Magnetic Field via Relativity

by Dr. Colton, Physics 442 (last updated: Winter 2020)

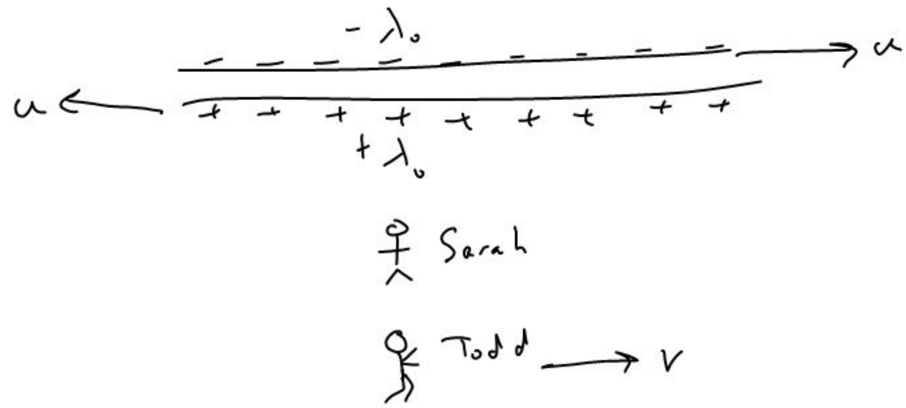
Background

This handout is similar to an example in Griffiths; however my notation is a little different to better match what we have been doing so far, so his equations may not be exactly the same.

Let's pretend magnetic fields and their associated forces had never been discovered. The point of this handout is to show that given an electric field in one frame and the knowledge of how the *forces* transform from frame to frame, would necessarily lead one to predict the existence of a magnetic field and the Lorentz force in a second frame.

The setup: moving line charges

The setup: suppose two infinite parallel line charges are very close to each other, one positively charged and the other negatively charged with charge densities λ_0 and $-\lambda_0$ in their own rest frames. Relative to Sarah, the top row of charges is moving to the right at speed u and the bottom row of charges is moving to the left also at speed u. Meanwhile, Todd is moving to the right relative to Sarah at speed v.



Charge densities

Sarah and Todd will measure higher charge densities than λ_0 due to Lorentz contraction.

<u>Sarah's charge densities</u>

$\lambda_{Sarah} = \gamma_u \lambda_0$ for each line (with one positive and the other negative), so

$$\lambda_{Sarah} = \frac{\lambda_0}{\sqrt{1-u^2/c^2}}$$ (Hold that thought for about a page.)

Because two the lines have equal and opposite charge densities, and assuming the lines are essentially infinitely close together, their fields cancel out and so there is no **E** field in her frame.

As a side note, because each line charge produces a leftward current of λu, Sarah will see a total current of:

$I_{Sarah} = -2\lambda_{Sarah}u$ (Hold that thought until the last section.)

Todd's charge densities

In Todd's frame, the two line charges are moving at different speeds. The $+\lambda$ line is bunched up more than the $-\lambda$ line, so he will see a net positive charge.

$-\lambda$ line: velocity is $\frac{u-v}{1-uv/c^2}$

gamma factor is $\frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{u-v}{1-uv/c^2}\right)^2}}$

charge density is $\frac{-\lambda_0}{\sqrt{1-\frac{1}{c^2}\left(\frac{u-v}{1-uv/c^2}\right)^2}}$

$+\lambda$ line: velocity is $-\frac{u+v}{1+uv/c^2}$

gamma factor is $\frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{u+v}{1+uv/c^2}\right)^2}}$

charge density is $\frac{+\lambda_0}{\sqrt{1-\frac{1}{c^2}\left(\frac{u+v}{1+uv/c^2}\right)^2}}$

The net charge density is the sum of the $-\lambda$ line and the $+\lambda$ line:

$$\lambda_{net,Todd} = \frac{-\lambda_0}{\sqrt{1-\frac{1}{c^2}\left(\frac{u-v}{1-uv/c^2}\right)^2}} + \frac{+\lambda_0}{\sqrt{1-\frac{1}{c^2}\left(\frac{u+v}{1+uv/c^2}\right)^2}}$$

We'll now skip about five lines of algebra where we could foil out the stuff in the square roots, add the 1's by including them via common denominators, flip the denominators of the square roots into the numerators of each term (at which time the two terms end up having the same denominators and can be combined), and simplify.

Result:

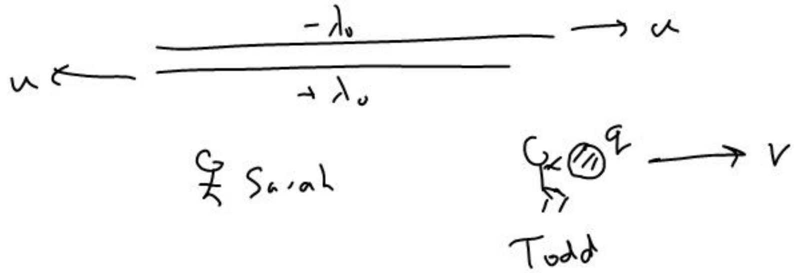
$$\lambda_{net,Todd} = \frac{2\lambda_0 uv/c^2}{\sqrt{(1-u^2/c^2)(1-v^2/c^2)}}$$

Remembering that $\lambda_{Sarah} = \frac{\lambda_0}{\sqrt{1-u^2/c^2}}$ (we were holding that thought), we can write:

$$\lambda_{net,Todd} = \frac{2\lambda_{Sarah} uv/c^2}{\sqrt{1-v^2/c^2}}$$

Todd with a charge

Now, let's say that Todd is holding a charge, q, as he runs past.



He sees an electric field from the line charges (acting like a single line charge with charge density $\lambda_{net,Todd}$), and so his charge experiences a force of $\mathbf{F} = q\mathbf{E}$.

From Gauss's law, the field from a line charge is known to be $\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$.

(Minor detail: Because the relativistic equations assume motion in the x-direction, I'm thinking of cylindrical coordinates in terms of the x-axis instead of the z-axis as is customary.)

The force experienced by q in Todd's frame is therefore:

$$\mathbf{F}_{Todd} = \frac{q}{2\pi\epsilon_0 s}\left(\frac{2\lambda_{Sarah} uv/c^2}{\sqrt{1 - v^2/c^2}}\right)\hat{\mathbf{s}}$$

Transformation of forces

What force does Sarah see? We have already worked out the equations for transformation of forces to answer this question. As a review, those equations were:

$$F_{x,Todd} = \frac{F_{x,Sarah} - \beta_v \mathbf{F}_{Sarah} \cdot \mathbf{u}}{1 - \beta_v u_{x,Sarah}/c}$$

$$F_{y,Todd} = \frac{F_{y,Sarah}}{\gamma_v(1 - \beta_v u_{x,Sarah}/c)}$$

$$F_{z,Todd} = \frac{F_{z,Sarah}}{\gamma_v(1 - \beta_v u_{x,Sarah}/c)}$$

To do the reverse transformation, i.e given Todd's force and wanting Sarah's force, you can just make all of the β_v's the negative of how they are listed. Since this force on q is in the s-direction, perpendicular to the relative velocities between the two people, either of the transverse equations could be used for s.

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$$F_{s,Sarah} = \frac{F_{s,Todd}}{\gamma_v(1 + \beta_v u_{x,Todd}/c)}$$

Todd is holding the charge at rest so $u_{x,Todd}$ (the velocity of the charge in Todd's frame) is zero, and the equation becomes.

$$F_{s,Sarah} = \frac{F_{s,Todd}}{\gamma_v}$$

We can plug in for $1/\gamma_v = \sqrt{1 - v^2/c^2}$ and for F_{Todd} as found above, and write:

$$F_{s,Sarah} = \sqrt{1 - v^2/c^2} \left(\frac{q}{2\pi\epsilon_0 s} \frac{2\lambda_{Sarah} uv/c^2}{\sqrt{1 - v^2/c^2}} \right)$$

Now simplify by cancelling the $\sqrt{1 - v^2/c^2}$ factor, and we have:

$$\mathbf{F}_{Sarah} = \frac{q}{2\pi\epsilon_0 s} \frac{2\lambda_{Sarah} uv}{c^2} \,\hat{\mathbf{s}}$$

As previously mentioned, there is no electric field in Sarah's frame. And yet the charge experiences a force! There must be some kind of <u>new field</u> which is producing this force!

The new field

Some facts about this force which Sarah sees the charge experience are apparent:

- It is proportional to the magnitude of the charge, q.
- It is proportional to the velocity of the charge, v.
- It is in the $\hat{\mathbf{s}}$ direction, which is perpendicular to the direction of the charge's velocity.

We can make two substitutions to simplify the force equation further, namely $I_{Sarah} = -2\lambda_{Sarah} u$ (remember I said to hold that thought) and $\mu_0 = 1/(\epsilon_0 c^2)$; then we have:

$$\mathbf{F}_{Sarah} = -qv \left(\frac{\mu_0 I_{Sarah}}{2\pi s} \right) \hat{\mathbf{s}}$$

Since $\hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}} = -\hat{\mathbf{s}}$, we can now define the new field as seen in Sarah's frame, along with the force that it produces on a charge q moving at speed v, to be:

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}}$$

and

$$\boxed{\mathbf{F}_{Sarah} = q\mathbf{v} \times \mathbf{B}}$$

Even had people never previously known about magnetic fields, they would have theoretically predicted them after Einstein's theory of relativity and the Lorentz transformations were used to derive the transformation of forces.