The Maxwell Stress Tensor Theorem

by Dr. Colton, Physics 442 (last updated: Winter 2025)

Goal

The goal here is to figure out an equation that is analogous to Poynting's Theorem, but involving momentum instead of energy. As a reminder, Poynting's Theorem has these two forms:

$$-\frac{\partial U}{\partial t} = \frac{\partial W}{\partial t} + \oint \mathbf{S} \cdot d\mathbf{a}$$

$$-\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S}$$

where U and u are the energy and energy density stored in the fields, W and w are the work and work per volume done on charges, and **S** is the Poynting vector.

The new equations will have this form:

$$-\frac{\partial(\textit{stored momentum})}{\partial t} = \textit{force on charges} + \oint(-\overleftrightarrow{\mathbf{T}}) \cdot d\mathbf{a}$$

$$-\frac{\partial(\textit{momentum density})}{\partial t} = \textit{force density} + \nabla \cdot \left(-\overleftrightarrow{\mathbf{T}}\right)$$

where $\overleftrightarrow{\mathbf{T}}$ is the Maxwell Stress Tensor and $-\overleftrightarrow{\mathbf{T}}$ represents outward momentum flow like **S** represents outward energy flow. Because momentum and force are vectors, $\overleftrightarrow{\mathbf{T}}$ must be a tensor, which for now you can think of as being just a 3×3 matrix. I call these equations the "MST theorem" by analogy to Poynting's theorem, but I haven't seen that term used elsewhere.

Derivation

We will arrive at this equation by starting with the force on the charges, which is given by the Lorentz force equation:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}.$$

Let's define force per volume to be **f**. Since $q/volume$ is ρ and a charge density moving at speed **v** becomes a current density with $\rho\mathbf{v} = \mathbf{J}$, we therefore have:

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}.$$

We can use Gauss's law and Ampere's law to substitute $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$ and $\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$, to obtain:

$$\mathbf{f} = \varepsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}$$

$$\mathbf{f} = \varepsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \varepsilon_0 \left(\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}.$$

Let's arbitrarily add in $\frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B}$ to the right hand side to increase the symmetry between **E** and **B**, which we can do because the divergence of **B** is zero—so it's just adding in nothing. Let's also reverse the order of the cross product of the second term on the right hand side. That gives us:

$$\mathbf{f} = \varepsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B} - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) - \varepsilon_0\left(\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}. \tag{1}$$

Let's now tackle the last term of on right hand side of (1). It relates to $\partial \mathbf{S}/\partial t$, as we will see as follows:

$$\frac{\partial \mathbf{S}}{\partial t} = \frac{1}{\mu_0}\frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t}$$

$$\mu_0\frac{\partial \mathbf{S}}{\partial t} = \left(\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B} + \mathbf{E} \times \left(\frac{\partial \mathbf{B}}{\partial t}\right)$$

$$\mu_0\frac{\partial \mathbf{S}}{\partial t} = \left(\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B} + \mathbf{E} \times (-\nabla \times \mathbf{E}) \tag{2}$$

where the last step was done via Faraday's law and the step before that using the derivative product rule.

We can now see that the last term of (1) closely relates to the first term on the right hand side of (2), and equals the following:

$$-\varepsilon_0\left(\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B} = -\varepsilon_0\left(\mu_0\frac{\partial \mathbf{S}}{\partial t} - \mathbf{E} \times (-\nabla \times \mathbf{E})\right),$$

which simplifies to

$$-\varepsilon_0\left(\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B} = -\varepsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t} - \varepsilon_0\mathbf{E} \times (\nabla \times \mathbf{E}). \tag{3}$$

Substituting (3) into (1), we have:

$$\mathbf{f} = \varepsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B} - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) - \varepsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t} - \varepsilon_0\mathbf{E} \times (\nabla \times \mathbf{E}) \tag{4}$$

Nearing the end, we have a term now that is $\mathbf{E} \times (\nabla \times \mathbf{E})$, and a similar term for **B**. We can use Griffiths' Product Rule #4 which gives an equation for $\nabla(\mathbf{A} \cdot \mathbf{B})$; when **A** and **B** are the same field, let's call it **A**, PR4 results in this:

$$\nabla(A^2) = 2\mathbf{A} \times (\nabla \times \mathbf{A}) + 2(\mathbf{A} \cdot \nabla)\mathbf{A}$$

$$\mathbf{A} \times (\nabla \times \mathbf{A}) = \frac{1}{2}\nabla(A^2) - (\mathbf{A} \cdot \nabla)\mathbf{A}$$

We use that expression in (4) for **E** and also for **B**, and reorder the terms to obtain the final result for f:

$$\boxed{\mathbf{f} = \varepsilon_0\left((\nabla \cdot \mathbf{E})\mathbf{E} - \frac{1}{2}\nabla(E^2) + (\mathbf{E} \cdot \nabla)\mathbf{E}\right) + \frac{1}{\mu_0}\left((\nabla \cdot \mathbf{B})\mathbf{B} - \frac{1}{2}\nabla(B^2) + (\mathbf{B} \cdot \nabla)\mathbf{B}\right) - \varepsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t}} \tag{5}$$

This is the MST equation, although not how it's typically written.

Standard Forms

What does $(\mathbf{E}\cdot\nabla)\mathbf{E}$ even mean, anyway? It's a shortcut notation for this:

$$(\mathbf{E}\cdot\nabla)\mathbf{E} = \left(E_x\frac{\partial}{\partial x} + E_y\frac{\partial}{\partial y} + E_z\frac{\partial}{\partial z}\right)\left(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}\right)$$

which is actually 9 terms once you "FOIL" it out!

There are also other derivatives acting on the various components of **E**, and also for **B**, and (5) can be made to look quite a bit simpler if we put it in a vector form, like this:

$$\boxed{\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}\begin{pmatrix} 3\times 3\ matrix \\ called \\ T_{ij} \end{pmatrix} - \varepsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t}}$$

Or using a shortcut notation, like this:

$$\boxed{\mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \varepsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t}}$$

That is the standard "differential form" of the MST equation.

Grouping the various components of (5) together and doing some algebra which I won't show, the 3×3 matrix which I've called T_{ij}, also known as the "Maxwell Stress Tensor", turns out to be given by the following formula for the $(i,j)^{\text{th}}$ component:

$$\boxed{T_{ij} = \varepsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)}$$

Here δ_{ij} refers to the Kronecker delta function, which is 1 if $i = j$ (for the diagonal components of the matrix) and 0 otherwise (for the off-diagonal components).

By integrating over a volume of space and applying the divergence theorem we have these other two standard "integral forms" of the MST equation which gives the force rather than force density:

$$\boxed{\mathbf{F} = \oint \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} - \varepsilon_0\mu_0\frac{\partial}{\partial t}\int \mathbf{S}d\tau}$$

$$\boxed{\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \oint \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} da - \varepsilon_0\mu_0\frac{\partial}{\partial t}\int \mathbf{S}d\tau}$$

I've explicitly written out the matrix elements and have also written $d\mathbf{a}$ in terms of a unit vector normal to the surface, **n**, i.e. $d\mathbf{a} = \mathbf{n}da$. When actually using the MST equation to deduce forces on charges, that last form will be the most useful. Just be sure to draw a closed "Maxwellian surface" (like a Gaussian surface) surrounding the charges whose force you want to know, so that you are very clear about the surface integral, and use a different **n** for each part of the closed surface as appropriate.

Conservation of Momentum

The MST equation is a statement of conservation of momentum. This can most easily be seen by considering a region of space where there is no medium (and hence no force), where it becomes:

$$-\frac{\partial}{\partial t}(\varepsilon_0\mu_0\mathbf{S}) = \nabla\cdot\left(-\overleftrightarrow{\mathbf{T}}\right)$$

Compare that to these other two previous conservation laws…

Conservation of charge (equation of continuity):

$$-\frac{\partial\rho}{\partial t} = \nabla\cdot\mathbf{J}$$

Conservation of energy (Poynting's theorem, in the absence of charges):

$$-\frac{\partial u}{\partial t} = \nabla\cdot\mathbf{S}$$

Just like ρ is charge density and u is energy density, we can recognize that $\varepsilon_0\mu_0\mathbf{S}$ is momentum density, which the units bear out.

Just like **J** represents an outward flow of charge (charge per time, per area) and **S** is an outward flow of energy (energy per time, per area), $-\overleftrightarrow{\mathbf{T}}$ represents an outward flow of momentum. Careful with the sign! With regards to what the specific matrix components represent, I really like this sentence from the 3rd edition of Griffiths: "Specifically, $-T_{ij}$ is the momentum in the i direction crossing a surface oriented in the j direction, per unit area, per unit time."

Conclusion

In summary: momentum is conserved in the absence of the fields exerting forces on charges. When the fields do exert forces on charges, then by rewriting the order of the MST terms to match what I said was the goal at the start, like this

$$-\frac{\partial}{\partial t}\left(\int\varepsilon_0\mu_0\mathbf{S}d\tau\right) = \mathbf{F} + \oint(-\overleftrightarrow{\mathbf{T}})\cdot d\mathbf{a}$$

$$-\frac{\partial(\boldsymbol{stored\ momentum})}{\partial t} = \boldsymbol{force\ on\ charges} + \oint(-\overleftrightarrow{\mathbf{T}})\cdot d\mathbf{a},$$

we can say that stored momentum in a volume of space can be lost either by (1) using it to exert a force on charges within that volume, or (2) by having an overall outward flow of momentum from the region.