Complex wave number, index of refraction, and relative permittivity
by Dr. Colton, Physics 442 (last updated: Summer 2021)

Notation

When we start talking about *complex wave numbers*, we also soon have *complex indices of refraction* and *complex relative permittivities*, and there is a dizzying amount of varying—and overlapping!!—notation that people use for these three quantities. To be as clear as possible and avoid any ambiguity, I myself will use the following symbols in class and on homework assignments and exams.

$$\tilde{k} = \text{complex wave vector}$$

$$k_{real} = \text{real part of } \tilde{k}, \text{called } k \text{ in Griffiths and some other sources}$$

$$k_{imag} = \text{imaginary part of } \tilde{k}, \text{called } \kappa \text{ in Griffiths and some other sources}$$

$$\tilde{n} = \text{complex index of refraction, not directly used in Griffiths,}$$
$$\text{also called } \mathcal{N} \text{ by some sources (e.g. Peatross and Ware)}$$

$$n_{real} = \text{real part of } \tilde{n}, \text{also called } n \text{ by nearly all sources (including Griffiths)}$$

$$n_{imag} = \text{imaginary part of } \tilde{n}, \text{not directly used in Griffiths,}$$
$$\text{also called } k \text{ by nearly all sources (!)}$$
$$\text{and } \kappa \text{ by some (e. g. Peatross and Ware along with some others)}$$
$$\rightarrow \text{ to add insult to injury, many sources define the imaginary index of}$$
refraction k as the *negative* of n_{imag}. I believe this has to do with representing complex traveling waves as $e^{i(kz-\omega t)}$ vs $e^{i(\omega t - kz)}$. As near as I can tell, the former is used by most physicists and the latter by most engineers.

$$\tilde{\epsilon}_r = \text{complex relative permittivity aka complex dielectric constant}$$
$$\text{also called } \epsilon \text{ by many sources and } K \text{ or } \tilde{K} \text{ by others}$$

$$\epsilon_{r,real} = \text{real part of } \tilde{\epsilon}_r, \text{not directly used in Griffiths,}$$
$$\text{also called } \epsilon_r, \epsilon_1 \text{ or } \epsilon' \text{ by many sources and } K, K_\mathrm{r} \text{ or } K' \text{ by others}$$

$$\epsilon_{r,imag} = \text{imaginary part of } \tilde{\epsilon}_r, \text{not directly used in Griffiths,}$$
$$\text{also called } \epsilon_2 \text{ or } \epsilon'' \text{ by many sources and } K_\mathrm{i} \text{ or } K'' \text{ by others}$$

Basic relationships

The complex index of refraction $\tilde{n}$ and complex wave number $\tilde{k}$ are related through this equation:

$$\boxed{\tilde{n} = \frac{c}{\omega}\tilde{k}}$$

Equating the real and imaginary parts of that equation (and remembering that ω must be real) results in:

$$n_{real} = \frac{c}{\omega} k_{real}$$

$$n_{imag} = \frac{c}{\omega} k_{imag}$$

Note in particular that $\tilde{n}$ and $\tilde{k}$ will always have the same complex angle.

The complex relative permittivity $\tilde{\epsilon}_r$ and complex index of refraction $\tilde{n}$ are related through this equation:

$$\boxed{\tilde{\epsilon}_r = \tilde{n}^2}$$

Plugging in $\tilde{\epsilon}_r = \epsilon_{r,real} + i\epsilon_{r,imag}$ and $\tilde{n} = n_{real} + in_{imag}$ and equating the real and imaginary parts of that equation results in:

$$\epsilon_{r,real} = n_{real}^2 - n_{imag}^2$$

$$\epsilon_{r,imag} = 2n_{real}n_{imag}$$

(As per the "add insult to injury" comment above, you will see many sources write that last equation as $\epsilon'' = -2nk$.)

It's "left as an exercise for the reader" to derive the inverse equations:

$$n_{real} = \sqrt{\frac{|\tilde{\epsilon}_r| + \epsilon_{r,real}}{2}}$$

$$n_{imag} = \sqrt{\frac{|\tilde{\epsilon}_r| - \epsilon_{r,real}}{2}}$$

where the absolute values mean the magnitude of the complex number, i.e. $|\tilde{\epsilon}_r| = \sqrt{\epsilon_{r,real}^2 + \epsilon_{r,imag}^2}$.

The complex relative permittivity $\tilde{\epsilon}_r$ and complex wave number $\tilde{k}$ are related through this equation:

$$\boxed{\tilde{\epsilon}_r = \frac{c^2}{\omega^2} \tilde{k}^2}$$

Note in particular that $\tilde{\epsilon}_r$ and $\tilde{k}^2$ will always have the same complex angle.

Why is the complex wave number important?

The complex wave number governs the wavelength in the material as well as the decay of the wave's amplitude. Assuming travel in the z-direction for simplicity, plane waves have the following form:

$$\tilde{E} = E_0 e^{i(kz-\omega t)}$$

When the wave number is complex, this becomes:

$$\tilde{E} = E_0 e^{i(\tilde{k}z-\omega t)}$$
$$\tilde{E} = E_0 e^{i((k_{real}+ik_{imag})z-\omega t)}$$
$$\tilde{E} = E_0 e^{-k_{imag}z} e^{i(k_{real}z-\omega t)}$$

Taking the real part, we have:

$$E = E_0 e^{-k_{imag}z} \cos(k_{real}z - \omega t)$$

You can see from that, that the real part of $\tilde{k}$ is related to the oscillations in space as per the usual definition of wave number, through the $\cos(k_{real}z - \omega t)$ term:

$$\boxed{\lambda = \frac{2\pi}{k_{real}}}$$

The imaginary part of $\tilde{k}$ is related to how the amplitude of the wave decays as it penetrates the material, through the $e^{-k_{imag}z}$ term. The inverse of k_{imag} has units of meters, and is often called the *skin depth*, δ.

$$\boxed{\delta = \frac{1}{k_{imag}}}$$

The decay of the wave is also characterized by the *absorption coefficient*, α, which describes the fall off in intensity according to $I = I_0 e^{-\alpha z}$. The absorption coefficient is typically what you would be able to measure in an experiment. Because intensity varies with amplitude *squared*, we have:

$$\alpha = 2k_{imag}$$

Equations for how the absorption coefficient relates to complex $\tilde{n}$ and complex $\tilde{\epsilon}_r$ can be obtained via the "basic relationships" equations above; three particularly useful such equations are the following:

$$n_{imag} = \frac{c}{2\omega}\alpha$$
$$\epsilon_{r,real} = \frac{c^2}{\omega^2}k_{real}^2 - \frac{c^2}{4\omega^2}\alpha^2$$
$$\epsilon_{r.imag} = \frac{c^2}{\omega^2}k_{real}\alpha$$

Since the ratio c/ω equals $\lambda_{vacuum}/2\pi$, you'll also often see those equations written in terms of the vacuum wavelength, such as: $n_{imag} = \frac{\lambda_{vacuum}}{4\pi}\alpha$.

To sum up: a big piece of the picture is therefore, when there is absorption (non-zero α) complex numbers very naturally enter the equations.

Caution: what about complex impedance?

The equation for the complex index of refraction, $\tilde{n}$, was obtained simply via analogy with the equation from the non-absorptive case, $n = \frac{c}{\omega} k$, and led directly to a complex relative permittivity. However, this is not the only way to incorporate complex numbers (and hence absorption) into the equations. An alternate method is to use a complex conductivity instead of a complex relative permittivity. Typically either one or the other is done, but not both. Therefore we will assume for now that the conductivity from Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$ is a real quantity. However, there will be times, as in the Ohm's law homework problem from HW 1, where it is more natural to talk about a complex conductivity.

Griffiths section 9.4.1 and the connection to complex ϵ_r

In my opinion Griffiths section 9.4.1 (both editions) is unclear in many respects. Let's pick up the story with Eq. 9.124 (both editions), since everything seems OK until that point. Here's that equation:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \qquad \text{Eq. 9.124}$$

That equation comes from plugging plane wave solutions into the modified wave equations for $\mathbf{E}$ or $\mathbf{B}$. The modified wave equations in turn, come from Maxwell's equations for linear materials with $\rho_f = 0$, and with Ohm's law, $\mathbf{J_f} = \sigma\mathbf{E}$. The second term on the RHS (imaginary component) comes from the free current term in Ampere's law, and the first term on the RHS (real component) comes from the displacement current term in Ampere's law.

<u>Caution</u>: in this equation, ϵ ($= \epsilon_0\epsilon_r$) and σ are both considered to be real numbers. In other words, in that equation you must NOT use the complex ϵ_r nor the complex σ that we have just been talking about. This confused me for several years and my earlier versions of this handout got it wrong!

For simplicity, let's assume $\mu = \mu_0$, write μ_0 as $\frac{1}{\epsilon_0 c^2}$, and write ϵ as $\epsilon_0\epsilon_{r,real}$. Griffiths Eq. 9.124 can then be written as the following:

$$\tilde{k}^2 = \frac{\omega^2}{c^2}\epsilon_{r,real} + \frac{i\sigma\omega}{\epsilon_0 c^2}$$
$$\tilde{k}^2 = \frac{\omega^2}{c^2}\left(\epsilon_{r,real} + i\frac{\sigma}{\epsilon_0\omega}\right)$$

Since in the absence of absorption (complex numbers), the stuff in the parentheses would simply be ϵ_r, another way to incorporate complex numbers into the equations, and hence the effects of absorption, is to *define* the complex relative permittivity as the following:

$$\boxed{\tilde{\epsilon}_r = \epsilon_{r,real} + i\frac{\sigma}{\epsilon_0\omega}}$$

The imaginary part of the complex permittivity can then be read off the equation as:

$$\epsilon_{r,imag} = \frac{\sigma}{\epsilon_0 \omega}$$

where to reiterate, σ must be a *real* conductivity. You will see this equation given by a lot of sources. Or, combining with the previous equation, $\epsilon_{r.imag} = \frac{c^2}{\omega^2} k_{real}\alpha$, we can write it in another potentially useful form:

$$\sigma = \frac{k_{real}\alpha}{\mu_0 \omega}$$

This therefore is the connection between absorption and conductivity: if a material is not at all conductive at a given frequency, it will also not have any absorption, nor will it have imaginary components of the relative permittivity, index of refraction, nor wave number.

Solving Griffiths Eq. 9.124

The square root of a complex number is given by taking the square root of its magnitude and dividing its angle in half. (Minor detail that each number actually has two different square roots, like $\sqrt{3}$ and $-\sqrt{3}$ are each square roots of the number 3, but let's not worry about that.) Griffiths takes the square root of Eq. 9.124 in that fashion and arrives at these answers, Eq. 9.126:

$$k_{real} = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1\right]^{1/2}$$

$$k_{imag} = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{1/2}$$

Caution: this equation assumes that $\tilde{k}^2$ is in the first quadrant, which also forces its square root $\tilde{k}$ to be in the first quadrant, as depicted in the complex plane figure on the right. However, as is discussed in limiting case 2 below, that is not always the case. **If $\widetilde{k^2}$ is not in the first quadrant, then you can't use Griffiths' Eq. 9.126.**
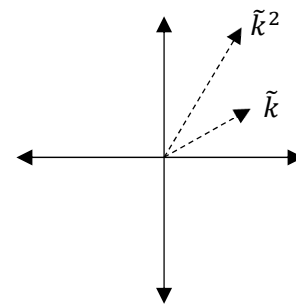
Three limiting cases

The textbook *Foundations of Electromagnetic Theory* by Reitz, Milford, and Christy (RMC), 4th edition, page 428, gives situations where the following three approximations may be employed and says "Most often one or another of the following approximations are valid." As a reminder, here is Eq. 9.124 again.
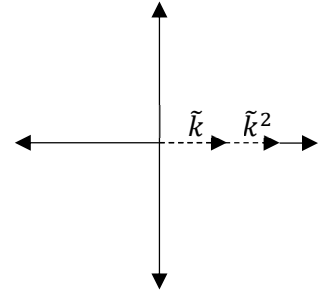
$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

<u>Case 1</u>. $\tilde{k}^2$ is mostly real, and positive.

If the real term is much larger than the imaginary term, then:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i(\text{small})$$
$$\tilde{k}^2 \approx \mu\epsilon\omega^2$$

In this case $\tilde{k}^2$ is on the x-axis of the complex plane, and its square root $\tilde{k}$ is also on the x-axis (real).

$$\tilde{k} \approx \sqrt{\mu\epsilon\omega^2}$$

That means:

$$\boxed{\begin{aligned} k_{real} &\approx \sqrt{\mu\epsilon\omega^2} \\ k_{imag} &\approx 0 \end{aligned}}$$

In other words, $\tilde{k}$ is nearly purely real, and positive; which therefore means that $\tilde{\epsilon}_r$ and $\tilde{n}$ are also both nearly purely real, and positive. This is the situation for insulators down to very low frequencies (essentially DC). The definition of an insulator is that $\sigma \ll \epsilon\omega$, assuming ϵ_r is positive.

<u>Case 2</u>. $\tilde{k}^2$ is mostly real, and negative.

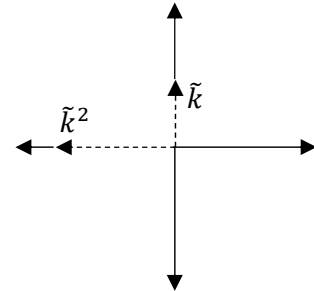If the real term is much larger than the imaginary term, but NEGATIVE (because ϵ_r is negative), then:

$$\tilde{k}^2 = -\mu|\epsilon|\omega^2 + i(\text{small})$$
$$\tilde{k}^2 \approx -\mu|\epsilon|\omega^2$$

Then $\tilde{k}^2$ is on the negative x-axis, and its square root $\tilde{k}$ is on the y-axis (purely imaginary). Specifically:

$$\tilde{k} \approx i\sqrt{\mu|\epsilon|\omega^2}$$

That means:

$$\boxed{\begin{aligned} k_{real} &\approx 0 \\ k_{imag} &\approx \sqrt{\mu|\epsilon|\omega^2} \end{aligned}}$$

Since $\tilde{k}$ is nearly purely imaginary; $\tilde{n}$ is also nearly purely imaginary. To have this case, we need $\sigma \ll |\epsilon|\omega$, and ϵ_r is negative. While having a small σ doesn't seem like a conductor, one can show that this case is typically true for conductors at frequencies in the upper infrared or higher. In other words, it's not so much that σ is small, but rather that ω is large.

<u>Case 3</u>. $\tilde{k}^2$ is mostly imaginary.

If the imaginary term is much larger than the real term, then:

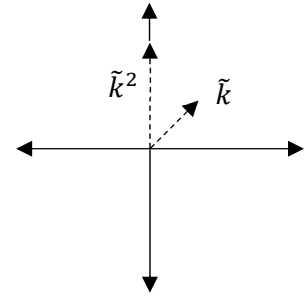$$\tilde{k}^2 = (\text{small real}) + i\mu\sigma\omega$$

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$$\tilde{k}^2 \approx i\mu\sigma\omega$$

Then $\tilde{k}^2$ is on the y-axis, and its square root $\tilde{k}$ is at 45°. Specifically:

$$\tilde{k} \approx \sqrt{\frac{\mu\sigma\omega}{2}} + \sqrt{\frac{\mu\sigma\omega}{2}}\, i$$

That means:

$$\boxed{\begin{aligned} k_{real} &\approx \sqrt{\frac{\mu\sigma\omega}{2}} \\[1em] k_{imag} &\approx \sqrt{\frac{\mu\sigma\omega}{2}} \end{aligned}}$$

Since $\tilde{k}$ is a complex number at 45°, $\tilde{n}$ is also a complex number at 45°. To have this case, we need $\sigma \gg \epsilon\omega$ (i.e. small frequencies). This is typically true for conductors at microwave frequencies and below.

According to RMC:

> The dividing frequency between the latter two cases is $\omega \approx 1/\tau$ where τ is the collision [aka damping] time for the free electrons. For pure metals at room temperature, $1/\tau \approx 10^{14}$ s^{-1}. Physically, in cases (1) and (2) the displacement current predominates and in case (3) the conduction current predominates. Where the skin depth is important in electrical problems, usually case (3) applies... and the "wave" is very strongly attenuated in terms of the wavelength.