plots the real and imaginary parts of $\tilde{n} = \sqrt{\tilde{\varepsilon}_r}$ from this equation with $\varepsilon_\infty = 1$ and $\omega_p = 50\gamma$, at frequencies close to ω_p; and also the reflectivity R from this equation for $\omega_p = 0.25$ (chosen to match ω_p in Fig. 14) and four values of damping: $\gamma = 0,\ 0.001,\ 0.01,\ 0.1$. Notice the abrupt crossover between being nearly purely imaginary (highly *reflective*) below the plasma frequency to nearly purely real (highly *transparent*) above the plasma frequency. Most metals have plasma frequencies in the UV, so if your eyes were sensitive to the far UV rather than to visible light, you would be able to see right through metals!

Finally, as an example where you need to use an ε_∞ not equal to 1, Figure 14 shows some data on the infrared reflectivity of indium antimonide (InSb) on the left, also with my own plot of R on the right using $\varepsilon_\infty = 12$, $\omega_p = 0.25$, and four damping values: $\gamma = 0,\ 0.001, 0.01, 0.1$. Notice how nicely my plot matches the experimental data; but also, how different my plot here is compared to Figure 13 in terms of where the reflectivity abruptly dips, even though ω_p is still 0.25. The differences arise from setting $\varepsilon_\infty = 12$.