

# Complex wave number, index of refraction, and relative permittivity

by Dr. Colton, Physics 442 (last updated: Jul 2025)

with thanks to Carson Tenney for help thinking through Griffiths (9.126) and issues related to complex  $\tilde{\sigma}_f$  and  $\tilde{\epsilon}_r$ .

## Introduction

Complex wave numbers and other related quantities arise in the context of waves in materials in two capacities: (1) a need for decaying oscillations to represent fields which damp out inside the material, i.e. a regular cosine wave multiplied by an exponential decay, and (2) phase shifts between fields, for example the electric field  $\mathbf{E}$  and an induced polarization field  $\mathbf{P}$ , which creates a complex susceptibility and hence a complex permittivity. And, oddly enough, those two things are intimately connected.

## Notation

This handout is primarily about *complex wave numbers*, *complex indices of refraction*, and *complex relative permittivities*. Unfortunately there is a dizzying amount of varying—and overlapping!!—notation that people use for these three quantities. To be as clear as possible and avoid any ambiguity, I myself will use the following symbols, but I have listed other common notations.

$\tilde{k}$  = complex wave number

$k_{real}$  = real part of  $\tilde{k}$ , called  $k$  in Griffiths and some other sources

$k_{imag}$  = imaginary part of  $\tilde{k}$ , called  $\kappa$  in Griffiths and some other sources

$\tilde{n}$  = complex index of refraction, not directly used in Griffiths,  
also called  $\mathcal{N}$  by some sources (e.g. Peatross and Ware)

$n_{real}$  = real part of  $\tilde{n}$ , also called  $n$  by nearly all sources (including Griffiths)

$n_{imag}$  = imaginary part of  $\tilde{n}$ , not directly used in Griffiths,  
also called  $k$  by nearly all sources (!)

and  $\kappa$  by some (e.g. Peatross and Ware along with some others)

→ to add insult to injury, many sources define the imaginary index of refraction  $k$  as the *negative* of  $n_{imag}$ . I believe this has to do with representing complex traveling waves as  $e^{i(kz-\omega t)}$  vs  $e^{i(\omega t-kz)}$ . As near as I can tell, the former is used by most physicists and the latter by most engineers.

$\tilde{\epsilon}_r$  = complex relative permittivity aka complex dielectric constant  
also called  $\epsilon$  by many sources and  $K$  or  $\tilde{K}$  by others

$\epsilon_{r,real}$  = real part of  $\tilde{\epsilon}_r$ , not directly used in Griffiths,  
also called  $\epsilon_r$ ,  $\epsilon_1$  or  $\epsilon'$  by many sources and  $K$ ,  $K_r$  or  $K'$  by others

$\epsilon_{r,imag}$  = imaginary part of  $\tilde{\epsilon}_r$ , not directly used in Griffiths,  
also called  $\epsilon_2$  or  $\epsilon''$  by many sources and  $K_i$  or  $K''$  by others

## Why have a complex wave number?

In conductors and in insulators close to resonant transitions (see the Lorentz oscillator model handout), the fields damp out as they penetrate the material. Using a complex wave number allows us to represent decaying waves as regular complex exponentials. Assuming travel in the  $z$ -direction for simplicity, plane waves have the following form:

$$\tilde{E} = E_0 e^{i(kz - \omega t)} \quad (1)$$

When the wave number is complex, this becomes:

$$\begin{aligned} \tilde{E} &= E_0 e^{i(\tilde{k}z - \omega t)} \\ \tilde{E} &= E_0 e^{i((k_{real} + ik_{imag})z - \omega t)} \\ \tilde{E} &= E_0 e^{-k_{imag}z} e^{i(k_{real}z - \omega t)} \end{aligned}$$

Taking the real part, we have:

$$E = E_0 e^{-k_{imag}z} \cos(k_{real}z - \omega t) \quad (2)$$

Notice here that in order to represent a damped, rightward moving wave,  $k_{imag}$  and  $k_{real}$  must both be positive quantities:  $\tilde{k}$  is in the first quadrant.

The real part of  $\tilde{k}$  is therefore related to the oscillations in space as per the usual definition of wave number, through the  $\cos(k_{real}z - \omega t)$  term:

$$\boxed{\lambda = \frac{2\pi}{k_{real}}} \quad (3)$$

The imaginary part of  $\tilde{k}$  is related to how the amplitude of the wave decays as it penetrates the material, through the  $e^{-k_{imag}z}$  term. The inverse of  $k_{imag}$  has units of meters, and is often called the *skin depth*,  $\delta$ .

$$\boxed{\delta = \frac{1}{k_{imag}}} \quad (4)$$

The decay of the wave is also characterized by the *absorption coefficient*,  $\alpha$ , which describes the fall off in intensity according to  $I = I_0 e^{-\alpha z}$ . Because intensity varies with amplitude *squared*, we have:

$$\boxed{\alpha = 2k_{imag}} \quad (5)$$

## Basic relationships between $\tilde{k}$ , $\tilde{n}$ , and $\tilde{\epsilon}_r$

The relationship between complex index of refraction  $\tilde{n}$  and complex wave number  $\tilde{k}$  is defined through this equation:

$$\boxed{\tilde{n} = \frac{c}{\omega} \tilde{k}} \quad (6)$$

Equating the real and imaginary parts of that equation (and remembering that  $\omega$  must be real) results in:

$$\left. \begin{aligned} n_{real} &= \frac{c}{\omega} k_{real} \\ n_{imag} &= \frac{c}{\omega} k_{imag} \end{aligned} \right\} \quad (7)$$

Note in particular that  $\tilde{n}$  and  $\tilde{k}$  will always have the same complex angle, and since  $\tilde{k}$  must be in the first quadrant ( $k_{real}$  and  $k_{imag}$  are both positive),  $\tilde{n}$  must also be in the first quadrant.

You'll also sometimes see these equations expressed in terms of the absorption coefficient  $\alpha$ ; for example  $n_{imag} = \frac{c}{\omega} k_{imag}$  becomes:

$$n_{imag} = \frac{c}{2\omega} \alpha \quad (8)$$

And since the ratio  $c/\omega$  equals  $\lambda_{vacuum}/2\pi$ , we also have:

$$n_{imag} = \frac{\lambda_{vacuum}}{4\pi} \alpha \quad (9)$$

Insulators: In materials with no free current (i.e. insulators), the complex relative permittivity  $\tilde{\epsilon}_r$  and complex index of refraction  $\tilde{n}$  are related through this equation:

$$\boxed{\tilde{\epsilon}_r = \tilde{n}^2} \quad (\text{for insulators}) \quad (10)$$

Also, since  $\tilde{n}$  must be in the first quadrant, then  $\tilde{\epsilon}_r$  must be either in the first or second quadrants (for insulators). We'll return to that fact below.

Plugging in  $\tilde{\epsilon}_r = \epsilon_{r,real} + i\epsilon_{r,imag}$  and  $\tilde{n} = n_{real} + in_{imag}$  and equating the real and imaginary parts of that equation results in:

$$\left. \begin{aligned} \epsilon_{r,real} &= n_{real}^2 - n_{imag}^2 \\ \epsilon_{r,imag} &= 2n_{real}n_{imag} \end{aligned} \right\} \quad (\text{for insulators}) \quad (11)$$

(However, as per the “add insult to injury” comment on page 1, you will see some sources write that last equation as  $\epsilon'' = -2nk$ .)

It's “left as an exercise for the reader” to derive the inverse equations:

$$\left. \begin{aligned} n_{real} &= \sqrt{\frac{|\tilde{\epsilon}_r| + \epsilon_{r,real}}{2}} \\ n_{imag} &= \sqrt{\frac{|\tilde{\epsilon}_r| - \epsilon_{r,real}}{2}} \end{aligned} \right\} \quad (\text{for insulators}) \quad (12)$$

where the absolute values mean the magnitude of the complex number, i.e.  $|\tilde{\epsilon}_r| = \sqrt{\epsilon_{r,real}^2 + \epsilon_{r,imag}^2}$ .

The complex wave number  $\tilde{k}$  and complex relative permittivity  $\tilde{\epsilon}_r$  are related through this equation:

$$\boxed{\tilde{k}^2 = \frac{\omega^2}{c^2} \tilde{\epsilon}_r} \quad (\text{for insulators}) \quad (13)$$

### Complex response functions

As mentioned in the Introduction, phase shifts between fields create complex response functions. We will assume linear, isotropic materials. Fundamental among those are:

$$\mathbf{P} = \epsilon_0 \tilde{\chi}_e \mathbf{E} \quad (14)$$

which is the definition of complex susceptibility, leading to:

$$\mathbf{D} = \epsilon_0 \tilde{\epsilon}_r \mathbf{E} \quad (15)$$

which is the definition of complex permittivity, and due to  $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ , the two are connected in the usual way:

$$\tilde{\epsilon}_r = 1 + \tilde{\chi}_e \quad (16)$$

Similarly the magnetic susceptibility and permeability,  $\chi_r$  and  $\mu_r$ , could also potentially be complex response functions but for simplicity we won't consider that possibility.

In conductors there can be a phase shift between electric field and free current, so Ohm's law becomes:

$$\mathbf{J}_f = \tilde{\sigma}_f \mathbf{E} \quad (17)$$

Here I've labeled the complex conductivity  $\tilde{\sigma}_f$  because as we will see below, in insulators there is an analogous relationship between  $\mathbf{J}_{polar}$  and  $\mathbf{E}$ .

As we saw in an early 442 homework problem, within the context of the scattering model (scattering time  $\tau$ ), the complex conductivity has this frequency dependence:

$$\tilde{\sigma}_f = \frac{\sigma_0}{1 - i\omega\tau} \quad (18)$$

### The damped wave equation: Griffiths (9.124)<sup>1</sup>

In my opinion Griffiths Section 9.4.1 is one of the most problematic sections of the entire textbook, which is a large part of what motivated this handout. But the derivation of the damped wave equation (9.124) is good. Start with Maxwell's equations in a material, assume  $\rho_f = 0$ , and incorporate Ohm's law as follows:

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<sup>1</sup> All equation numbers refer to the 5th edition.

$$\begin{aligned}
& \nabla \cdot \mathbf{D} = \rho_f \\
\rightarrow & \nabla \cdot (\tilde{\epsilon} \mathbf{E}) = 0 \\
\rightarrow & \nabla \cdot \mathbf{E} = 0
\end{aligned} \tag{19}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{20}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{21}$$

$$\begin{aligned}
& \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \\
\rightarrow & \nabla \times \left( \frac{\mathbf{B}}{\mu} \right) = \tilde{\sigma}_f \mathbf{E} + \tilde{\epsilon} \frac{\partial \mathbf{E}}{\partial t} \\
\rightarrow & \nabla \times \mathbf{B} = \mu \tilde{\sigma}_f \mathbf{E} + \mu \tilde{\epsilon} \frac{\partial \mathbf{E}}{\partial t}
\end{aligned} \tag{22}$$

Taking the curl of either Faraday's law (or Ampere's law) leads to the following equation for  $\mathbf{E}$  (or  $\mathbf{B}$ ), which is the damped wave equation, Griffiths (9.124).

$$\nabla^2 \mathbf{E} = \mu \tilde{\epsilon} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu \tilde{\sigma}_f \frac{\partial \mathbf{E}}{\partial t} \tag{23}$$

Assuming plane wave solutions with complex  $\tilde{k}$ , the  $\nabla^2$  operator produces a  $-\tilde{k}^2$  and the time derivatives each produce a  $-i\omega$ , so the damped wave equation leads to this equation, which is Griffiths (9.126).

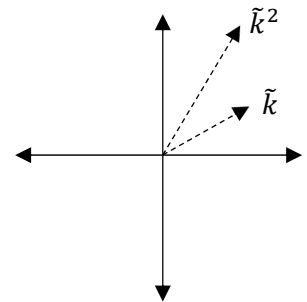
$$\boxed{\tilde{k}^2 = \mu \tilde{\epsilon} \omega^2 + i\mu \tilde{\sigma}_f \omega} \tag{24}$$

HOWEVER, in deriving this equation Griffiths apparently forgets that  $\tilde{\epsilon}$  and  $\tilde{\sigma}_f$  are in general complex quantities and presents them without tildes; and in analyzing this equation he assumes that the first term is always real and the second term always imaginary. This is incorrect—in general the first term could have both real and imaginary parts to it if  $\tilde{\epsilon} = 1 + \tilde{\chi}_e$  is complex due to a phase shift between  $\mathbf{E}$  and  $\mathbf{P}$ , and the second term could similarly have both imaginary and real parts to it if  $\tilde{\sigma}_f$  is complex due to a phase shift between  $\mathbf{E}$  and  $\mathbf{J}_f$ .

### Solving Griffiths Eq. 9.126 when $\tilde{k}^2$ is in the first quadrant

The square root of a complex number is given by taking the square root of its magnitude and dividing its angle in half. (Minor detail that each number actually has two different square roots, like  $\sqrt{3}$  and  $-\sqrt{3}$  are each square roots of the number 3, but let's not worry about that.)

If we assume, like Griffiths does, that the first term in (9.126) is purely real and the second term is purely imaginary, then  $\tilde{k}^2$  is in the first quadrant and taking the square root of  $\tilde{k}^2$  in that fashion yields these answers which are Griffiths (9.128):



$$\begin{aligned}
 k_{real} &= \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma_f}{\epsilon\omega}\right)^2} + 1 \right]^{\frac{1}{2}} \\
 k_{imag} &= \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma_f}{\epsilon\omega}\right)^2} - 1 \right]^{1/2}
 \end{aligned}
 \tag{25}$$

Important: If  $\tilde{k}^2$  is not in the first quadrant, then you can't use these equations (Griffiths (9.128)).

### Solving Griffiths Eq. 9.126: three limiting cases

The textbook *Foundations of Electromagnetic Theory* by Reitz, Milford, and Christy (RMC), 4<sup>th</sup> edition, page 428, gives situations where the following three approximations may be employed and says “Most often one or another of the following approximations are valid.”

Case 1.  $\tilde{k}^2$  is mostly real, and positive: insulators away from resonance, and conductors above the plasma frequency

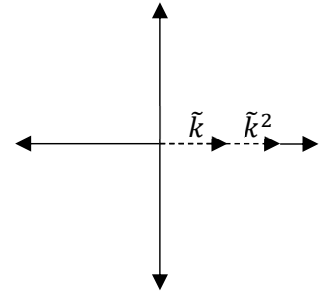
For insulators far from their resonant frequencies, the first term of Griffiths (9.126) will be much larger than the second term and  $\tilde{\epsilon}$  will be a positive, real number (so I will drop the tilde). Then we have:

$$\begin{aligned}
 \tilde{k}^2 &= \mu\epsilon\omega^2 + i(\text{small}) \\
 \tilde{k}^2 &\approx \mu\epsilon\omega^2
 \end{aligned}
 \tag{26}$$

In this case  $\tilde{k}^2$  is on the x-axis of the complex plane, and its square root  $\tilde{k}$  is also on the x-axis (real),  $\tilde{k} \approx \sqrt{\mu\epsilon\omega^2}$ .

That means:

$$\begin{aligned}
 k_{real} &\approx \sqrt{\mu\epsilon\omega^2} \\
 k_{imag} &\approx 0
 \end{aligned}
 \tag{27}$$



In other words, in this case  $\tilde{k}$  is nearly purely real, and positive; which therefore means that  $\epsilon_r$  and  $\tilde{n}$  are also both nearly purely real, and positive. For an insulator the scattering time  $\tau$  will also be quite short, so  $\tilde{\sigma}_f$  will be mostly real and we can drop the tilde. The condition that the first term is much larger than the second term means that  $\sigma_f \ll \epsilon\omega$ , which in some sense the definition of what it means to be an insulator.

This case also holds for conductors at frequencies above the so-called “plasma frequency”, typically in the UV, when they become transparent (see the Lorentz oscillator model handout).

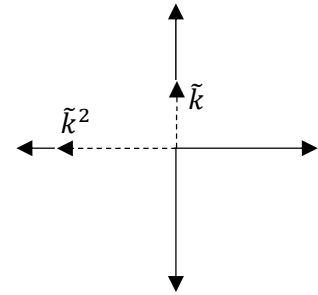
Case 2.  $\tilde{k}^2$  is mostly real, and negative: conductors at optical frequencies.

Remember how I said that whereas  $\tilde{k}$  and  $\tilde{n}$  must be in the first quadrant,  $\tilde{\epsilon}_r$  can be either in the first or second quadrants? Well, this case is an example of it being in the second quadrant. As we will see in the Lorentz oscillator model handout, for conductors in the near infrared or visible light range (which are high frequencies, but below the plasma frequency),  $\tilde{\epsilon}$  can become a negative real number (so I will drop the tilde again).

If the first term in Griffiths (9.126) is much larger in magnitude than the second term, but NEGATIVE (because  $\epsilon$  is negative), then:

$$\begin{aligned}\tilde{k}^2 &= -\mu|\epsilon|\omega^2 + i(\text{small}) \\ \tilde{k}^2 &\approx -\mu|\epsilon|\omega^2\end{aligned}\tag{28}$$

Then  $\tilde{k}^2$  is on the negative x-axis, and its square root  $\tilde{k}$  is on the y-axis (purely imaginary). Specifically,  $\tilde{k} \approx i\sqrt{\mu|\epsilon|\omega^2}$ .



That means:

$$\boxed{\begin{aligned}k_{real} &\approx 0 \\ k_{imag} &\approx \sqrt{\mu|\epsilon|\omega^2}\end{aligned}}\tag{29}$$

Since  $\tilde{k}$  is nearly purely imaginary;  $\tilde{n}$  is also nearly purely imaginary.

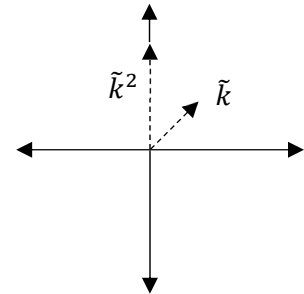
To have the first term be much larger in magnitude than the second term requires  $|\tilde{\sigma}_f| \ll |\epsilon|\omega$ . While having a small  $|\tilde{\sigma}_f|$  doesn't seem like a conductor, recall that this is true for optical and near IR frequencies, which are large. In other words, it's not so much that  $|\tilde{\sigma}_f|$  is small, but rather that  $\omega$  is large.

Case 3.  $\tilde{k}^2$  is mostly imaginary: conductors in microwave and lower frequencies.

For conductors in the microwave and lower frequencies,  $\tilde{\sigma}_f$  will be mostly real (so we can lose the tilde), and  $\sigma_f \gg |\tilde{\epsilon}|\omega$  because that's the defining feature of a conductor. Therefore the second term of Griffiths (9.126) will be much larger than the first term. Then we have:

$$\begin{aligned}\tilde{k}^2 &= (\text{small real}) + i\mu\sigma_f\omega \\ \tilde{k}^2 &\approx i\mu\sigma_f\omega\end{aligned}\tag{30}$$

Then  $\tilde{k}^2$  is on the y-axis, and its square root  $\tilde{k}$  is at 45°. Specifically,  $\tilde{k} \approx \sqrt{\frac{\mu\sigma_f\omega}{2}} + \sqrt{\frac{\mu\sigma_f\omega}{2}}i$ .



That means:

$$\boxed{\begin{aligned} k_{real} &\approx \sqrt{\frac{\mu\sigma_f\omega}{2}} \\ k_{imag} &\approx \sqrt{\frac{\mu\sigma_f\omega}{2}} \end{aligned}} \quad (31)$$

Since  $\tilde{k}$  is a complex number at  $45^\circ$ ,  $\tilde{n}$  is also a complex number at  $45^\circ$ .

According to RMC:

The dividing frequency between the latter two cases is  $\omega \approx 1/\tau$  where  $\tau$  is the collision [aka damping] time for the free electrons. For pure metals at room temperature,  $1/\tau \approx 10^{14} \text{ s}^{-1}$ . Physically, in cases (1) and (2) the displacement current predominates and in case (3) the conduction current predominates. Where the skin depth is important in electrical problems, usually case (3) applies... and the “wave” is very strongly attenuated in terms of the wavelength.

### An “Ohm’s law” for polarization current

Speaking of polarization current... the polarization current is defined as  $\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$ . Using  $\mathbf{P} = \epsilon_0 \tilde{\chi}_e \mathbf{E}$  we can rewrite this as:

$$\mathbf{J}_p = \epsilon_0 \tilde{\chi}_e \frac{\partial \mathbf{E}}{\partial t} \quad (32)$$

and since we have a harmonic time dependence,  $\frac{\partial}{\partial t} \rightarrow -i\omega$ :

$$\mathbf{J}_p = -i\epsilon_0 \tilde{\chi}_e \omega \mathbf{E} \quad (33)$$

Thus we have an Ohm’s law-type behavior where the induced current density is proportional to the electric field. We can therefore define the conductivity as the proportionality constant, which is:

$$\tilde{\sigma}_p = -i\epsilon_0 \tilde{\chi}_e \omega \quad (34)$$

where I have used the subscript  $p$  to differentiate it from the normal conductivity which is based on free current. This type of conductivity is only present with AC fields. Also, although it is complex, it is not as fundamental of a response function as  $\tilde{\chi}_e$ ,  $\tilde{\epsilon}_r$ , or  $\tilde{\sigma}_f$ , because its value is determined by  $\tilde{\chi}_e$  and  $\omega$ .

In general we could say that the total conductivity has components from both  $\tilde{\sigma}_f$  and  $\tilde{\sigma}_p$ :

$$\tilde{\sigma}_{tot} = \tilde{\sigma}_f + \tilde{\sigma}_p \quad (35)$$

But as far as I can tell this is not typically done, since polarization current (and hence  $\tilde{\sigma}_p$ ) is typically a feature of insulators whereas free current (and hence  $\tilde{\sigma}_f$ ) is typically a feature of conductors. At any rate Griffiths doesn’t talk about this second type of conductivity at all, although I did find some discussion of it in RMC and some sources just call  $\tilde{\sigma}_p$  “the conductivity” when it comes to insulators.