Most important equations from Physics 442

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Chapter 7 (the 442 half)

Polarization current:

$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$

$\mathbf{J}_{\mathbf{tot}} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p$

Ampere's law with Maxwell's fix, for materials: $\nabla \times \mathbf{H} = \mathbf{J}_\mathbf{f} + \frac{\partial \mathbf{D}}{\partial t}$

Ohm's law: $\mathbf{J} = \sigma \mathbf{E}$

Scattering model: $\sigma = \frac{ne^2\tau}{m}$

Inductance:

Mutual inductance: $M_{12} = \frac{\Phi_{loop\,2}}{I_1}$

Self inductance: $L = \frac{\Phi_{loop\,1}}{I_1}$

EMF: $\mathcal{E} = -L\frac{dI}{dt}$

Chapter 8

Energy:

Stored energy: $U = \int \left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) d\tau$

Energy density: $u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$

Energy density in materials: $u = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$

Poynting vector:

Basic: $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$

In matter: $\mathbf{S} = \mathbf{E}\times\mathbf{H}$

Poynting's theorem:

Basic: $\nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J} - \frac{\partial u}{\partial t}$

Alternate: $\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -\nabla\cdot\mathbf{S}$

Integral form: $\oint \mathbf{S}\cdot d\mathbf{a} + \frac{\partial W}{\partial t} = -\frac{\partial U}{\partial t}$

With no work (continuity eq. for energy): $\frac{\partial u}{\partial t} = -\nabla\cdot\mathbf{S}$

In matter: $\frac{\partial u}{\partial t} = \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}$

Momentum density $\mathbf{g} = \frac{1}{c^2}\mathbf{S} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0\mathbf{E}\times\mathbf{B}$

Maxwell stress tensor: $T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$

Maxwell force equation (my name; I think it's unnamed in the book):

Basic: $\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0\mu_0 \frac{\partial}{\partial t}\int_V \mathbf{S}\, d\tau$

Alternate: $\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \oint_S \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} da - \epsilon_0\mu_0 \frac{\partial}{\partial t}\int_V \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} d\tau$

Per volume: $\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \mu_0\epsilon_0 \frac{\partial \mathbf{S}}{\partial t}$

With no force (continuity eq. for momentum): $\frac{\partial \mathbf{g}}{\partial t} = -\nabla \cdot \left(-\overleftrightarrow{\mathbf{T}}\right)$

Chapter 9

Wave equation:

1D: $\frac{\partial^2 f}{dz^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

3D: $\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

Speed of light: $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$

Electric and magnetic fields in free space:

$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$

$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0$

Wave quantity relationships:

Wavelength, frequency, speed: $v = \lambda f$

Wavelength and wavevector: $\lambda = \frac{2\pi}{k}$

Frequency, etc.: $f = \frac{1}{T} = \frac{kv}{2\pi}$

Angular frequency: $\omega = 2\pi f = kv$

Inhomogeneous wave equation in linear, isotropic matter (assuming Ohm's law):

Electric field: $\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}$

Magnetic field: $\nabla^2\mathbf{B} = \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{B}}{\partial t}$

Plane waves:

Electric field: $\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{\mathbf{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

Alternate: $\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{\mathrm{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}}$

Magnetic field: $\tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{c}\,\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t)$

Alternate: $\tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{c}\tilde{\mathrm{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{k}} \times \hat{\mathbf{n}}$

Intensity:

Plane wave in free space: $\langle S\rangle = \frac{1}{2}c\epsilon_0 E_0^2$

Plane wave in matter: $\langle S\rangle = \frac{1}{2}nc\epsilon_0 E_0^2$

Radiation pressure: $P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E^2 = \frac{I}{c}$

Index of refraction:

Basic: $n = \sqrt{\epsilon_r \mu_r} \approx \sqrt{\epsilon_r}$

Wave speed: $v = \frac{c}{n} = \frac{\omega}{k}$

Laws of geometrical optics:

Reflection: $\theta_I = \theta_R$

Refraction (Snell's law): $\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$

Fresnel equations:

Definitions of quantities:

$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$

$\beta = \frac{\mu_1 n_2}{\mu_2 n_1} \approx \frac{n_2}{n_1}$

$r = \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}}$

$t = \frac{\tilde{E}_{0T}}{\tilde{E}_{0I}}$

$R = \frac{P_R}{P_I} = r^2$

$T = \frac{P_T}{P_I} = \alpha\beta t^2 = 1 - R$

normal incidence (assuming $\mu_r = 1$ for both materials):

$r = \frac{n_1 - n_2}{n_1 + n_2}$

$t = \frac{2n_1}{n_1 + n_2}$

$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$

$T = \frac{4n_1 n_2}{(n_1 + n_2)^2}$

p polarization:

$r = \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{\alpha - \beta}{\alpha + \beta}$

$t = \frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2}{\alpha + \beta}$

$R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2$

$T = \frac{4\alpha\beta}{(\alpha + \beta)^2}$

s polarization:

$r = \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1 - \alpha\beta}{1 + \alpha\beta}$

$t = \frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2}{1 + \alpha\beta}$

$R = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2$

$T = \frac{4\alpha\beta}{(1 + \alpha\beta)^2}$

Brewster's angle: $\tan\theta_B = \frac{n_2}{n_1}$

Conductors:

Complex permittivity, complex n: $\tilde{\epsilon}_r = \tilde{n}^2$

Complex k: $\frac{\tilde{k}}{\omega} = \frac{\tilde{n}}{c}$
$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$
Skin depth: $\delta = \frac{1}{k_{imag}}$
$\tilde{\epsilon}_r = \epsilon_{r,real} + i\frac{\sigma}{\epsilon_0\omega}$

Lorentz oscillator model:
Susceptibility: $\tilde{\chi} = \frac{qE_0}{m}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}$
Plasma frequency: $\omega_p = \sqrt{\frac{Nq^2}{m\epsilon_0}}$
Dielectric constant: $\tilde{\epsilon}_r = 1 + \frac{\omega_p^2}{\omega_0^2-\omega^2-i\omega\gamma}$
Applied to conductors: $\tilde{\epsilon}_r = 1 - \frac{\omega_p^2}{\omega^2+i\omega\gamma}$

Waveguides:
Rectangular waveguides: $k = \sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}$
Cylindrical waveguides: $k = \sqrt{\frac{\omega^2}{c^2} - \frac{u_{\alpha m}^2}{R^2}}$
$u_{\alpha m}$ is Bessel function zero for TM, Bessel function derivative zero for TE

Chapter 10

Fields from potentials:
$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$
$\mathbf{B} = \nabla \times \mathbf{A}$

Gauge transformations:
$\mathbf{A}' = \mathbf{A} + \nabla\lambda$
$V' = V - \frac{\partial\lambda}{\partial t}$

Coulomb vs Lorentz gauges:
Coulomb:
Definition: $\nabla \cdot \mathbf{A} = 0$

Poisson's eq: $\nabla^2 V = -\rho/\epsilon_0$
Eq. for **A**: $\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla V)$
Lorentz:
Definition: $\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$
Inhomogeneous wave eq.: $\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$
IWE, alternate notation: $\Box^2 V = -\frac{\rho}{\epsilon_0}$
Inhomogeneous wave eq.: $\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}$
IWE, alternate notation: $\Box^2\mathbf{A} = -\mu_0\mathbf{J}$

Assuming Lorentz gauge from here on out…

Potentials from charge & current densities:

Retarded time: $t_r = t - \frac{\mathscr{r}}{c}$

$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r},t_r)}{\mathscr{r}}\,d\tau'$

$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r},t_r)}{\mathscr{r}}\,d\tau'$

Liénard–Wiechert potentials for point charges:

$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})}$

$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$

Fields of a moving point charge:

General formulas:

Defn. of $\mathbf{u}$: $\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$

$\mathbf{E}(\mathbf{r},t) = \frac{q\,\mathscr{r}}{4\pi\epsilon_0(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})]$

$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\,\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r},t)$

Constant velocity formulas:

Distance vector: $\mathbf{R}$ = from *current* location of charge to field point

θ = angle between $\mathbf{v}$ and $\mathbf{R}$

$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}$

$\mathbf{B}(\mathbf{r},t) = \frac{1}{c^2}\,\mathbf{v} \times \mathbf{E}$

Chapter 11

Radiated power:

Relationship to Poynting vector: $P = \int \langle \mathbf{S} \rangle \cdot d\mathbf{a}$

From oscillating electric dipole: $\langle \mathbf{S} \rangle \sim \frac{\sin^2\theta}{r^2}$; $P = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$

From oscillating magnetic dipole: $\langle \mathbf{S} \rangle \sim \frac{\sin^2\theta}{r^2}$; $P = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$

From arbitrary localized source: $P = \frac{\mu_0 \ddot{p}^2}{6\pi c}$

From point charges (Larmor formula): $\langle \mathbf{S} \rangle \sim \frac{\sin^2\theta}{\mathscr{r}^2}$; $P = \frac{\mu_0 q^2 a^2}{6\pi c}$

Liénard correction to Larmor formula: $\frac{dP}{d\Omega} \sim \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}$; $P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\mathbf{v}\times\mathbf{a}}{c}\right|^2\right)$

Abraham-Lorentz radiation reaction force: $\mathbf{F} = \frac{\mu_0 q^2 \dot{\mathbf{a}}}{6\pi c}$

Chapter 12

Definitions:

$\beta = v/c$
$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

Einstein velocity addition: $\beta_{13} = \frac{\beta_{12}+\beta_{23}}{1+\beta_{12}\beta_{23}}$

Velocity addition alternate notation, including transverse formulas:
(important note: prime frame is moving to the right with respect to unprimed frame)

$u_x' = \frac{u_x - v}{1 - v u_x/c^2}$
$u_y' = \frac{u_y}{\gamma(1 - v u_x/c^2)}$
$u_z' = \frac{u_z}{\gamma(1 - v u_x/c^2)}$

Time dilation: $\Delta t_2 = \frac{1}{\gamma}\Delta t_1$
Length contraction: $\Delta x_2 = \gamma \Delta x_1$

Lorentz transformations: (important note: all γ's here are γ_v)

$$\begin{pmatrix} ct \\ x \end{pmatrix}_2 = \begin{pmatrix} \gamma & \pm\gamma\beta \\ \pm\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}_1$$

Defn. of $\mathbf{\Lambda}$ matrix: $\mathbf{\Lambda} = \Lambda^\mu_\nu = \begin{pmatrix} \gamma & \pm\gamma\beta & 0 & 0 \\ \pm\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Master list of all 4-vectors which transform according to $\mathbf{\Lambda}$: (important note: all γ's here are γ_u)

Position: $x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$

Velocity: $\eta^\mu = \begin{pmatrix} \gamma c \\ \gamma u_x \\ \gamma u_y \\ \gamma u_z \end{pmatrix}$

Momentum: $p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$

Relativistic energy, is conserved: $E = \gamma m c^2$
Relativistic momentum, is conserved: $p_x = \gamma m u_x$, etc.

Acceleration: $a^\mu = \begin{pmatrix} \gamma^4 \frac{\mathbf{u}\cdot\mathbf{a}}{a} \\ \gamma^2 \left(a_x + \frac{u_x^2 a_x}{c^2 - u^2}\right) \\ \gamma^2 \left(a_y + \frac{u_y^2 a_y}{c^2 - u^2}\right) \\ \gamma^2 \left(a_z + \frac{u_z^2 a_z}{c^2 - u^2}\right) \end{pmatrix}$

Minkowski force: $K^\mu = m a^\mu$

Potential: $A^\mu = \begin{pmatrix} V/c \\ A_x \\ A_y \\ A_z \end{pmatrix}$

Current density: $J^\mu = \begin{pmatrix} \rho c \\ J_x \\ J_y \\ J_z \end{pmatrix}$

where $J_x = \rho u_x$, etc.

Covariant/contravariant relationship: $a_\mu = (-a^0,\ a^1, a^2,\ a^3\ )$

Relativistic energy-momentum relation: $E^2 = (pc)^2 + (mc^2)^2$

Field transformations: (all γ's here are γ_v; prime frame is moving to the right)

$E'_x = E_x$
$E'_y = \gamma(E_y - vB_z)$
$E'_z = \gamma(E_z + vB_y)$
$B'_x = B_x$
$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$
$B'_z = \gamma\left(B_z - \frac{v}{c^2}B_y\right)$

List of 4-tensors:

Field tensor: $F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$

Dual tensor: $G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y & E_x/c & 0 \end{pmatrix}$

Stress tensor: $T^{\mu\nu} = \begin{pmatrix} u & S_x/c & S_y/c & S_z/c \\ S_x/c & -T_{1,1} & -T_{1,2} & -T_{1,3} \\ S_y/c & -T_{2,1} & -T_{2,2} & -T_{2,3} \\ S_z/c & -T_{3,1} & -T_{3,2} & -T_{3,3} \end{pmatrix}$

Lorentz transformations: $F^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma F^{\lambda\sigma}$ (similarly for $G^{\mu\nu}$, $T^{\mu\nu}$)

Important equations in tensor form:

Maxwell equations (first half): $\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$
Maxwell equations (second half): $\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$
Lorentz force: $K^\mu = q\eta_\nu F^{\mu\nu}$
Fields from potentials: $F^{\mu\nu} = \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu}$
Lorentz gauge condition: $\frac{\partial A^\mu}{\partial x^\mu} = 0$

Potentials from sources: $\Box^2 A^\mu = -\mu_0 J^\mu$
Equation of continuity for charge: $\frac{\partial J^\mu}{\partial x^\mu} = 0$
Equation of continuity for energy & momentum: $\frac{\partial T^{\mu\nu}}{\partial x^\mu} = 0$