TM Modes of a Cylindrical Waveguide

by Dr. Colton, Physics 442 (last updated: Winter 2020)

Calculating TM Modes

Using Mathematica, we can calculate the first 16 TM modes for a rectangular waveguide. I'm using a dimensions of $R = 10$ cm which was chosen arbitrarily.

Here are the cutoff frequencies of the first 16 modes; they are shown first in table form and then in list form in ascending order.

```
In[44]:= uαn[α_, n_] = BesselJZero[α, n];

         (* for dimension of R = 10 cm *)
         R = 0.10;
         c = 3*^8;
         wcutoff[alpha_, n_] := uαn[alpha, n] c / R
         cutofftable = Table[wcutoff[alpha, n], {alpha, 0, 3}, {n, 1, 4}] ;
         cutofftable // MatrixForm
         cutofftable // Flatten // Sort
```

Out[49]//MatrixForm=

$$\begin{pmatrix} 7.21448\times10^{9} & 1.65602\times10^{10} & 2.59612\times10^{10} & 3.53746\times10^{10} \\ 1.14951\times10^{10} & 2.10468\times10^{10} & 3.05204\times10^{10} & 3.99711\times10^{10} \\ 1.54069\times10^{10} & 2.52517\times10^{10} & 3.48595\times10^{10} & 4.43879\times10^{10} \\ 1.91405\times10^{10} & 2.92831\times10^{10} & 3.90456\times10^{10} & 4.86704\times10^{10} \end{pmatrix}$$

Out[50]= $\{7.21448\times10^{9}, 1.14951\times10^{10}, 1.54069\times10^{10}, 1.65602\times10^{10}, 1.91405\times10^{10}, 2.10468\times10^{10}, 2.52517\times10^{10}, 2.59612\times10^{10}, 2.92831\times10^{10}, 3.05204\times10^{10}, 3.48595\times10^{10}, 3.53746\times10^{10}, 3.90456\times10^{10}, 3.99711\times10^{10}, 4.43879\times10^{10}, 4.86704\times10^{10}\}$

The $k(\omega)$ dispersion relations for the first 16 modes are as follows:

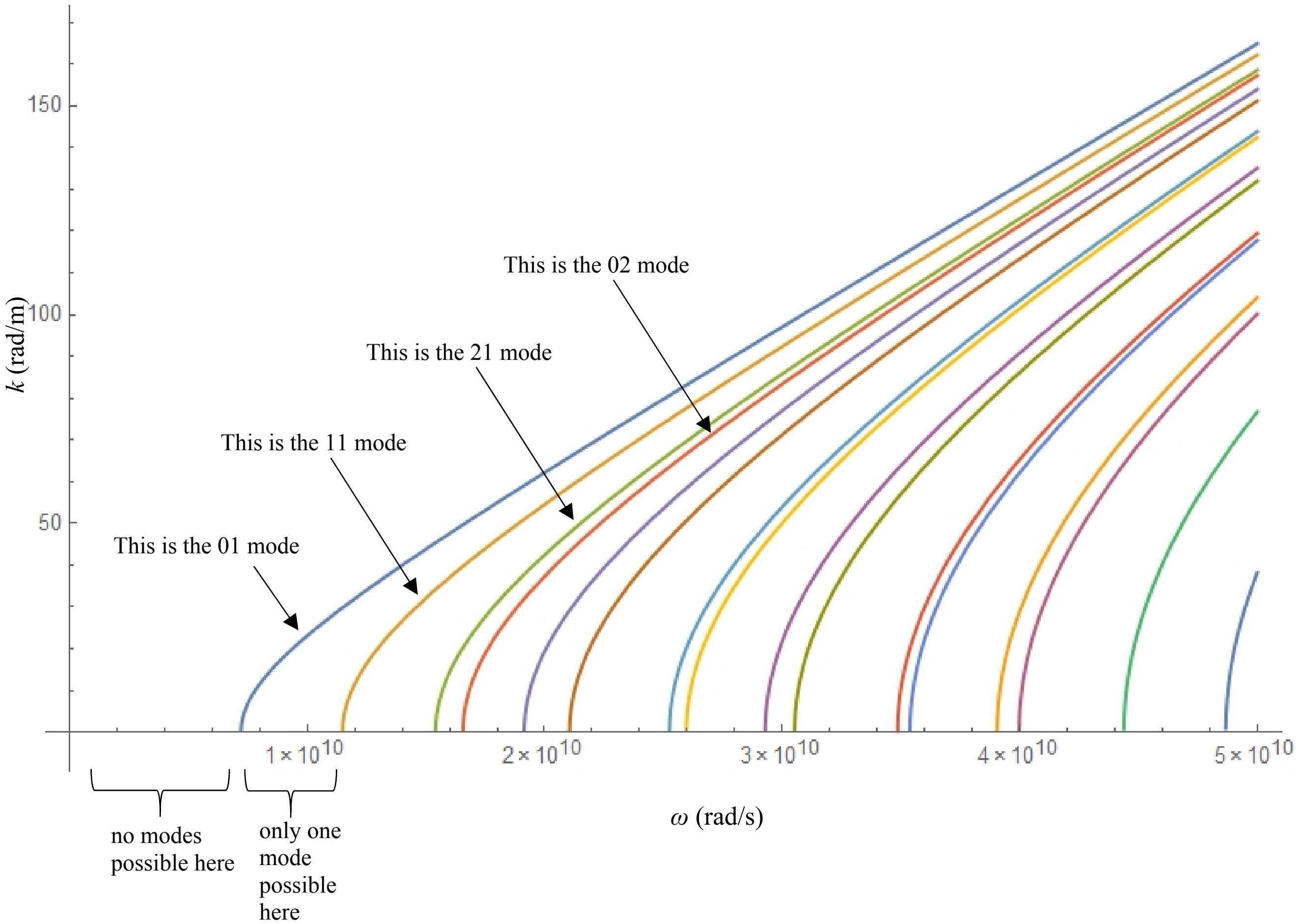
```
In[15]:= k[w_, alpha_, n_] := Sqrt[w^2 / c^2 - wcutoff[alpha, n]^2 / c^2]
         Table[k[w, alpha, n], {alpha, 0, 3}, {n, 1, 4}] // Flatten // Sort // Reverse
```

Out[16]=

$$\left\{\sqrt{-578.319+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-1468.2+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-2637.46+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-3047.13+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-4070.65+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-4921.85+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-7085.+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-7488.7+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-9527.76+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-10\,349.9+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-13\,502.1+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-13\,904.+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-16\,939.5+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-17\,752.1+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-21\,892.+\frac{w^2}{90\,000\,000\,000\,000\,000}}, \sqrt{-26\,320.1+\frac{w^2}{90\,000\,000\,000\,000\,000}}\right\}$$

$k(\omega)$ dispersion relation plots

For a given mode its dispersion relation is set by one of the following curves.

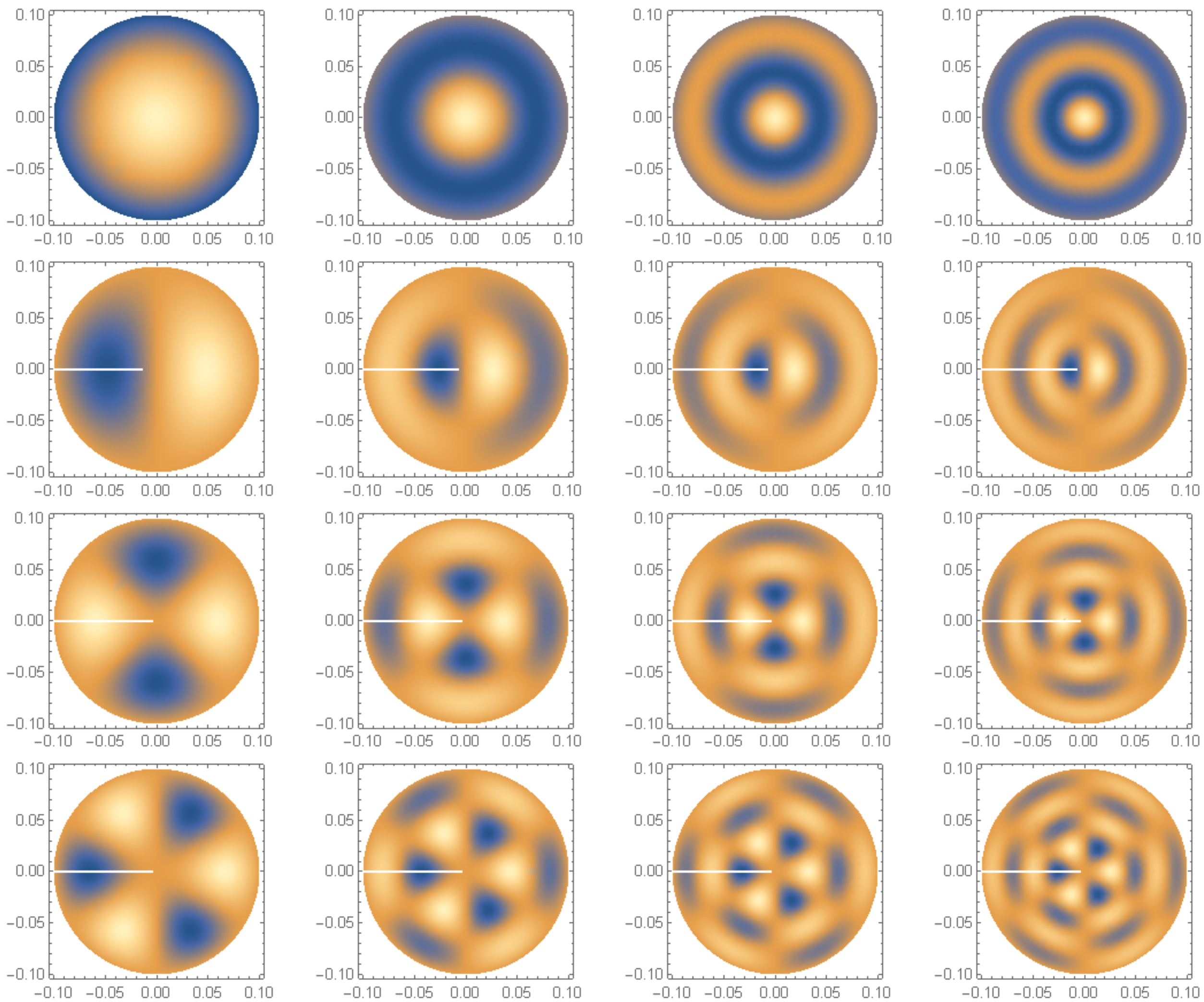


For a given mode its dispersion relation is set by these curves. Note that these are the FIRST 16 modes, in the sense that α goes from 0 to 3 and n goes from 1 to 4, but they are not necessarily the LOWEST 16 modes. For example, the $(\alpha = 4, n = 1)$ mode is lower than many of these that are shown (with its $\omega_{cutoff} = 2.28 \times 10^{10}$ rad/s).

E_z plots

Recall that the governing field for the TM modes is the z component of the electric field (because the magnetic field has no z-component). Here are plots of E_z for the first 16 modes. Aside from the upper left one, which has a node at the boundary and a single antinode in the middle, tannish white is the positive antinode and blue is the negative antinode.

```
Table[DensityPlot[Evaluate[f[x, y, alpha, n]], {x, -0.1, 0.1}, {y, -0.1, 0.1},
   RegionFunction → Function[{x, y, z}, x^2 + y^2 < 0.1^2], PlotRange → All] , {alpha, 0, 3}, {n, 1, 4}] //
 TableForm
```



$B_z = 0$ by definition, and all of the other nonzero components of the fields, namely $E_x, E_y, B_x,$ and B_y, can be calculated from E_z.